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On Commutativity of Rings and Banach Algebras with Homoderivations

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Abstract: By imposing specific algebraic conditions on homoderivations, this research explores their role in promoting commutativity within prime rings and Banach algebras.

Keywords: Prime ring, Banach algebra, homoderivation.

1 Introduction

Let R represents a ring with center Z. The primeness of R gives if for $w,s \in R, wRs = (0)$, then $w = 0$ or $s = 0$. The symbol $[l,t]$ denotes the commutator $lt - tl$ and $l \circ t$ is the anti-commutator $lt + tl$. Let β be an additive mapping of R. We say β is a derivation of R if $\beta(tl) = \beta(t)l + t\beta(l) \quad \forall t,l \in R$ and β is termed a homoderivation of R if it fulfills the relation $\beta(tl) = \beta(t)\beta(l) + \beta(t)l + t\beta(l) \quad \forall t,l \in R$. For an algebra A over a field K, we additionally require that β is linear, that is $\beta(\lambda t) = \lambda\beta(t) \quad \forall t \in A$ and $\lambda \in K$.

Several results in ring theory and Banach algebra have established profound connections between the commutativity of algebraic structures and certain algebraic identities involving derivations and some related maps.

In 1992, Giri and Dhoble [8] demonstrated that a semiprime ring R becomes commutative if, for fixed integers $n,m > 1$ and $[t^n,l^m] \in Z$ or $(t^n \circ l^m) \in Z$ $\forall t,l \in R$. This result was extended by Yood [13], who considered a Banach algebra A which is semiprime has center Z and two non-empty open subsets T_1, T_2 of A. He showed if $\forall t \in T_1, l \in T_2$ and $m = m(t,l) > 1$, $n = n(t,l) > 1$ are integers such that $[t^m,l^n] \in$ or $t^m \circ l^n \in Z$. Then A is commutative. Daif and Bell [7] examined the impact of derivations on semiprime rings. They established that if a derivation β satisfies $\beta([t,l]) = [t,l] \quad \forall t,l \in R$ or $\beta([t,l]) = -[t,l] \quad \forall t,l \in R$, then the ring R must be commutative. This result was studied by Hongan [11] when $\beta([t,l]) - [t,l] \in Z \quad \forall t,l$ in an ideal M of R. Bell [3] generalized these results when R is prime with non-trivial center and characteristic zero or greater than some integer $r > 1$, R is commutative provided that there exists $\beta \neq 0$ a derivation of R satisfying $\beta([t^r,l]) - [t,l^r] \in Z \quad \forall t,l \in R$. More recently in 2016, Ali et al. [1] expanded upon findings of Bell by identifying several sufficient conditions for commutativity in 2-torsion-free semiprime rings involving derivations and powers of elements. They showed the commutativity of R when one of the next holds $\quad \forall t,l \in R$

(i)$\beta([t^m,l^n]) - [t^m,l^n] \in Z$,
(ii)$\beta([t^m,l^n]) + [t^m,l^n] \in Z$,
(iii)$\beta \neq 0$ and $\beta([t^m,l^n]) \in Z$,
(iv)$\beta \neq 0$ and $\beta(t^m \circ l^n) \in Z$.

They also presented analogous results in the context of Banach algebras, showing that a semiprime Banach algebra A with center Z is commutative when a continuous linear derivation β of R satisfies any of the following $\quad \forall t \in T_1, l \in T_2$, where $n_1, n_2 > 1$ are integers depending on t,l (i) $\beta([t^{n_1},l^{n_2}]) - [t^{n_1},l^{n_2}] \in Z$ or $\beta([t^{n_1},l^{n_2}]) + [t^{n_1},l^{n_2}] \in Z$, (ii) $\beta([t^{n_1},l^{n_2}]) \in Z$ or $\beta(t^{n_1} \circ l^{n_2}) \in Z$.

This paper's objective is to look into how the presence of a homoderivation affects the commutativity of rings and Banach algebras. We seek to establish sufficient conditions under which a ring or algebra admitting a homoderivation becomes commutative. In particular, we examine the commutativity of a prime ring R with a

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homoderivation β and one of the following satisfies $\forall a,b \in R$
(i) $\beta([a^m,b^m]) - [a^m,b^m] \in Z$, (ii) $\beta([a^m,b^m]) \in Z$, (iii) $\beta(a^m \circ b^n) - (a^m \circ b^n) \in Z$, and (iv) $\beta(a^m \circ b^n) \in Z$, where $m,n \in \mathbb{Z}^+$ are fixed. Also, we proved the following results: Let A be a prime Banach algebra, T_1, T_2 be non-empty open subsets of A and $m > 1$ is an integer depending on a,b. If β is a continuous linear homoderivation of A, then A is commutative when one of the next holds $\forall a \in T_1, b \in T_2$

(i)$\beta([a^m,b^m]) - [a^m,b^m] \in Z$ or $\beta(a^m \circ b^m) - (a^m \circ b^m) \in Z$,
(ii)$\beta([a^m,b^m]) \in Z$ or $\beta(a^m \circ b^m) \in Z$.

Our results are inspired by the work in [1,2,3,7,8,11,13].

2 On Homoderivations in Prime Rings

Throughout this section, R is a prime ring, S is a Lie ideal of R, char(R) $\neq 2$, $m,n \in \mathbb{Z}^+$ are fixed and β is a homoderivation of R.

In what follows, we examine the influence of homoderivations on the structure of Lie ideals.

Lemma 21*If $\beta([t,l]) - [t,l] \in Z \quad \forall t,l \in S$, then $S \subseteq Z$.*

Proof. $\forall t,l \in S$

$$\beta([t,l]) - [t,l] \in Z. \tag{2.1}$$

If $\beta = 0$, then $[t,l] \in Z$. Thus $[S,S] \subseteq Z$ and [10, Lemma 1] implies $S \subseteq Z$. Accordingly, one can assume that $\beta \neq 0$. Putting $[l,g]$ instead of l in (2.1) leads to

$$[\beta(t), \beta([l,g])] + [\beta(t), [l,g]] \in Z \quad \forall t,l,g \in S. \tag{2.2}$$

The relation (2.2) becomes

$$[\beta(t), \beta([l,g]) - [l,g]] + 2[\beta(t), [l,g]] \in Z.$$

Using (2.1) leads to

$$[\beta(t), [l,g]] \in Z. \tag{2.3}$$

Replacing t by $[t,p]$ in (2.3) gives $[\beta[t,p],[l,g]] \in Z \quad \forall t,p,l,g \in S$. By (2.1), we arrive at $[[t,p],[l,g]] \in Z$, i.e., $[[S,S],[S,S]] \subseteq Z$. So $[S,S] \subseteq Z$ which implies $S \subseteq Z$.

Corollary 22*If $\beta(t) - t \in Z \quad \forall t \in S$, then $S \subseteq Z$.*

Theorem 23*If $\beta([a^m,b^m]) - [a^m,b^m] \in Z \quad \forall a,b \in R$, then R is commutative.*

Proof. $\forall a,b \in R$

$$\beta([a^m,b^m]) - [a^m,b^m] \in Z. \tag{2.4}$$

If $\beta = 0$, then $[a^m,b^m] \in Z$. For $m = 1$, [10, Lemma 1] gives the commutativity of R. If m is larger than 1, then [8, Theorem 1.1] gives the required result. Suppose $\beta \neq 0$. Consider W is the additive subgroup generated by $\{s^m | s \in R\}$. Hence $\beta([a,b]) - [a,b] \in Z \quad \forall a,b \in W$. By [6, Main Theorem] and since char(R) $\neq 2$, then either $s^m \in Z \quad \forall s \in R$ or W has $L \not\subseteq Z$ a Lie ideal. If $s^m \in Z \quad \forall s \in R$, then R is commutative. Assuming the second instance happened, then by [4, Lemma 1] there exists $(0) \neq I$ an ideal of R and $(0) \neq [I,R] \subseteq L$. Thus $\beta([a,b]) - [a,b] \in Z \quad \forall a,b \in [I,I]$. Assume $S = [I,I]$ and Lemma 21 is applied, yielding $I \subseteq Z$. So, [12, 9 Lemma 3] leads to the desired outcome.

Theorem 24*If $\beta \neq 0$ and $\beta([a^m,b^m]) \in Z \quad \forall a,b \in R$, then R is commutative.*

Proof. $\forall a,b \in R$

$$\beta([a^m,b^m]) \in Z. \tag{2.5}$$

Applying the same reasoning as in the Theorem 23 proof yields either R is commutative or $\beta([a,b]) \in Z \quad \forall a,b \in [I,I]$. Now let $S = [I,I]$ and applying [9, Lemma 4.2.4] and [12, Lemma 3], we obtain the desired outcome.

Theorem 25*If $\beta(a^m \circ b^n) - (a^m \circ b^n) \in Z \quad \forall a,b \in R$, then R is commutative.*

Proof. $\forall a,b \in R$

$$\beta(a^m \circ b^n) - (a^m \circ b^n) \in Z. \tag{2.6}$$

If $\beta = 0$, then $(a^m \circ b^n) \in Z$. Replacing b by a implies that $a^k \in Z \quad \forall a \in R$, where $k = m+n$. That is, $[a^k, s^r] = 0 \quad \forall a,s \in R$, where k,r positive integers each greater than 1. By [8, Theorem 1.1], R is commutative. Therefore, assuming $\beta \neq 0$. For $a = b$, we get $\beta(a^k) - a^k \in Z \quad \forall a \in R$, where $k = m+n$. Suppose W is the additive subgroup generated by $\{s^k | s \in R\}$. Hence, $\beta(a) - a \in Z \quad \forall a \in W$. By [6, Main Theorem], we find R is commutative or $\beta(a) - a \in Z \quad \forall a \in [I,I]$. Suppose $S = [I,I]$ and applying Corollary 22, $I \subseteq Z$. So, [12, 9 Lemma 3] implies R is commutative.

Theorem 26*If $\beta \neq 0$ and $\beta(a^m \circ b^n) \in Z \quad \forall a,b \in R$, then R is commutative.*

*Proof.*Applying the same reasoning of Theorem 25 proof, R is commutative or $\beta(a) \in Z \quad \forall a \in [I,I]$. Now for $S = [I,I]$, applying [9, Lemma 4.2.3] and [12, Lemma 3], we arrive at the desired outcome.

3 On Homoderivations in Banach Algebras

Throughout A denotes a prime Banach algebra over a complex field $\mathbb{C}$ with center Z, T_1, T_2 are non-empty open subsets of A, $m > 1$ is an integer depending on a and b and β is a continuous linear homoderivation of A.

In our proof, we apply the next result.

Lemma 31[5] *Suppose $p(t)=\sum_{r=1}^{n} b_r t^r$ is a polynomial in real variable t for infinite values of t and each $b_r \in A$. If $p(t)\in W$, then each $b_r \in W$, where W is a closed linear subspace of A.*

We begin with the following theorem.

Theorem 32 *If $\beta([a^m,b^m])-[a^m,b^m]\in Z$ or $\beta(a^m\circ b^m)-(a^m\circ b^m)\in Z \quad \forall a\in T_1, b\in T_2$, then A is commutative.*

Proof. Let $\beta=0$. Then $[a^m,b^m]\in Z$ or $(a^m\circ b^m)\in Z$ $\forall a\in T_1, b\in T_2$. Then by [13, Theorem 2], A is commutative. Now, suppose $\beta\neq 0$ and fix $a\in T_1$.

For each fix m, we define $Y_m=\{b\in A : \beta([a^m,b^m])-[a^m,b^m]\notin Z$ and $\beta(a^m\circ b^m)-(a^m\circ b^m)\notin Z\}$. We will demonstrate that each Y_m is open in A. So, we demonstrate that Y_m^c the complement of Y_m is closed. Taking a sequence $(s_k)\in Y_m^c$ and $s_k\to s$ as $k\to\infty$, and we must demonstrate that $s\in Y_m^c$. Since $s_k\in Y_m^c$, then

$$\beta([a^m,s_k^m])-[a^m,s_k^m]\in Z, \tag{3.1}$$

or

$$\beta(a^m\circ s_k^m)-(a^m\circ s_k^m)\in Z. \tag{3.2}$$

Taking limit on k yields

$$\lim_{k\to\infty}(\beta([a^m,s_k^m])-[a^m,s_k^m])\in Z, \tag{3.3}$$

or

$$\lim_{k\to\infty}(\beta(a^m\circ s_k^m)-(a^m\circ s_k^m))\in Z. \tag{3.4}$$

Since β is continuous, then .

$$\beta([a^m,\lim_{k\to\infty}s_k^m])-[a^m,\lim_{k\to\infty}s_k^m]\in Z, \tag{3.5}$$

or

$$\beta(a^m\circ\lim_{k\to\infty}s_k^m)-(a^m\circ\lim_{k\to\infty}s_k^m)\in Z. \tag{3.6}$$

Since $s_k\to s$ as $k\to\infty$, then

$$\beta([a^m,s^m])-[a^m,s^m]\in Z, \tag{3.7}$$

or

$$\beta(a^m\circ s^m)-(a^m\circ s^m)\in Z. \tag{3.8}$$

This implies that $s\in Y_m^c$. Hence Y_m^c is closed so each Y_m is open. If every Y_m is dense, then their intersection is also dense, according to [5, Baire category theorem], which runs counter to the existence of T_1 and T_2. So there is $r\in\mathbb{Z}^+$ such that Y_r is not dense in A. Hence, there exists a non-empty open subset T_3 in the complement of Y_r, thus $\forall b\in T_3$ either $\beta([a^r,b^r])-[a^r,b^r]\in Z$ or $\beta(a^r\circ b^r)-(a^r\circ b^r)\in Z$. Let $b_0\in T_3$ and $u\in A$, $b_0+tu\in T_3$ for all sufficiently small real t. Therefore, for each t, we find

$$\beta([a^r,(b_0+tu)^r])-[a^r,(b_0+tu)^r]\in Z \tag{3.9}$$

or

$$\beta(a^r\circ(b_0+tu)^r)-(a^r\circ(b_0+tu)^r)\in Z. \tag{3.10}$$

Then at least one of (3.9) or (3.10) must satisfy for infinitely many t. Consider (3.9) satisfies for these t. Now,

$$\begin{aligned}&\beta([a^r,(b_0+tu)^r])-[a^r,(b_0+tu)^r]=\\&\beta([a^r,B_{r,0}(b_0,u)])-[a^r,B_{r,0}(b_0,u)]\\&+(\beta([a^r,B_{r-1,1}(b_0,u)])-[a^r,B_{r-1,1}(b_0,u)])t\\&+\ldots+(\beta([a^r,B_{1,r-1}(b_0,u)])\\&-[a^r,B_{1,r-1}(b_0,u)])t^{r-1}+(\beta([a^r,B_{0,r}(b_0,u)])\\&-[a^r,B_{0,r}(b_0,u)])t^r.\end{aligned} \tag{3.11}$$

$B_{i,j}(b_0,u)$ refers to the sum of all terms such that b_0, u appear exactly i, j times respectively in the expansion of $(b_0+tu)^r$, where i and j are non-negative integers and $i+j=r$. Equation (3.11) is a polynomial in t and the coefficient of t^r is $\beta([a^r,u^r])-[a^r,u^r]$. Therefore, we obtain $\beta([a^r,u^r])-[a^r,u^r]\in Z$. In the same way, if (3.10) satisfies for these t, then we find $\beta(a^r\circ u^r)-(a^r\circ u^r)\in Z$. Thus, given $a\in T_1$ there is $r\in\mathbb{Z}^+$ and $r=r(u)$ so that $\forall u\in A$, we have $\beta([a^r,u^r])-[a^r,u^r]\in Z$ or $\beta(a^r\circ u^r)-(a^r\circ u^r)\in Z$. Let $A_1=\{u\in A|\beta([a^r,u^r])-[a^r,u^r]\in Z\}$ and $A_2=\{u\in A|\beta(a^r\circ u^r)-(a^r\circ u^r)\in Z\}$. It follows that A must equal $A_1\cup A_2$, and each $A_n,(n=1,2)$ is closed (as previously shown). According to [5, Baire category theorem], at least one of A_1 and A_2 must have a non-empty open subset of A. Suppose A_1 has a non-empty open subset T_4 of A. Let $f_0\in T_4$ and $z\in A$. Then $f_0+tz\in T_4$ for sufficiently small t, we find $\beta([a^r,(f_0+tz)^r])-[a^r,(f_0+tz)^r]\in Z$. As mentioned above, this can be expressed as a polynomial in t, where the coefficient of t^r is $\beta([a^r,z^r])-[a^r,z^r]$. By Lemma 31, we get $\beta([a^r,z^r])-[a^r,z^r]\in Z \quad \forall z\in A$. Likewise, if A_2 has a non-empty open subset, then $\beta(a^r\circ z^r)-(a^r\circ z^r)\in Z \quad \forall z\in A$. Thus, considering $a\in T_1$ there is $r\in\mathbb{Z}^+$ so that either $\beta([a^r,z^r])-[a^r,z^r]\in Z$ or $\beta(a^r\circ z^r)-(a^r\circ z^r)\in Z \quad \forall z\in A$. Now, we reverse the role of T_1 and T_2 in the settings mentioned above. Continuing as before yields $\beta([a^r,z^r])-[a^r,z^r]\in Z$ or $\beta(a^r\circ z^r)-(a^r\circ z^r)\in Z \quad \forall a,z\in A$. Hence by Theorem 23 and Theorem 25, A is commutative.

Drawing upon Theorems 24 and 26, we are able to establish the next result through a method analogous to that employed in the proof of Theorem 32.

Theorem 33 *If $\beta\neq 0$ and $\beta([a^m,b^m])\in Z$ or $\beta(a^m\circ b^m)\in Z \quad \forall a\in T_1, b\in T_2$, then A is commutative.*

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