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Certain Properties of Fractional Differential Variation Function

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Abstract: Prajapati and Kachhia introduced the concept of functions of bounded fractional differential variation and posed an open question: "Is the class of functions of bounded fractional differential variation on a closed bounded interval a Banach algebra?" The authors provided a solution to this problem in their previous work titled "Functions of bounded fractional differential variation: A new Banach algebra". This paper continues the work initiated in the aforementioned article. In this study, the authors define a fractional differential variation function for a function f in the space $\mathrm{BFDV}[a,b]$. Additionally, we discuss several inequalities within $\mathrm{BFDV}[a,b]$, which are utilized to explore various properties of the fractional differential variation function that are inherited from the Caputo fractional derivative of its parent function.

Keywords: Functions of bounded variation, Caputo fractional derivatives, functions of bounded fractional differential variation, variation function, fractional differential variation function.

1 Introduction

The concept of a function of bounded variation was introduced by C. Jordan in 1881. Functions of bounded variation have garnered significant interest across various fields, including real analysis, functional analysis, measure theory, integration theory, operator theory, Fourier analysis, nonlinear analysis, and some branches of mathematical physics.

For a function f that is of bounded variation on the interval $[a,b]$, the variation function, $\mathrm{V}_f(x)$, is defined as the total variation of f over the interval $[a,x]$. Huggins [3] and Russell [4] have extensively studied the variation function. The variation function plays a vital role in understanding the Jordan Decomposition Theorem, which states that any real function of bounded variation can be expressed as the difference of two increasing functions.

In 2016, Prajapati and Kachhia [1] introduced a new extension of Jordan's original definition of functions of bounded variation to Caputo fractional differentiable functions. Prajapati and Kachhia [1] defined functions of bounded fractional differential variation. The class of functions of bounded fractional differential variation on a closed bounded interval $[a,b]$ form a Banach algebra (Kumar and Prajapati [2]).

In this paper, the concepts of fractional differential variation function and differential variation function are introduced in section 2. Section 3 discusses various inequalities that are important for examining the properties of the fractional differential variation function, such as Lipschitz continuity, absolute continuity, and Hölder continuity, all of which are addressed in section 4.

Notations :
P : A partition of $[a,b]$
${}^cD_a^{\alpha} f$: Caputo fractional derivative of order α on $[a,b]$
$\mathrm{C}[a,b]$: The class of real valued continuous functions on $[a,b]$
$\mathrm{V}(f,P)$: The variation of f with respect to P
$\mathrm{V}(f,[a,b])$: The total variation of f on $[a,b]$
V_f : The variation function
$\mathrm{BV}[a,b]$: The class of functions of bounded variation on $[a,b]$.
$\mathrm{FDV}_l(f,P)$: The left fractional differential variation of f with respect to P

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$\mathrm{FDV}_r(f,P)$: The right fractional differential variation of f with respect to P
$\mathrm{FDV}_l(f,[a,b])$: The total left fractional differential variation of f on $[a,b]$
$\mathrm{FDV}_r(f,[a,b])$: The total right fractional differential variation of f on $[a,b]$
$\mathrm{FDV}(f,[a,b])$: The total fractional differential variation of f on $[a,b]$
$\mathrm{BFDV}[a,b]$: The class of functions of bounded fractional differential variation on $[a,b]$
FDV^f : The fractional differential variation function

Definition 1. *(Caputo [5], Podlubny [6], Kilbas et al. [7]) Let $\alpha \in \mathbb{C}$ and $\Re(\alpha) \geq 0$. If $n-1 < \Re(\alpha) < n$ and $f \in \mathrm{AC}^n[a,b]$, then* Caputo fractional derivative *of order α of f on $[a,b]$ is defined as*

$$({}^cD_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (n-1 < \Re(\alpha) < n).$$

The following mean value theorem for Caputo fractional derivatives will be useful for further study.

Theorem 1. *(Odibat and Shawagfeh [8], Diethelm [9]) Let $0 < \alpha \leq 1$ and $f \in \mathrm{C}[a,b]$ be such that ${}^cD_a^\alpha f \in \mathrm{C}[a,b]$. Then, there exists some $c \in [a,b]$ such that*

$$\Gamma(\alpha+1)\left(\frac{f(b)-f(a)}{(b-a)^\alpha}\right) = ({}^cD_a^\alpha f)(c). \tag{1}$$

Functions of bounded variation is defined in ([10], [11], [12]). $\mathrm{BV}[a,b]$ is a Banach algebra relative to the norm $||f||_{\mathrm{BV}} = ||f||_\infty + \mathrm{V}(f)$ (Carothers [11] and Appell *et al.* [12])

For $f \in \mathrm{BV}[a,b]$, define the *variation function* [3,4] $\mathrm{V}_f : [a,b] \to \mathbb{R}$ by

$$\mathrm{V}_f(x) = \mathrm{V}(f,[a,x]). \tag{2}$$

Definition 2. *(Appell et al. [12]) $f : [a,b] \to \mathbb{R}$ is said to be* Lipschitz continuous *on $[a,b]$ if there exists a constant $M > 0$ such that*

$$|f(x) - f(y)| \leq M|x-y|,$$

for all $x,y \in [a,b]$.

Definition 3. *(Appell et al. [12]) $f : [a,b] \to \mathbb{R}$ is said to be* Hölder continuous *if there exists constant $M > 0$ and $\beta > 0$ such that*

$$|f(x) - f(y)| \leq M|x-y|^\beta,$$

for all $x,y \in [a,b]$.

Definition 4. *(Bhatt et al. [13]) Let $f : [a,b] \to \mathbb{R}$ be a function and $P := \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Define the* left differential variation *of f with respect to P by*

$$\mathrm{DV}_l(f,P) = \sum_{k=1}^{n} \left| \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - f'(x_{k-1}) \right|,$$

and the right differential variation *of f with respect to P by*

$$\mathrm{DV}_r(f,P) = \sum_{k=1}^{n} \left| \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - f'(x_k) \right|.$$

Define the total left differential variation *of f on $[a,b]$ by*

$$\mathrm{DV}_l(f) = \sup\{\mathrm{DV}_l(f,P) : \textit{P a partition of [a,b]}\},$$

and the total right differential variation *of f on $[a,b]$ by*

$$\mathrm{DV}_r(f) = \sup\{\mathrm{DV}_r(f,P) : \textit{P a partition of [a,b]}\}.$$

f is said to be of bounded left differential variation *on $[a,b]$ if $\mathrm{DV}_l(f) < \infty$ and f is said to be of* bounded right differential variation *on $[a,b]$ if $\mathrm{DV}_r(f) < \infty$. f is said to be of* bounded differential variation *on $[a,b]$ if $\mathrm{DV}_l(f) < \infty$ and $\mathrm{DV}_r(f) < \infty$.*

The total differential variation *of f on $[a,b]$ is defined as*

$$\mathrm{DV}(f) = \mathrm{DV}_l(f) + \mathrm{DV}_r(f).$$

$\mathrm{BDV}[a,b]$ *denote the class of functions of bounded differential variation on $[a,b]$.*

Instead of using first-order derivatives, Prajapati and Kachhia [1] utilized Caputo fractional derivatives to define the function of bounded fractional differential variation as follows:

The real number α satisfying $0 < \alpha \leq 1$ is fixed throughout the paper.

Definition 5. *(Prajapati and Kachhia [1]) Let* $0 < \alpha \leq 1$. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a Caputo fractional differentiable function and* $P := \{x_0, x_1, \ldots, x_n\}$ *be a partition of* $[a,b]$. *The* left fractional differential variation *of* f *with respect to* P *is defined as*

$$\mathrm{FDV}_l(f,P) = \sum_{k=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_k) - f(x_{k-1})}{(x_k - x_{k-1})^{\alpha}} \right) - ({}^{c}D_a^{\alpha} f)(x_{k-1}) \right|,$$

and the right fractional differential variation *of* f *with respect to* P *is defined as*

$$\mathrm{FDV}_r(f,P) = \sum_{k=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_k) - f(x_{k-1})}{(x_k - x_{k-1})^{\alpha}} \right) - ({}^{c}D_a^{\alpha} f)(x_k) \right|.$$

The total left fractional differential variation *of* f *on* $[a,b]$ *is defined as*

$$\mathrm{FDV}_l(f) = \sup\{\mathrm{FDV}_l(f,P) : \text{ P a partition of [a,b]}\},$$

and the total right fractional differential variation *of* f *on* $[a,b]$ *is defined as*

$$\mathrm{FDV}_r(f) = \sup\{\mathrm{FDV}_r(f,P) : \text{ P a partition of [a,b]}\}.$$

f *is said to be of* bounded left fractional differential variation *on* $[a,b]$ *if* $\mathrm{FDV}_l(f) < \infty$. *Similarly,* f *is said to be of* bounded right fractional differential variation *on* $[a,b]$ *if* $\mathrm{FDV}_r(f) < \infty$. f *is said to be of* bounded fractional differential variation *of on* $[a,b]$ *if* $\mathrm{FDV}_l(f) < \infty$ *and* $\mathrm{FDV}_r(f) < \infty$. *The* total fractional differential variation *of* f *on* $[a,b]$ *is defined as*

$$\mathrm{FDV}(f) = \mathrm{FDV}_l(f) + \mathrm{FDV}_r(f). \tag{3}$$

$\mathrm{BFDV}[a,b]$ *denote the class of functions of bounded fractional differential variation on* $[a,b]$ *for all* α, $0 < \alpha \leq 1$.

Remark. For $\alpha = 1$, we have $\mathrm{FDV}_l(f,[a,b]) = \mathrm{DV}_l(f,[a,b])$, $\mathrm{FDV}_r(f,[a,b]) = \mathrm{DV}_r(f,[a,b])$, and $\mathrm{FDV}(f,[a,b]) = \mathrm{DV}(f,[a,b])$.

The results from Proposition 1 to Lemma 2 are important for further study.

Proposition 1. *(Kumar and Prajapati [2]) If* $f\colon [a,b] \to \mathbb{R}$ *is a Caputo fractional differentiable function of order* α, $0 < \alpha \leq 1$, *and* $r \in \mathbb{R}$, *then*

$$\mathrm{FDV}(rf,[a,b]) = |r|\,\mathrm{FDV}(f,[a,b]).$$

Proposition 2. *(Kumar and Prajapati [2]) If* $f\colon [a,b] \to \mathbb{R}$ *is a Caputo fractional differentiable function of order* α, $0 < \alpha \leq 1$, *and* $a < c < b$, *then*

$$\mathrm{FDV}(f,[a,b]) = \mathrm{FDV}(f,[a,c]) + \mathrm{FDV}(f,[c,b]).$$

Lemma 1. *(Kumar and Prajapati [2]) If* $f\colon [a,b] \to \mathbb{R}$ *is a Caputo fractional differentiable function of order* α, $0 < \alpha \leq 1$, *then*

$$\mathrm{V}({}^{c}D_a^{\alpha} f,[a,b]) \leq \mathrm{FDV}(f,[a,b]).$$

Lemma 2. *(Kumar and Prajapati [2]) If* $f\colon [a,b] \to \mathbb{R}$ *is a Caputo fractional differentiable function of order* α, $0 < \alpha \leq 1$, *then*

$$\mathrm{FDV}_l(f) \leq \mathrm{V}({}^{c}D_a^{\alpha}(f))$$

and

$$\mathrm{FDV}_r(f) \leq \mathrm{V}({}^{c}D_a^{\alpha}(f)).$$

2 Fractional Differential Variation Function

In this section, the authors define the fractional differential variation function and the differential variation function. These functions are useful for studying the properties of functions of bounded fractional differential variation. Refer to the following Jordan-type Decompositions for applications of these functions.

Theorem 2. *(Kumar and Prajapati [14]) f is a function of bounded fractional differential variation on $[a,b]$ if and only if ${}^cD_a^\alpha f$ can be represented in the form ${}^cD_a^\alpha f = g - h$, where both g and h are monotonically increasing functions.*

Theorem 3. *(Kumar and Prajapati [14]) If f is a function of bounded fractional differential variation on $[a,b]$, then there exists monotonically non-decreasing functions p and q on $[a,b]$ such that*

(a) ${}^cD_a^\alpha f = p - q$
(b) $\mathrm{FDV}^f = p + q$.

Definition 6. *Let f be a function of bounded fractional differential variation on $[a,b]$. The* left fractional differential variation function *and* right fractional differential variation function*, $\mathrm{FDV}_l^f : [a,b] \to \mathbb{R}$ and $\mathrm{FDV}_r^f : [a,b] \to \mathbb{R}$, are defined by*

$$\mathrm{FDV}_l^f(x) = \mathrm{FDV}_l(f,[a,x]) \quad \text{and} \quad \mathrm{FDV}_r^f(x) = \mathrm{FDV}_r(f,[a,x]),$$

respectively.

The fractional differential variation function*, $\mathrm{FDV}^f : [a,b] \to \mathbb{R}$, is defined by*

$$\mathrm{FDV}^f(x) = \mathrm{FDV}_l^f(x) + \mathrm{FDV}_r^f(x).$$

Substituting $\alpha = 1$ in Definition 6 gives Definition 7.

Definition 7. *Let $f \in \mathrm{BDV}[a,b]$. The* left differential variation function *and* right differential variation function*, $\mathrm{DV}_l^f : [a,b] \to \mathbb{R}$ and $\mathrm{DV}_r^f : [a,b] \to \mathbb{R}$, are defined by*

$$\mathrm{DV}_l^f(x) = \mathrm{DV}_l(f,[a,x]) \quad \text{and} \quad \mathrm{DV}_r^f(x) = \mathrm{DV}_r(f,[a,x]),$$

respectively.

The differential variation function*, $\mathrm{FDV}^f : [a,b] \to \mathbb{R}$, is defined by*

$$\mathrm{DV}^f(x) = \mathrm{DV}_l^f(x) + \mathrm{DV}_r^f(x).$$

For $\alpha = 1$, examples of fractional differential variation function are constructed in Examples 1 and 2.

Example 1. Let $a > 0$. We have (see, e.g., [[15], Example 3.5])

$$\mathrm{DV}(\frac{1}{x},[a,b]) = \frac{1}{a^2} - \frac{1}{b^2},$$

for $f(x) = \frac{1}{x}$ defined on $[a,b]$, we get

$$\mathrm{DV}^f(x) = \frac{1}{a^2} - \frac{1}{x^2}.$$

Example 2. We have (see, e.g., [[15], Example 3.6])

$$\mathrm{DV}(x^n,[a,b]) = n\left|b^{n-1} - a^{n-1}\right|,$$

for $f(x) = x^n$ defined on $[a,b]$, we get

$$\mathrm{DV}^f(x) = n\left|x^{n-1} - a^{n-1}\right|.$$

3 Some inequalities in $\mathrm{BFDV}[a,b]$

This section discusses various inequalities. These inequalities are helpful to study various properties (Lipschitz, absolute, Hölder continuous) of FDV^f discussed in section 4. Proofs of Lemma 3 and Lemma 4 are left to the reader. Lemma 3 shows that total fractional differential variation, FDV, is monotonically increasing with respect to intervals and is proved using the fact that FDV is additive with respect to intervals. Lemma 3 is used to show that the fractional differential variation function, FDV^f, is a monotonically increasing function in Lemma 4.

Lemma 3. *Let $f\colon [a,b]\to\mathbb{R}$ be a Caputo fractional differentiable function of order α, $0<\alpha\leq 1$ and $[c,d]\subseteq[a,b]$. Then*

$$\mathrm{FDV}(f,[c,d])\leq \mathrm{FDV}(f,[a,b]).$$

Lemma 4. *If $f\in \mathrm{BFDV}[a,b]$, then FDV^f is a monotonically increasing function.*

For $f\in \mathrm{BV}[a,b]$, f is bounded by $|f(a)|+\mathrm{V}(f,[a,b])$ [[12], Chapter 1]. This property of the total variation gives a bound on ${}^cD_a^\alpha f$ for $f\in \mathrm{BFDV}[a,b]$.

Proposition 3.. *If $f\in \mathrm{BFDV}[a,b]$, then ${}^cD_a^\alpha f$ is bounded with*

$$||{}^cD_a^\alpha f||_\infty \leq \mathrm{FDV}(f,[a,b]).$$

Proof. The total variation of ${}^cD_a^\alpha f$ on $[a,b]$ gives

$$||{}^cD_a^\alpha f||_\infty \leq |{}^cD_a^\alpha f(a)|+\mathrm{V}({}^cD_a^\alpha f,[a,b]),$$

since ${}^cD_a^\alpha f(a)=0$,

$$||{}^cD_a^\alpha f||_\infty \leq \mathrm{V}({}^cD_a^\alpha f,[a,b]),$$

using Lemma 1,

$$||{}^cD_a^\alpha f||_\infty \leq \mathrm{FDV}(f,[a,b]).$$

The variation function V_f is already studied in detail ([3], [4]). Propositions 4 and 5 establish relationships between the variation function V_f and the fractional differential variation function FDV^f using inequalities. The variation function V_f becomes important for studying the fractional differential variation function FDV^f.

Proposition 4. *If $f\in \mathrm{BFDV}[a,b]$, then for all $x,y\in[a,b]$*

$$|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)|\leq 2|\mathrm{V}_{{}^cD_a^\alpha f}(x)-\mathrm{V}_{{}^cD_a^\alpha f}(y)|.$$

Proof. Without loss of generality assume that $y<x$. Since
$\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)$ is a non-negative quantity, we have

$$\begin{aligned}|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| &= \mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)\\ &= \mathrm{FDV}(f,[a,x])-\mathrm{FDV}(f,[a,y]),\end{aligned}$$

from Proposition 2, it follows that

$$\begin{aligned}|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| &= \mathrm{FDV}(f,[a,y])+\mathrm{FDV}(f,[y,x])-\mathrm{FDV}^f(f,[a,y])\\ &= \mathrm{FDV}(f,[y,x])\\ &= \mathrm{FDV}_l(f,[y,x])+\mathrm{FDV}_r(f,[y,x]),\end{aligned}$$

using Lemma 2, we have

$$\begin{aligned}|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| &\leq \mathrm{V}({}^cD_a^\alpha f,[y,x])+\mathrm{V}({}^cD_a^\alpha f,[y,x])\\ &= 2\,\mathrm{V}({}^cD_a^\alpha f,[y,x])\\ &= 2\Big(\mathrm{V}({}^cD_a^\alpha f,[a,y])+\mathrm{V}({}^cD_a^\alpha f,[y,x])-\mathrm{V}({}^cD_a^\alpha f,[a,y])\Big),\end{aligned}$$

the equation $\mathrm{V}({}^cD_a^\alpha f,[a,y])+\mathrm{V}({}^cD_a^\alpha f,[y,x])=\mathrm{V}({}^cD_a^\alpha f,[a,x])$ (cf., e.g., [[3], Lemma 2]) gives

$$\begin{aligned}|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| &\leq 2\Big(\mathrm{V}({}^cD_a^\alpha f,[a,x])-\mathrm{V}({}^cD_a^\alpha f,[a,y])\Big)\\ &= 2\Big(\mathrm{V}_{{}^cD_a^\alpha f}(x)-\mathrm{V}_{{}^cD_a^\alpha f}(y)\Big)\\ &= 2|\mathrm{V}_{{}^cD_a^\alpha f}(x)-\mathrm{V}_{{}^cD_a^\alpha f}(y)|.\end{aligned}$$

Proposition 5. *If $f \in \mathrm{BFDV}[a,b]$, then for all $x,y \in [a,b]$*

$$|\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)| \le |\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)|.$$

Proof. Without loss of generality assume that $y < x$. Since $\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)$ is a non-negative quantity, we have

$$\begin{aligned} |\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)| &= \mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y) \\ &= \mathrm{V}({}^cD_a^\alpha f,[a,x]) - \mathrm{V}({}^cD_a^\alpha f,[a,y]), \end{aligned}$$

using $\mathrm{V}({}^cD_a^\alpha f,[a,x]) = \mathrm{V}({}^cD_a^\alpha f,[a,y]) + \mathrm{V}({}^cD_a^\alpha f,[y,x])$ (cf., e.g., [[3], Lemma 2]), we obtain

$$\begin{aligned} |\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)| &= \mathrm{V}({}^cD_a^\alpha f,[a,y]) + \mathrm{V}({}^cD_a^\alpha f,[y,x]) - \mathrm{V}({}^cD_a^\alpha f,[a,x]) \\ &= \mathrm{V}({}^cD_a^\alpha f,[y,x]), \end{aligned}$$

by Lemma 1,

$$\begin{aligned} |\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)| &\le \mathrm{FDV}(f,[y,x]) \\ &= \mathrm{FDV}(f,[a,y]) + \mathrm{FDV}(f,[y,x]) - \mathrm{FDV}(f,[a,y]), \end{aligned}$$

Proposition 2 combined with the fact that $\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)$ is a non-negative quantity gives that

$$\begin{aligned} |\mathrm{V}_{^cD_a^\alpha f}(x) - \mathrm{V}_{^cD_a^\alpha f}(y)| &\le \mathrm{FDV}(f,[a,x]) - \mathrm{FDV}(f,[a,y]) \\ &= \mathrm{FDV}^f(x) - \mathrm{FDV}^f(y) \\ &= \left|\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)\right|. \end{aligned}$$

Proposition 6 presents an inequality relating ${}^cD_a^\alpha f$ and FDV^f.

Proposition 6. *If $f \in \mathrm{BFDV}[a,b]$, then for all $x,y \in [a,b]$*

$$|({}^cD_a^\alpha f)(x) - ({}^cD_a^\alpha f)(y)| \le \left|\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)\right|.$$

Proof. Without loss of generality assume that $y < x$. By definition of variation of a function f with respect to the partition $P = \{y,x\}$ of $[y,x]$,

$$|({}^cD_a^\alpha f)(x) - ({}^cD_a^\alpha f)(y)| \le \mathrm{V}({}^cD_a^\alpha f,[y,x]),$$

Lemma 1 reduces the last inequality to

$$\begin{aligned} |({}^cD_a^\alpha f)(x) - ({}^cD_a^\alpha f)(y)| &\le \mathrm{FDV}(f,[y,x]) \\ &\le \mathrm{FDV}(f,[a,y]) + \mathrm{FDV}(f,[y,x]) - \mathrm{FDV}(f,[a,y]), \end{aligned}$$

using Proposition 2, we conclude that

$$\begin{aligned} |({}^cD_a^\alpha f)(x) - ({}^cD_a^\alpha f)(y)| &\le \mathrm{FDV}(f,[a,x]) - \mathrm{FDV}(f,[a,y]) \\ &= \mathrm{FDV}^f(x) - \mathrm{FDV}^f(y) \\ &= \left|\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)\right|. \end{aligned}$$

The triangle inequality for the total fractional differential variation FDV (see [[1], Proposition 2.5]) is given by

$$\mathrm{FDV}(f+g,[a,b]) \le \mathrm{FDV}(f,[a,b]) + \mathrm{FDV}(g,[a,b]).$$

Proposition 7 provides a reverse triangle inequality for the total fractional differential variation, FDV.

Proposition 7. *If $f,g \in \mathrm{BFDV}[a,b]$, then*

$$|\mathrm{FDV}(f,[a,b]) - \mathrm{FDV}(g,[a,b])| \le \mathrm{FDV}(f-g,[a,b]).$$

Proof. Using $\mathrm{FDV}(f-g+g,[a,b]) \leq \mathrm{FDV}(f-g,[a,b])+\mathrm{FDV}(g,[a,b])$ (cf., e.g., [[1], Proposition 2.5]) , we obtain

$$\begin{aligned}\mathrm{FDV}(f,[a,b]) &= \mathrm{FDV}(f-g+g,[a,b])\\ &\leq \mathrm{FDV}(f-g,[a,b])+\mathrm{FDV}(g,[a,b]),\end{aligned}$$

therefore,

$$\mathrm{FDV}(f,[a,b])-\mathrm{FDV}(g,[a,b]) \leq \mathrm{FDV}(f-g,[a,b]), \tag{4}$$

using $\mathrm{FDV}(g,[a,b]) \leq \mathrm{FDV}(g-f,[a,b])+\mathrm{FDV}(f,[a,b])$ (cf., e.g., [[1], Proposition 2.5]) and the homogeneity property (Proposition 1) of FDV reduces the previous inequality to

$$\mathrm{FDV}(g,[a,b]) \leq \mathrm{FDV}(f-g,[a,b])+\mathrm{FDV}(f,[a,b]),$$

therefore,

$$\mathrm{FDV}(g,[a,b])-\mathrm{FDV}(f,[a,b]) \leq \mathrm{FDV}(f-g,[a,b]), \tag{5}$$

inequalities (4) and (5) give

$$|\mathrm{FDV}(f,[a,b])-\mathrm{FDV}(g,[a,b])| \leq \mathrm{FDV}(f-g,[a,b]).$$

4 Certain properties of fractional differential variation function

In this section, we discuss various types of properties (Lipschitz, absolute, Hölder continuous) of FDV^f that it inherits from ${}^cD_a^\alpha f$.

Proposition 8. *Let $f \in \mathrm{BFDV}[a,b]$. Then FDV^f is continuous at x_0 if and only if ${}^cD_a^\alpha f$ is continuous at x_0.*

Proof. First assume that ${}^cD_a^\alpha f$ is continuous function at x_0. $V_{{}^cD_a^\alpha f}$ is continuous at x_0 (cf., e.g., [[3], Corollary 1.1]). Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that $|\mathrm{V}_{{}^cD_a^\alpha f}(x)-\mathrm{V}_{{}^cD_a^\alpha f}(x_0)| < \frac{\varepsilon}{2}$ whenever $|x-x_0| < \delta$ and $x \in [a,b]$. We see from Proposition 4 that $|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta$ and $x \in [a,b]$. Therefore, FDV^f is continuous at x_0.

Next, assume that FDV^f is continuous at x_0. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta$ and $x \in [a,b]$. By Proposition 6, $|({}^cD_a^\alpha f)(x)-({}^cD_a^\alpha f)(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta$ and $x \in [a,b]$. Therefore, ${}^cD_a^\alpha f$ is continuous at x_0.

Remark. Let $f \in \mathrm{BDV}[a,b]$. Then DV^f is continuous at x_0 if and only if f' is continuous at x_0.

Proposition 9. *Let $f \in \mathrm{BFDV}[a,b]$. Then FDV^f is Lipschitz continuous on $[a,b]$ if and only if ${}^cD_a^\alpha f$ is Lipschitz continuous on $[a,b]$.*

Proof. First assume that ${}^cD_a^\alpha f$ is Lipschitz continuous on $[a,b]$. $\mathrm{V}_{{}^cD_a^\alpha f}$ is Lipschitz continuous on $[a,b]$ (cf., e.g., [[3], Theorem 5]). There exists a positive real number K such that for all $x,y \in [a,b]$

$$|\mathrm{V}_{{}^cD_a^\alpha f}(y)-\mathrm{V}_{{}^cD_a^\alpha f}(x)| \leq K|y-x|,$$

by Proposition 4, it follows that for all $x,y \in [a,b]$

$$|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| \leq 2K|y-x|,$$

therefore, FDV^f is Lipschitz continuous on $[a,b]$.

Conversely, assume that FDV^f is Lipschitz continuous on $[a,b]$. Then there exists a positive real number $M > 0$ such that for all $x,y \in [a,b]$,

$$|\mathrm{FDV}^f(x)-\mathrm{FDV}^f(y)| \leq M|y-x|,$$

by Proposition 6, it follows that for all $x,y \in [a,b]$

$$|({}^cD_a^\alpha f)(x)-({}^cD_a^\alpha f)(y)| \leq M|y-x|.$$

Therefore, ${}^cD_a^\alpha f$ is Lipschitz continuous on $[a,b]$.

Remark. Let $f \in \mathrm{BDV}[a,b]$. Then DV^f is Lipschitz continuous on $[a,b]$ if and only if f' is Lipschitz continuous on $[a,b]$.

Proposition 10. *Let $f \in \mathrm{BFDV}[a,b]$. Then FDV^f is absolutely continuous on $[a,b]$ if and only if ${}^cD_a^\alpha f$ is absolutely continuous on $[a,b]$.*

Proof. First assume that ${}^cD_a^\alpha f$ is absolutely continuous on $[a,b]$. $\mathrm{V}_{{}^cD_a^\alpha f}$ is absolutely continuous on $[a,b]$ (cf., e.g., [[3], Theorem 6]). Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $\sum_{i=1}^{n} \left|\mathrm{V}_{{}^cD_a^\alpha f}(b_i) - \mathrm{V}_{{}^cD_a^\alpha f}(a_i)\right| < \frac{\varepsilon}{2}$ whenever $\{(a_i,b_i)\}$ is a disjoint collection of open intervals in (a,b) satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$. By Proposition 4, it follows that $\sum_{i=1}^{n} \left|\mathrm{FDV}^f(b_i) - \mathrm{FDV}^f(a_i)\right| < 2 \times \frac{\varepsilon}{2} = \varepsilon$ whenever $\{(a_i,b_i)\}$ is a disjoint collection of open intervals in (a,b) satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Therefore, FDV^f is absolutely continuous on $[a,b]$.

Now, assume that FDV^f is absolutely continuous. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $\sum_{i=1}^{n} \left|\mathrm{FDV}^f(b_i) - \mathrm{FDV}^f(a_i)\right| < \varepsilon$ whenever $\{(a_i,b_i)\}$ is a disjoint collection of open intervals in (a,b) satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$. By Proposition 6, it follows that $\sum_{i=1}^{n} |({}^cD_a^\alpha f)(b_i) - ({}^cD_a^\alpha f)(a_i)| < \varepsilon$ whenever $\{(a_i,b_i)\}$ is a disjoint collection of open intervals in (a,b) satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Therefore, ${}^cD_a^\alpha f$ is absolutely continuous on $[a,b]$.

Remark. Let $f \in \mathrm{BDV}[a,b]$. Then DV^f is absolutely continuous on $[a,b]$ if and only if f' is absolutely continuous on $[a,b]$.

Proposition 11. *Let $f \in \mathrm{BFDV}[a,b]$. Then $\mathrm{FDV}^f \in \mathrm{BV}[a,b]$ if and only if ${}^cD_a^\alpha f \in \mathrm{BV}[a,b]$.*

Proof. First assume that ${}^cD_a^\alpha f \in \mathrm{BV}[a,b]$. Then, $\mathrm{BV}_{{}^cD_a^\alpha f} \in \mathrm{V}[a,b]$ (cf., e.g., [[3], Theorem 2]) from which it follows that $\mathrm{V}\left(\mathrm{V}_{{}^cD_a^\alpha f},[a,b]\right) < \infty$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Proposition 4 gives

$$\sum_{i=1}^{n} \left|\mathrm{FDV}^f(x_i) - \mathrm{FDV}^f(x_{i-1})\right| \le 2\sum_{i=1}^{n} \left|\mathrm{V}_{{}^cD_a^\alpha f}(x_i) - \mathrm{V}_{{}^cD_a^\alpha f}(x_{i-1})\right|,$$

therefore,

$$\mathrm{V}(\mathrm{FDV}^f, P) \le 2\,\mathrm{V}(\mathrm{V}_{{}^cD_a^\alpha f}, P),$$

taking supremum over all partitions of $[a,b]$,

$$\mathrm{V}(\mathrm{FDV}^f,[a,b]) \le 2\,\mathrm{V}(\mathrm{V}_{{}^cD_a^\alpha f},[a,b]) < \infty.$$

Hence, $\mathrm{FDV}^f \in \mathrm{BV}[a,b]$.

Now, consider that FDV^f is a function of bounded variation on $[a,b]$. Therefore, $\mathrm{V}(\mathrm{FDV}^f,[a,b]) < \infty$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. By Proposition 6,

$$\sum_{i=1}^{n} |({}^cD_a^\alpha f)(x_i) - ({}^cD_a^\alpha f)(x_{i-1})| \le \sum_{i=1}^{n} \left|\mathrm{FDV}^f(x_i) - \mathrm{FDV}^f(x_{i-1})\right|,$$

this gives,

$$\mathrm{V}({}^cD_a^\alpha f, P) \le \mathrm{V}(\mathrm{FDV}^f, P),$$

taking supremum over all partitions P of $[a,b]$,

$$\mathrm{V}({}^cD_a^\alpha f,[a,b]) \le \mathrm{V}(\mathrm{FDV}^f,[a,b]) < \infty.$$

Hence, ${}^cD_a^\alpha f \in \mathrm{BV}[a,b]$.

Remark. Let $f \in \mathrm{BDV}[a,b]$. Then $\mathrm{DV}^f \in \mathrm{BV}[a,b]$ if and only if $f' \in \mathrm{BV}[a,b]$.

Proposition 11 gives the following bound on the total variation of FDV^f on $[a,b]$

$$\mathrm{V}({}^cD_a^\alpha f,[a,b]) \le \mathrm{V}(\mathrm{FDV}^f,[a,b]) \le 2\,\mathrm{V}(\mathrm{V}_{{}^cD_a^\alpha f},[a,b]).$$

Proposition 12. *Let $f \in \mathrm{BFDV}[a,b]$. If FDV^f is Hölder continuous on $[a,b]$, then ${}^cD_a^\alpha f$ is Hölder continuous on $[a,b]$.*

Proof. As per given hypothesis, there exists positive real numbers M and α such that for all $x, y \in [a,b]$,

$$\left|\mathrm{FDV}^f(x) - \mathrm{FDV}^f(y)\right| \leq M\,|x-y|^\alpha,$$

from Proposition 6, it follows that

$$|({}^cD_a^\alpha f)(x) - ({}^cD_a^\alpha f)(y)| \leq M\,|x-y|^\alpha, \qquad (x, y \in [a,b]).$$

Hence, ${}^cD_a^\alpha f$ is Hölder continuous.

Remark. Let $f \in \mathrm{BDV}[a,b]$. If DV^f is Hölder continuous on $[a,b]$, then f' is Hölder continuous on $[a,b]$.

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