Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110311

Exploring Impulsive Fractional Derivatives with Mixed-Delay Integral Boundary Constraints

Kottakkaran Sooppy Nisar[1,2,*], *Muhannad Ibrahim Al-Shartab*[1], *Chokkalingam Ravichandran* [3], *Abdel-Haleem Abdel-Aty*[4], *and Mohamed R. Eid*[5]

[1]Department of Mathematics, College of Science and Humanities in Al Kharj, Prince Sattam Bin Abdulaziz University, Al Kharj, 11942, Saudi Arabia
[2]Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Jordan
[3]Department of Mathematics, Kongunadu Arts and Science College, Coimbatore - 641 029, Tamil Nadu, India
[4]Department of Physics, College of Sciences, University of Bisha, Bisha 61922, Saudi Arabia
[5]Center for Scientific Research and Entrepreneurship, Northern Border University, Arar 73213, Saudi Arabia

Received: 3 Aug. 2024, Revised: 12 Feb. 2025, Accepted: 17 May 2025
Published online: 1 Jul. 2025

Abstract: This paper presents a systematic approach to investigating the existence, uniqueness, and stability of fractional impulsive differential equations through the Atangana-Baleanu derivative, facilitating the application of Schafer's fixed-point theorem and the Banach fixed-point theorem. Also, the stability results are thoroughly analyzed, with integral boundary conditions and mixed delays enriching the depth and rigor of the investigation.

Keywords: Impulsive equations, Fractional calculus, Fixed point techniques(FPT), Delay terms, AB-derivative, Boundary condition.

1 Introduction

Fractional calculus provides a broader mathematical framework that combines differential & integral operations, extending the conventional scope of calculus [1,2,3]. The idea of fractional derivatives originated concurrently with the evolution of classical derivative concepts. In 1695, the idea of fractional order was proposed initially by Hospital & Leibniz, which gradually attracted the attention of researchers. This type of fractional derivative has captured the interest of prominent mathematicians such as Liouville, Riemann, Euler, Grüinwald, Laplace, Letnikov, among others. The field of fractional calculus, which has its roots in the 19th century, encompasses the study of fractional differential equations (FDEs), as well as fractional dynamics & geometry. Applications of fractional studies span numerous fields, such as bioengineering, mechanical & electrical engineering, control systems, robotics, statistical physics, chemistry, optics, acoustics, as well as areas like viscoelasticity & rheology, among others[4,5,6]. Fractional-order models offer enhanced accuracy compared to traditional integer-order models, providing deeper insights into complex dynamics. In pursuit of refining these models, many researchers have introduced diverse formulations of fractional derivatives, notably including the ABC (Atangana-Baleanu Caputo) derivative, which has become a focal point in advancing the study of fractional calculus [7].
In numerous real-world applications, we encounter systems that are subjected to brief disturbances,referred to as impulsive conditions, where the duration of which is often negligible compared to the overall time scale of the process. Such impulsive effects can result in the occurrence of jump discontinuities in the solutions of the corresponding equations at specific moments in time. This phenomenon highlights the sensitivity of dynamic systems to transient influences, emphasizing the importance of understanding the effects of even minor disruptions on system behavior[8]. Kamran et al. [9] studied the Bagley-Torvik equation by applying the Laplace transform, utilizing the ABC derivative as part of their analysis. Partohaghighi et al. [10] propose an advection-dispersion model utilizing ABC derivative & Chebyshev polynomials to address transport of pollutants in a non-integer framework. Numerous researchers have delved into the AB-fractional derivative, exploring its applications and impact across a wide range of analytical frameworks, such as

* Corresponding author e-mail: n.sooppy@psau.edu.sa

existence proofs [11,12,13,14], stability studies [15,16], & model development [17,18].
Building on the findings of [19], we explore the initial boundary problem (IBP) for impulsive fractional differential equations (FDEs) within the ABC framework, incorporating mixed delays in the specified form,

$$\left.\begin{array}{ll} ({}_0^{ABC}D_t^{\iota})\rho(t) & = g(t,\rho(t),\rho(mt),\, {}_0^{ABC}D_t^{\iota}\rho(t))),\ t\neq t_k,\ t\in[0,1],\ 0<m<1,\\ p\rho(0)+q\rho'(0) & = \int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z},\ p\rho(1)+q\rho'(1)=\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z},\ 1<\iota\leq 2\\ \triangle\rho(t_k) & = F_k(\rho(t_k)),\ \triangle\rho'(t_k)=\bar{F}_k(\rho(t_k)),\ k=1,,...\mathfrak{N} \end{array}\right\} \quad (1.1)$$

here ${}_0^{ABC}D_t^{\iota}$ be ABC fractional derivative, $g:[0,1]\times R^3\to R$ is a nonlinear fn & the fns $h_1,h_2:R\to R$ where each of them exhibits continuity, $p>0,q\geq 0$ specifies the real constants.
In the 2^{nd} section, we presented key results, definitions, & theorems. 3^{rd} section focused on hypotheses, lemmas, & analytical findings related to the fractional differential equations exhibiting impulsive condition characterized by integral boundary conditions & mixed delays, utilizing FPT. Stability analysis of the problem is examined in 4^{th} section.

2 Facts

Definition 1.[19] *Suppose $u\in H^1(a,b), b>a$, so that the derivative of ABC is represented in the following form [7]:*

$${}_0^{ABC}D^{\delta}(u(\tilde{\upsilon}))=\frac{\mathcal{N}(\delta)}{1-\delta}\int_0^{\tilde{\upsilon}}u'(\breve{s})E_{\delta}[\frac{(\tilde{\upsilon}-\breve{s})^{\delta}}{\delta-1}]d\breve{s}\ \delta\in[0,1] \quad (2.1)$$

here $E(\delta)$ denotes the Mittag-Leffler function & $\mathcal{N}(0)=\mathcal{N}(1)=1$ represents a fn being normalized.
Associate AB integral is

$$({}_0^{AB}I^{\delta}u)(\tilde{\upsilon})=\frac{1-\delta}{\mathcal{N}(\delta)}u(\tilde{\upsilon})+\frac{\delta}{\mathcal{N}(\delta)}({}_0I^{\delta}u)(\tilde{\upsilon})$$

The derivatives of R-L, R-L AB, Caputo & ABC are discussed in [1,11,12,13].

Proposition 1.[20,21] *Let $0\leq\delta\leq 1$, so that*

$$\begin{aligned}({}_0^{AB}I^{\delta}({}_0^{ABC}D^{\delta}u))(\tilde{\upsilon}) &= u(\tilde{\upsilon})-u(0)\left(E_{\delta}(\lambda\tilde{\upsilon}^{\delta})-\frac{\delta}{1-\delta}E_{\delta,\delta+1}(\lambda\tilde{\upsilon}^{\delta})\right)\\ &= u(\tilde{\upsilon})-u(0).\end{aligned}$$

Theorem 1.(Schaefer's FPT) [22] *To demonstrate that $\mathfrak{D}:\mathcal{W}\to\mathcal{W}$ has at least one FP within a closed, bounded, & convex subset $\mathcal{W}$ of a Banach space, we note that if $\mathfrak{D}$ is compact, then there exists at least one point $x\in\mathcal{W}$ s.t. $\mathfrak{D}(x)=x$.*

3 Main results

Lemma 1. *Let $\iota\in(1,2]$, $\delta:[0,1]\to R$ exhibits continuity, then $\rho\in\mathcal{W}$ serves as a soln to the subsequent problem:*

$$\left.\begin{array}{ll} ({}_0^{ABC}D_t^{\iota})\rho(t) & = \delta(t),\ t\in[0,1],\ 0<m<1,\ t\neq t_k\\ p\rho(0)+q\rho'(0) & = \int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z},\ p\rho(1)+q\rho'(1)=\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z},\ 1<\iota\leq 2\\ \triangle\rho(t_k) & = F_k(\rho(t_k)),\ \triangle\rho'(t_k)=\bar{F}_k(\rho(t_k)),\ k=1,,...\mathfrak{N} \end{array}\right\} \quad (3.1)$$

The soln of (3.1) is

$$\rho(t)=\begin{cases}\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^t\delta(\mathfrak{z})d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_0^t(t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z}+\mathfrak{G}^*,\ t\in[0,t_1];\\ \frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^{t_k}\delta(\mathfrak{z})d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t(t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z}\\ +\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\iota_{j=1}^k\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z}\\ +\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^k(t-t_j)\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z}\\ +\Sigma_{j=1}^kF_j(\rho(t_j))+\Sigma_{j=1}^k(t-t_j)\bar{F}_j(\rho(t_j))+\mathfrak{G}^*,\ t\in(t_k,1],\ k=1,,...\mathfrak{N};\end{cases} \quad (3.2)$$

Proof. Let $t \in [0,t_1]$, ρ be a solution of (1.1) & $a_1, a_2 \in \mathscr{R}$, we get

$$\left.\begin{aligned} \rho(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^t \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^t (t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} - a_1 - a_2 t \\ \rho'(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t) + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_0^t (t-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} - a_2. \end{aligned}\right\} \tag{3.3}$$

For $t \in (t_1, t_2]$

$$\left.\begin{aligned} \rho(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^{t_1} \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_1}^t (t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} - d_1 - d_2(t-t_1) \\ \rho'(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t_1) + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_1}^t (t-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} - d_2. \end{aligned}\right\} \tag{3.4}$$

Substitute t_1 in t in (3.3), we get

$$\rho(t_1^-) = \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^{t_1} \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^{t_1} (t_1-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} - a_1 - a_2 t_1$$
$$\rho'(t_1^-) = \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t_1) + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_0^{t_1} (t_1-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} - a_2.$$

Substitute t_1 in t in (3.4), we get

$$\rho(t_1^+) = \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^{t_1} \delta(\mathfrak{z})d\mathfrak{z} - d_1$$
$$\rho'(t_1^+) = \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t_1) - d_2.$$

Through Impulsive conditions, we obtain

$$\begin{aligned} &\triangle\rho(t_1) = \rho(t_1^+) - \rho(t_1^-) = F_1(\rho(t_1)) \\ &\triangle\rho'(t_1) = \rho'(t_1^+) - \rho'(t_1^-) = \bar{F}_1(\rho(t_1)) \\ &\text{i.e.,} -d_1 = \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^{t_1} (t_1-s)^{(\iota-1)}\delta(\mathfrak{z})d\mathfrak{z} - a_1 - a_2 t_1 + F_1(\rho(t_1)) \\ &\& -d_2 = \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_0^{t_1} (t_1-s)^{(\iota-2)}\delta(\mathfrak{z})d\mathfrak{z} - a_2 + \bar{F}_1(\rho(t_1)) \end{aligned}$$

Substitute d_1 & d_2 in (3.4), we get

$$\begin{aligned} \rho(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^{t_1} \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_1}^t (t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} \\ &+ \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^{t_1} (t_1-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \frac{(t-t_1)(\iota-1)}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_0^{t_1} (t_1-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \\ &+ F_1(\rho(t_1)) + (t-t_1)\bar{F}_1(\rho(t_1)) - a_1 - a_2 t \\ \rho'(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t_1) + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_1}^t (t-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \\ &+ \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_0^{t_1} (t_1-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} + \bar{F}_1(\rho(t_1)) - a_2. \end{aligned}$$

Similarly for $t \in (t_k, 1]$, one has

$$\left.\begin{aligned} \rho(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\int_0^{t_k} \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t (t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)} \\ &\times \iota_{j=1}^k \int_{t_{j-1}}^{t_j} (t_j-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \frac{(t-t_1)(\iota-1)}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^k (t-t_j) \\ &\int_{t_{j-1}}^{t_j} (t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \quad + \Sigma_{j=1}^k F_j(\rho(t_j)) + \Sigma_{j=1}^k (t-t_j)\bar{F}_1(\rho(t_1)) - a_1 - a_2 t, \\ &t \in (t_k, 1],\ k = 1,2,\ldots\mathfrak{N} \\ \rho'(t) &= \frac{2-\delta}{\mathscr{N}(\iota-1)}\delta(t_k) + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^t (t-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)} \\ &\times \Sigma_{j=1}^k \int_{t_{j-1}}^{t_j} (t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} + \Sigma_{j=1}^k \bar{F}_j(\rho(t_j)) - a_2, t \in (t_k, 1],\ k = 1,2,\ldots\mathfrak{N}. \end{aligned}\right\} \tag{3.5}$$

Using boundary conditions in (3.5), we get

$$
\begin{aligned}
a_1 = & -\frac{1}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} - \frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\int_0^{t_k}\delta(\mathfrak{z})d\mathfrak{z} - \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} \\
& - \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} - \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \times \Sigma_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} - \frac{q}{p}\Sigma_{j=1}^{k}F_j(\rho(t_j)) - \frac{q}{p}\Sigma_{j=1}^{k}(1-t_j)\bar{F}_j(\rho(t_j)) \\
& - \frac{q}{p^2}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} - \frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}\delta(t_k) - \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \\
& - \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} - \frac{q^2}{p^2}\Sigma_{j=1}^{k}\bar{F}_j(\rho(t_j)) \\
& + \frac{q}{p^2}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z}
\end{aligned}
$$

and

$$
\begin{aligned}
a_2 = & \frac{(2-\iota)}{\mathcal{N}(\iota-1)}\int_0^{t_k}\delta(\mathfrak{z})d\mathfrak{z} + \frac{(\iota-1)}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \frac{(\iota-1)}{\mathcal{N}(\iota-1)\Gamma(\iota)} \\
& \times \Sigma_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \frac{(\iota-1)}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \\
& + \Sigma_{j=1}^{k}F_j(\rho(t_j)) + \Sigma_{j=1}^{k}(1-t_j)\bar{F}_j(\rho(t_j)) + \frac{1}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} + \frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\delta(t_k) \\
& + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \times \Sigma_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} + \frac{q}{p}\Sigma_{j=1}^{k}\bar{F}_j(\rho(t_j)) - \frac{1}{p}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z}.
\end{aligned}
$$

Sub a_1 & a_2, we get

$$
\therefore \rho(t) = \begin{cases}
\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^t \delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_0^t(t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} + \mathfrak{G}^*, \ t\in[0,t_1]; \\
\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^{t_k}\delta(\mathfrak{z})d\mathfrak{z} + \frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t(t-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} \\
+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\delta(\mathfrak{z})d\mathfrak{z} \\
+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{k}(t-t_j)\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\delta(\mathfrak{z})d\mathfrak{z} \\
+\Sigma_{j=1}^{k}F_j(\rho(t_j)) + \Sigma_{j=1}^{k}(t-t_j)\bar{F}_j(\rho(t_j)) + \mathfrak{G}^*, \ t\in(t_k,1], \ k=1,,\ldots\mathfrak{N};
\end{cases}
$$

Corollary 1.*A result of Lemma, (1.1) has the following solution*

$$
\mathcal{I}\rho(t) = \begin{cases}
\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^t \Phi_\rho d\mathfrak{z} + \frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_0^t(t-\mathfrak{z})^{\iota-1}\Phi_\rho d\mathfrak{z} + \mathfrak{G}^*, \ t\in[0,t_1]; \\
\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^{t_k}\Phi_\rho d\mathfrak{z} + \frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t(t-\mathfrak{z})^{\iota-1}\Phi_\rho d\mathfrak{z} \\
+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\rho d\mathfrak{z} \\
+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{k}(t-t_j)\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho d\mathfrak{z} \\
+\Sigma_{j=1}^{k}F_j(\rho(t_j)) + \Sigma_{j=1}^{k}(t-t_j)\bar{F}_j(\rho(t_j)) + \mathfrak{G}^*, \ t\in(t_k,1], \ k=1,\ldots\mathfrak{N};
\end{cases}
$$

here

$$
\begin{aligned}
\mathfrak{G}^* = &\frac{1}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} + \frac{q(2-\iota)}{p\mathscr{N}(\iota-1)}\int_0^{t_k}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z}\\
&+\frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\\
&\times\Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q}{p}\Sigma_{j=1}^n F_j(\rho(t_j)) + \frac{q}{p}\Sigma_{j=1}^n(1-t_j)\bar{F}_j(\rho(t_j))\\
&+\frac{q}{p^2}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} + \frac{q^2(2-\iota)}{p^2\mathscr{N}(\iota-1)}\Phi_\rho(t_k) + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\\
&\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z}\\
&+\frac{q^2}{p^2}\Sigma_{j=1}^n\bar{F}_j(\rho(t_j)) - \frac{q}{p^2}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z}\\
&-\frac{(2-\iota)t}{\mathscr{N}(\iota-1)}\int_0^{t_k}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\\
&\times\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)}{\mathscr{N}(\iota-1)\Gamma(\iota-1)t}\Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z}\\
&-t\Sigma_{j=1}^n F_j(\rho(t_j)) - t\Sigma_{j=1}^n(1-t_j)\bar{F}_j(\rho(t_j)) - \frac{t}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} - \frac{q(2-\iota)t}{p\mathscr{N}(\iota-1)}\Phi_\rho(t_k)\\
&-\frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\\
&\times\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{qt}{p}\Sigma_{j=1}^n\bar{F}_j(\rho(t_j)) + \frac{t}{p}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z}.
\end{aligned}
$$

Assumptions

A$_1$ $g:[0,1]\times\mathscr{R}^3\to[0,\infty)$ is continuous, $\forall\,\rho,\bar{\rho}\in\mathscr{C}([0,\infty],\mathscr{R})$ & $\mathscr{L}_g>0, 0<\mathscr{N}_g<1$, then

$$
\begin{aligned}
|g(t,\rho(t),\rho(mt),\Phi_\rho(t)) - g(t,\bar{\rho}(t),\bar{\rho}(mt),\Phi_{\bar{\rho}}(t))| \le L_g\Big(&|\rho(t)-\bar{\rho}(t)|\\
&+|\rho(mt)-\bar{\rho}(mt)|\\
&+N_g|\Phi_\rho(t)-\Phi_{\bar{\rho}}(t)|\Big)
\end{aligned}
$$

A$_2$ There exist a constants $C_1, C_2>0$ such that

$$
\begin{aligned}
|F_k(\rho(t_m)) - F_k(\bar{\rho}(t_m))| &\le C_1|\rho(t_m)-\bar{\rho}(t_m)|\\
|\bar{F}_k(\rho(t_m)) - \bar{F}_k(\bar{\rho}(t_m))| &\le C_2|\rho(t_m)-\bar{\rho}(t_m)|
\end{aligned}
$$

A$_3$ There exists a constants $C_3, C_4>0$, $\rho\in R$ s.t.

$$
\begin{aligned}
|h_1(\rho(t)) - h_1(\bar{\rho}(t))| &\le C_3|\rho(t)-\bar{\rho}(t)|\\
|h_2(\rho(t)) - h_2(\bar{\rho}(t))| &\le C_4|\rho(t)-\bar{\rho}(t)|
\end{aligned}
$$

A$_4$ There exists a constants $C_5, C_6>0$, $\rho\in R$ s.t.

$$
\begin{aligned}
|h_1(\rho(t)))| &\le C_5\\
|h_2(\rho(t)))| &\le C_6
\end{aligned}
$$

A$_5$ There exists a constants $\theta_1,\theta_2,\theta_3\in C([0,1],R^+)$ such that

$$
\begin{aligned}
|g(t,\rho(t),\rho(mt),\,{}_0^{ABC}D_t^\iota\rho(t))| \le\ &\theta_1(t)+\theta_2(t)(|\rho(t)|+|\rho(mt)|)\\
&+\theta_3(t)|\,{}_0^{ABC}D_t^\iota\rho(t)|, t\in[0,1], \rho\in\mathfrak{W}
\end{aligned}
$$

A₆ $g, F_k, \bar{F}_k$ be continuous & $\exists$ a constant $B, B^*, M, M^* > 0$ such that

$$|F_k(\rho(t))| \leq \mathscr{B}\|\rho\|_{\mathscr{PC}} + \mathscr{B}^*$$
$$|\bar{F}_k(\rho(t))| \leq \mathscr{M}\|\rho\|_{\mathscr{PC}} + \mathscr{M}^*$$

Theorem 2. *Suppose (A_1)-(A_6) are met then (1.1) contains at least one solution.*

Proof. **Step 1:** To demonstrate the continuity of $\mathscr{I}$.
i.e., $\{\rho_n\} \in \mathscr{W}$ with $\rho_n \to \rho \in \mathfrak{W}$.

$$\begin{aligned}
|\mathscr{I}\rho_n(t) - \mathscr{I}\rho(t)| = {} & \frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{t}(t-\mathfrak{z})^{\iota-1} \\
& \times|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1} \\
& |\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)} \\
& \Sigma_{0<t_k<t}|t-t_k|\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \\
& \Sigma_{0<t_k<t}|F_k(\rho_n(t_k)) - F_k(\rho(t_k))| + \Sigma_{0<t_k<t}|t-t_k||\bar{F}_k(\rho_n(t_k)) - \bar{F}_k(\rho(t_k))| \\
& + \frac{1}{p}\int_0^1|h_1(\rho_n(\mathfrak{z})) - h_1(\rho(\mathfrak{z}))|d\mathfrak{z} + \frac{q(2-\iota)}{p\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\iota_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho_n(t_j)) - F_j(\rho(t_j))| + \frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\rho_n(t_j)) - \bar{F}_j(\rho(t_j))| \\
& + \frac{q}{p^2}\int_0^1|h_1(\rho_n(\mathfrak{z})) - h_1(\rho(\mathfrak{z}))|d\mathfrak{z} + \frac{q^2(2-\iota)}{p^2\mathscr{N}(\iota-1)}|\Phi_{\rho,n}(t_k) - \Phi_\rho(t_k)| \\
& + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q^2}{p^2}\Sigma_{j=1}^{\mathfrak{N}}|\bar{F}_j(\rho_n(t_j)) - \bar{F}_j(\rho(t_j))| + \frac{q}{p^2}\int_0^1|h_2(\rho_n(\mathfrak{z})) - h_2(\rho(\mathfrak{z}))|d\mathfrak{z} \\
& + \frac{(2-\iota)t}{\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1} \\
& \times|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1} \\
& |\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota-1)} \\
& \Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z}) - \Phi_\rho(\mathfrak{z})|d\mathfrak{z}
\end{aligned}$$

$$
\begin{aligned}
&+t\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho_n(t_j))-F_j(\rho(t_j))|+t\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\rho_n(t_j))-\bar{F}_j(\rho(t_j))|\\
&+\frac{t}{p}\int_0^1|h_1(\rho_n(\mathfrak{z}))-h_1(\rho(\mathfrak{z}))|d\mathfrak{z}+\frac{q(2-\iota)t}{p\mathscr{N}(\iota-1)}|\Phi_{\rho,n}(t_k)-\Phi_\rho(t_k)|\\
&+\frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_{\rho,n}(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{qt}{p}\Sigma_{j=1}^{n}|\bar{F}_j(\rho_n(t_j))-\bar{F}_j(\rho(t_j))|+\frac{t}{p}\int_0^1|h_2(\rho_n(\mathfrak{z}))-h_2(\rho(\mathfrak{z}))|d\mathfrak{z}.
\end{aligned}
$$

where $\Phi_{\rho_n}(t),\Phi_\rho(t)\in\mathfrak{W}$ satisfies $\Phi_{\rho_n}(t)=g(t,\rho_n(t),\rho_n(mt),\Phi_{\rho_n}(t))$ &
$\Phi_\rho(t)=g(t,\rho(t),\rho(mt),\Phi_\rho(t))$. Applying A_2, one can obtain

$$
\begin{aligned}
|\Phi_{\rho_n}(t)-\Phi_\rho(t)|&=|g(t,\rho_n(t),\rho_n(mt),\Phi_{\rho_n}(t))-g(t,\rho(t),\rho(mt),\Phi_\rho(t))|\\
&\le L_g\Big(|\rho_n(t)-\rho(t)|+|\rho_n(mt)-\rho(mt)|+N_g|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\Big)
\end{aligned}
$$

i.e., $\|\Phi_{\rho_n}-\Phi_\rho\|\le\frac{2L_g}{1-N_g}\|\rho_n-\rho\|_{PC}$. As $n\to\infty,\rho_n\to\rho\Rightarrow\Phi_{\rho_n}\to\Phi_\rho$ $\exists\, b>0$ s.t., $\forall\, t,|\Phi_{\rho,n}(t)|\le b$ & $|\Phi_\rho(t)|\le b$.

$$
\begin{aligned}
\text{Then}\quad &(t-\mathfrak{z})^{\iota-1}|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\le 2b(t-\mathfrak{z})^{\iota-1}\\
&(t_j-\mathfrak{z})^{\iota-1}|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\le 2b(t_j-\mathfrak{z})^{\iota-1}\\
&(t_j-\mathfrak{z})^{\iota-2}|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\le 2b(t_j-\mathfrak{z})^{\iota-2}\\
&(1-\mathfrak{z})^{\iota-1}|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\le 2b(1-\mathfrak{z})^{\iota-1}\\
&(1-\mathfrak{z})^{\iota-2}|\Phi_{\rho_n}(t)-\Phi_\rho(t)|\le 2b(1-\mathfrak{z})^{\iota-2}
\end{aligned}
$$

$\forall\, t\in[0,1]$ then $\mathfrak{z}\to 2b(t-\mathfrak{z})^{\iota-1},\mathfrak{z}\to 2b(t_j-\mathfrak{z})^{\iota-1},\mathfrak{z}\to 2b(t_j-\mathfrak{z})^{\iota-2},\mathfrak{z}\to 2b(1-\mathfrak{z})^{\iota-1},\mathfrak{z}\to 2b(1-\mathfrak{z})^{\iota-2}$ are integrable.
Applying the dominated convergent theorem of Lebesgue, $\|\mathscr{I}\rho_n(t)-\mathscr{I}\rho(t)\|\to 0$ as $t\to\infty$. i.e., $\mathscr{I}$ exhibits continuity.
Step 2: Demonstrate that $\mathscr{I}$ exhibits boundedness
Let $\forall\,\rho\in\mathscr{E}=\{\rho\in\mathscr{W}\}:\|\rho\|_{PC}\le r^*$.
To demonstrate $\|\mathscr{I}\rho\|_{\mathfrak{W}}\le\eta$.

For $t \in (t_k, 1)$, we have

$$
\begin{aligned}
|\mathscr{I}\rho(t)| = {} & \frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{t}(t-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)} \\
& \times \Sigma_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{0<t_k<t}|t-t_k| \\
& \times \int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \Sigma_{0<t_k<t}|F_k(\rho(t_k))| + \Sigma_{0<t_k<t}|t-t_k||\bar{F}_k(\rho(t_k))| \\
& + \frac{1}{p}\int_0^1|h_1(\rho(\mathfrak{z}))|d\mathfrak{z} + \frac{q(2-\iota)}{p\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)} \\
& \int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)} \times \Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho(t_j))| \\
& + \frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\rho(t_j))| + \frac{q}{p^2}\int_0^1|h_1(\rho(\mathfrak{z}))|d\mathfrak{z} + \frac{q^2(2-\iota)}{p^2\mathscr{N}(\iota-1)}|\Phi_\rho(t_k)| \\
& + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q^2(\iota-1)}{p^2\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{q^2}{p^2}\Sigma_{j=1}^{\mathfrak{N}}|\bar{F}_j(\rho(t_j))| \\
& + \frac{q}{p^2}\int_0^1|h_2(\rho(\mathfrak{z}))|d\mathfrak{z} + \frac{(2-\iota)t}{\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1} \\
& |\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + \frac{(\iota-1)t}{\mathscr{N}(\iota-1)\Gamma(\iota-1)} \\
& \Sigma_{j=1}^{\mathfrak{N}}(1-t_j) \times \int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} + t\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho(t_j))| + t\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\rho(t_j))| \\
& + \frac{t}{p}\int_0^1|h_1(\rho(\mathfrak{z}))|d\mathfrak{z} \\
& + \frac{q(2-\iota)t}{p\mathscr{N}(\iota-1)}|\Phi_\rho(t_k)| + \frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{q(\iota-1)t}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}d \times \Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z} \\
& + \frac{qt}{p}\Sigma_{j=1}^{n}|\bar{F}_j(\rho(t_j))| + \frac{t}{p}\int_0^1|h_2(\rho(\mathfrak{z}))|d\mathfrak{z}.
\end{aligned}
$$

By (H_6),

$$
\begin{aligned}
|\Phi_\rho(t)| &\le |g(t,\rho(t),\rho(mt),\Phi_\rho(t)| \\
&\le \theta_1(t) + \theta_2(t)(|\rho(t)| + |\rho(mt)|) + \theta_3(t)|\Phi_\rho(t)|.
\end{aligned}
$$

where $\theta_1^* = \max_{t\in J}|\theta_1(t)| < 1, \theta_2^* = \max_{t\in J}|\theta_2(t)| < 1, \theta_3^* = \max_{t\in J}|\theta_3(t)| < 1$. Taking maximum,

$$
\begin{aligned}
|\Phi_\rho(t)| &\le \theta_1^* + 2\theta_2^* r^* + \theta_3^*|\Phi_\rho(t)| \\
\Rightarrow \max_{t\in J}|\Phi_\rho(t)| &\le \frac{\theta_1^* + 2\theta_2^* r^*}{1-\theta_3^*} := \tau^*
\end{aligned}
$$

then the above inequality becomes

$$\begin{aligned}\|\mathscr{I}\rho(t)\| \leq \Big[&\frac{2-\iota}{p^2\mathscr{N}(\iota-1)}(2p^2+2pq+q^2)+\frac{\iota-1}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)}(2p+\mathfrak{N}p+(1+\mathfrak{N})(p+q))\\ &+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}(p^2+q^2(1+\mathfrak{N})+pq(1+2\mathfrak{N}))\Big]\tau^*+\frac{(Br^*+B^*)}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\ &+\frac{(Mr^*+M^*)}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_5}{p^2}(2p+q)+\frac{C_6}{p^2}(p+q):\eta\end{aligned}$$

$\therefore \mathscr{I}$ be bounded.

Step 3: To demonstrate equicontinuity of $\mathscr{I}$.

Assume $\omega_1,\omega_2 \in (t_k,t_{k+1}]$ s. t. $\omega_1<\omega_2$ & $\mathscr{E}$ exhibits boundedness in $\mathfrak{W}$ as in step 2. Now, for $\rho\in\mathscr{E}$, we have

$$\begin{aligned}|\mathscr{I}\rho(\omega_2)-\mathscr{I}\rho(\omega_1)| = &\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{\omega_1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_{\omega_1}^{\omega_2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\\ &\times\int_0^{\omega_1}(t-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{\omega_1}^{\omega_2}(t-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{0<t_k<\omega_2-\omega_1}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\times\Sigma_{0<t_k<\omega_2-\omega_1}|(\omega_2-t_k)-(\omega_1-t_k)|\\ &\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\Sigma_{0<t_k<\omega_2-\omega_1}|F_k(\rho(t_k))|\\ &+\Sigma_{0<t_k<\omega_2-\omega_1}|(\omega_2-t_k)-(\omega_1-t_k)||\bar{F}_k(\rho(t_k))|+\frac{(2-\iota)(\omega_2-\omega_1)}{\mathscr{N}(\iota-1)}\\ &\int_0^{t_k}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)(\omega_2-\omega_1)}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{(\iota-1)(\omega_2-\omega_1)}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)(\omega_2-\omega_1)}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\\ &\times\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+(\omega_2-\omega_1)\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho(t_j))|\\ &+(\omega_2-\omega_1)\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\rho(t_j))|+\frac{(\omega_2-\omega_1)}{p}\int_0^1|h_1(\rho(\mathfrak{z}))|d\mathfrak{z}\\ &+\frac{q(2-\iota)(\omega_2-\omega_1)}{p\mathscr{N}(\iota-1)}|\Phi_\rho(t_k)|+\frac{q(\iota-1)(\omega_2-\omega_1)}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\\ &|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)(\omega_2-\omega_1)}{p\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{q(\omega_2-\omega_1)}{p}\Sigma_{j=1}^{n}|\bar{F}_j(\rho(t_j))|+\frac{(\omega_2-\omega_1)}{p}\int_0^1|h_2(\rho(\mathfrak{z}))|d\mathfrak{z}\\ \leq &\Big[\frac{2-\iota}{p\mathscr{N}(\iota-1)}(p\omega_2+(\omega_2-\omega_1)(p+q))+\frac{\iota-1}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)}(2(\omega_2-\omega_1)^\iota\\ &+\omega_2^\iota-\omega_1^\iota+(\omega_2-\omega_1)(2+\mathfrak{N}))+\frac{(\iota-1)(\omega_2-\omega_1)}{\mathscr{N}(\iota-1)\Gamma(\iota)}((1+\mathfrak{N})(1+\frac{q}{p}))\Big]\tau^*\\ &+(1+\mathfrak{N})(\omega_2-\omega_1)(Br^*+B^*)+(\omega_2-\omega_1)(Mr^*+M^*)\\ &\times(|(\omega_2-t_k)-(\omega_1-t_k)|+\mathfrak{N}(1+\frac{q}{p}))+\frac{(\omega_2-\omega_1)}{p}(C_5+C_6)\end{aligned}\tag{3.6}$$

As $\omega_1\to\omega_2$, the R.H.S of (3.6) tends to 0. Using Arzela-Ascoli theorem, $\mathscr{I}$ exhibits complete continuity.

Step 4: A priori estimates: Assume $\mathscr{E}_\sigma:\{\rho\in\mathscr{W}:\rho=\sigma\mathscr{I}\rho,$ for $0<\sigma<1\}$. One must demonstrate that it exhibits boundedness. Assume $\rho\in\mathscr{E}_{\tau^*}$, so that $\rho=\tau^*\mathscr{I}\rho$ for some $0<\rho<1$.

Thus, as established in step 2, $\forall\, t\in[0,1]$, one can obtain

$$\begin{aligned}\|\mathscr{I}\rho\| \le & \frac{\sigma\tau^*}{\mathscr{N}(\iota-1)}\Big[\frac{2-\iota}{p^2}(2p^2+2pq+q^2)+\frac{\iota-1}{p\Gamma(\iota+1)}(2p+\mathfrak{N}p+(1+\mathfrak{N})(p+q))\\ &+\frac{\iota-1}{\Gamma(\iota)}(p^2+q^2(1+\mathfrak{N})+pq(1+2\mathfrak{N}))\Big]+\frac{(Br^*+B^*)}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\ &+\frac{(Mr^*+M^*)}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_5}{p^2}(2p+q)+\frac{C_6}{p^2}(p+q)\\ &\le\sigma\eta<\eta\end{aligned}$$

$\therefore \mathscr{E}_\sigma$ be bounded.
Therefore, it has been demonstrated that $\mathscr{I}$ possesses at least one FP, as established by Theorem 1.

Theorem 3.*If (A_1) & (A_3) & the inequality*

$$\begin{aligned}&\frac{2L_g}{(1+N_g)}\Big(\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathscr{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)}\\ &+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big)+\frac{C_1}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\ &+\frac{C_2}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_3}{p^2}(2p+q)+\frac{C_4}{p^2}(q+p)\le 1,\end{aligned} \tag{3.7}$$

If these conditions are met, then (1.1) *possesses a unique solution.*

*Proof.*Let $\rho,\bar{\rho}\in\mathscr{W}$, one can obtain

$$\begin{aligned}|\mathscr{I}\rho(t)-\mathscr{I}\bar{\rho}(t)| = & \frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{t}(t-\mathfrak{z})^{\iota-1}\\ &|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{0<t_k<t}|t-t_k|\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\\ &+\Sigma_{0<t_k<t}|F_k(\rho(t_k))-F_k(\bar{\rho}(t_k))|+\Sigma_{0<t_k<t}|t-t_k||\bar{F}_k(\rho(t_k))-\bar{F}_k(\bar{\rho}(t_k))|\\ &+\frac{1}{p}\int_0^1|h_1(\rho(\mathfrak{z}))-h_1(\bar{\rho}(\mathfrak{z}))|d\mathfrak{z}+\frac{q(2-\iota)}{p\mathscr{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\\ &+\frac{q(\iota-1)}{p\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar{\rho}}(\mathfrak{z})|d\mathfrak{z}\end{aligned}$$

$$
\begin{aligned}
&+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho(t_j))-F_j(\bar\rho(t_j))|+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar F_j(\rho(t_j))-\bar F_j(\bar\rho(t_j))|\\
&+\frac{q}{p^2}\int_0^1|h_1(\rho(\mathfrak{z}))-h_1(\bar\rho(\mathfrak{z}))|d\mathfrak{z}+\frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}|\Phi_{\rho,n}(t_k)-\Phi_\rho(t_k)|\\
&+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q^2}{p^2}\Sigma_{j=1}^{\mathfrak{N}}|\bar F_j(\rho(t_j))-\bar F_j(\bar\rho(t_j))|+\frac{q}{p^2}\int_0^1|h_2(\rho(\mathfrak{z}))-h_2(\bar\rho(\mathfrak{z}))|d\mathfrak{z}\\
&+\frac{(2-\iota)t}{\mathcal{N}(\iota-1)}\int_0^{t_k}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+t\Sigma_{j=1}^{\mathfrak{N}}|F_j(\rho(t_j))-F_j(\bar\rho(t_j))|+t\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar F_j(\rho(t_j))-\bar F_j(\bar\rho(t_j))|\\
&+\frac{t}{p}\int_0^1|h_1(\rho(\mathfrak{z}))-h_1(\bar\rho(\mathfrak{z}))|d\mathfrak{z}+\frac{q(2-\iota)t}{p\mathcal{N}(\iota-1)}|\Phi_\rho(t_k)-\Phi_{\bar\rho}(t_k)|\\
&+\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\rho(\mathfrak{z})-\Phi_{\bar\rho}(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{qt}{p}\Sigma_{j=1}^{n}|\bar F_j(\rho(t_j))-\bar F_j(\bar\rho(t_j))|+\frac{t}{p}\int_0^1|h_2(\rho(\mathfrak{z}))-h_2(\bar\rho(\mathfrak{z}))|d\mathfrak{z}.
\end{aligned}
$$

Consider

$$\Phi_\rho(t)=g(t,\rho(t),\rho(mt),\Phi_\rho(t)),\Phi_{\bar\rho}(t)=g(t,\bar\rho(t),\bar\rho(mt),\Phi_{\bar\rho}(t)).$$

Employing A_2, one can obtain

$$
\begin{aligned}
|\Phi_\rho(t)-\Phi_{\bar\rho}(t)|&=|g(t,\rho(t),\rho(mt),\Phi_\rho(t))-g(t,\bar\rho(t),\bar\rho(mt),\Phi_{\bar\rho}(t))|\\
&\le L_g\Big(|\rho(t)-\bar\rho(t)|+|\rho(mt)-\bar\rho(mt)|+N_g|\Phi_\rho(t)-\Phi_{\bar\rho}(t)|\Big)\\
\|\Phi_\rho-\Phi_{\bar\rho}\|_{PC}&\le\frac{2L_g}{1-N_g}\|\rho-\bar\rho\|_{PC}.
\end{aligned}
$$

From above inequality, we get

$$\begin{aligned}\|\mathscr{I}\rho(t)-\mathscr{I}\bar{\rho}(t)\|_{PC} \leq \Big[&\frac{2L_g}{(1+N_g)}\Big(\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathscr{N}(\iota-1)}\\&+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)}\\&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big)\\&+\frac{C_1}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\&+\frac{C_2}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_3}{p^2}(2p+q)\\&+\frac{C_4}{p^2}(q+p)\Big]\|\rho-\bar{\rho}\|_{PC}\\\|\mathscr{I}\rho(t)-\mathscr{I}\bar{\rho}(t)\|_{PC} \leq K&\|\rho-\bar{\rho}\|_{PC}\end{aligned}$$

$$\begin{aligned}K=&\frac{2L_g}{(1+N_g)}\Big(\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathscr{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)}\\&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big)+\frac{C_1}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\&+\frac{C_2}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_3}{p^2}(2p+q)+\frac{C_4}{p^2}(q+p)<1.\end{aligned}$$

$\mathscr{I}$ is contraction.
∴ According to Banach's FPT, the solution to (1.1) is unique. This concludes proof.

4 Stability

Theorem 4. *If (A_1) & (A_3) & the inequality (3.7) holds, Thus, (1.1) exhibits UH stability.*

Proof. Suppose $\chi \in \mathfrak{W}$ be the solution of inequality (6) in [6] & ρ be a solution (1.1) i.e., unique. So that applying Remark 1 in [6], for $t \in [0,1], t \neq t_k, k=1,,...\mathfrak{N}$, one can obtain

$$\left.\begin{aligned}({}_0^{ABC}D_t^{\iota})\chi(t) &= g(t,\chi(t),\chi(mt),{}_0^{ABC}D_t^{\iota}\chi(t))+y(t),\ t\in[0,1],\ 0<m<1,\ t\neq t_k\\ p\chi(0)+q\chi'(0) &= \textstyle\int_0^1 h_1(\chi(\mathfrak{z}))d\mathfrak{z},\ p\chi(1)+q\chi'(1)=\int_0^1 h_2(\chi(\mathfrak{z}))d\mathfrak{z},\ 1<\iota\leq 2\\ \triangle\chi(t_k) &= F_k(\chi(t_k)),\ \triangle\chi'(t_k)=\bar{F}_k(\chi(t_k)),\ k=1,,...\mathfrak{N}\end{aligned}\right\} \tag{4.1}$$

& the soln is

$$\chi(t)=\begin{cases}\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^t\Phi_\chi d\mathfrak{z}+\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^t y(\mathfrak{z})d\mathfrak{z}+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^t(t-\mathfrak{z})^{\iota-1}\Phi_\chi d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_0^t(t-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z}+\bar{\mathfrak{G}},\ t\in[0,t_1];\\\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{t_k}\Phi_\chi d\mathfrak{z}+\frac{2-\iota}{\mathscr{N}(\iota-1)}\int_0^{t_k}y(\mathfrak{z})d\mathfrak{z}+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t(t-\mathfrak{z})^{\iota-1}\Phi_\chi d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^t(t-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^k\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\chi d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^k\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^k(t-t_j)\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\chi d\mathfrak{z}\\+\frac{\iota-1}{\mathscr{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^k(t-t_j)\int_{t_j-1}^{t_j}(t_j-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z}\\+\Sigma_{j=1}^kF_j(\chi(t_j))+\Sigma_{j=1}^ky_j+\Sigma_{j=1}^k(t-t_j)\bar{F}_j(\chi(t_j))\\+\Sigma_{j=1}^k(t-t_j)y_j+\bar{\mathfrak{G}},\ t\in(t_k,1],\ k=1,,...\mathfrak{N};\end{cases}$$

here

$$
\begin{aligned}
\bar{\mathfrak{G}} = & \frac{1}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} + \frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\int_0^{t_k}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\int_0^{t_k} y(\mathfrak{z})d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)} \times \int_{t_k}^1 (1-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1 (1-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)} \\
& \times \Sigma_{j=1}^n \int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z} \\
& + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \times \Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} + \frac{q}{p}\Sigma_{j=1}^n F_j(\rho(t_j)) + \frac{q}{p}\Sigma_{j=1}^n y_j + \frac{q}{p}\Sigma_{j=1}^n(1-t_j) \\
& \bar{F}_j(\rho(t_j)) + \frac{q}{p}\Sigma_{j=1}^n(1-t_j)y_j + \frac{q}{p^2}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} + \frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}\Phi_\rho(t_k) \\
& + \frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}y(t_k) + \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} \\
& + \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} + \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} + \frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \times \Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} + \frac{q^2}{p^2}\Sigma_{j=1}^n\bar{F}_j(\rho(t_j)) + \frac{q^2}{p^2}\Sigma_{j=1}^n y_j - \frac{q}{p^2}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z} \\
& - \frac{(2-\iota)t}{\mathcal{N}(\iota-1)}\int_0^{t_k}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(2-\iota)t}{\mathcal{N}(\iota-1)}\int_0^{t_k}y(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)} \\
& \times\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)} \\
& \times \Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}y(\mathfrak{z})d\mathfrak{z} \\
& - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \times \Sigma_{j=1}^n(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} - t\Sigma_{j=1}^n F_j(\rho(t_j)) - t\Sigma_{j=1}^n y_j \\
& - t\Sigma_{j=1}^n(1-t_j)\bar{F}_j(\rho(t_j)) - t\Sigma_{j=1}^n(1-t_j)y_j - \frac{t}{p}\int_0^1 h_1(\rho(\mathfrak{z}))d\mathfrak{z} - \frac{q(2-\iota)t}{p\mathcal{N}(\iota-1)}\Phi_\rho(t_k) \\
& - \frac{q(2-\iota)t}{p\mathcal{N}(\iota-1)}y(t_k) - \frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} \\
& - \frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^1(1-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} - \frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)} \\
& \Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\Phi_\rho(\mathfrak{z})d\mathfrak{z} - \frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)} \times \Sigma_{j=1}^n\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}y(\mathfrak{z})d\mathfrak{z} \\
& - \frac{qt}{p}\Sigma_{j=1}^n\bar{F}_j(\rho(t_j)) - \frac{qt}{p}\Sigma_{j=1}^n y_j(\rho(t_j)) + \frac{t}{p}\int_0^1 h_2(\rho(\mathfrak{z}))d\mathfrak{z}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\chi(t)-\rho(t)| = {} & \frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^{t_k}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{t}(t-\mathfrak{z})^{\iota-1}\\
& \times|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1}\\
& |\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{0<t_k<t}|t-t_k|\\
& \int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\Sigma_{0<t_k<t}|F_k(\chi(t_k))-F_k(\rho(t_k))|\\
& +\Sigma_{0<t_k<t}|t-t_k||\bar{F}_k(\chi(t_k))-\bar{F}_k(\rho(t_k))|+\frac{1}{p}\int_0^1|h_1(\chi(\mathfrak{z}))-h_1(\rho(\mathfrak{z}))|d\mathfrak{z}\\
& +\frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\int_0^{t_k}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\\
& \int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}\\
& |\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\times\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\\
& |\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}|F_j(\chi(t_j))-F_j(\rho(t_j))|\\
& \quad+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\chi(t_j))-\bar{F}_j(\rho(t_j))|+\frac{q}{p^2}\int_0^1|h_1(\chi(\mathfrak{z}))-h_1(\rho(\mathfrak{z}))|d\mathfrak{z}\\
& +\frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}|\Phi_\chi(t_k)-\Phi_\rho(t_k)|+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}\\
& \times|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}\\
& \times|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{q^2}{p^2}\Sigma_{j=1}^{\mathfrak{N}}|\bar{F}_j(\chi(t_j))-\bar{F}_j(\rho(t_j))|\\
& +\frac{q}{p^2}\int_0^1|h_2(\chi(\mathfrak{z}))-h_2(\rho(\mathfrak{z}))|d\mathfrak{z}\\
& +\frac{(2-\iota)t}{\mathcal{N}(\iota-1)}\int_0^{t_k}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}\\
& \times|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
& +\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
& +t\Sigma_{j=1}^{\mathfrak{N}}|F_j(\chi(t_j))-F_j(\rho(t_j))|+t\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|\bar{F}_j(\chi(t_j))-\bar{F}_j(\rho(t_j))|\\
& +\frac{t}{p}\int_0^1|h_1(\chi(\mathfrak{z}))-h_1(\rho(\mathfrak{z}))|d\mathfrak{z}+\frac{q(2-\iota)t}{p\mathcal{N}(\iota-1)}|\Phi_\chi(t_k)-\Phi_\rho(t_k)|\\
& +\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
& +\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|\Phi_\chi(\mathfrak{z})-\Phi_\rho(\mathfrak{z})|d\mathfrak{z}\\
& +\frac{qt}{p}\Sigma_{j=1}^{n}|\bar{F}_j(\chi(t_j))-\bar{F}_j(\rho(t_j))|+\frac{t}{p}\int_0^1|h_2(\chi(\mathfrak{z}))-h_2(\rho(\mathfrak{z}))|d\mathfrak{z}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2-\iota}{\mathcal{N}(\iota-1)}\int_0^{t_k}|y(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{t}(t-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}+\frac{\iota-1}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\\
&\times\Sigma_{0<t_k<t}|t-t_k|\int_{t_k-1}^{t_k}(t_k-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}+\Sigma_{0<t_k<t}|y(t_k)|+\Sigma_{0<t_k<t}|t-t_k||y(t_k)|\\
&+\frac{q(2-\iota)}{p\mathcal{N}(\iota-1)}\int_0^{t_k}|y(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}+\frac{q(\iota-1)}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\\
&\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}|y_j|+\frac{q}{p}\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|y_j|\\
&\quad+\frac{q^2(2-\iota)}{p^2\mathcal{N}(\iota-1)}|y(t_k)|+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q^2(\iota-1)}{p^2\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}+\frac{q^2}{p^2}\Sigma_{j=1}^{\mathfrak{N}}|y_j|\\
&+\frac{(2-\iota)t}{\mathcal{N}(\iota-1)}\int_0^{t_k}|y(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-1}|y(\mathfrak{z})|d\mathfrak{z}+\frac{(\iota-1)t}{\mathcal{N}(\iota-1)\Gamma(\iota-1)}\\
&\times\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}+t\Sigma_{j=1}^{\mathfrak{N}}|y_j|+t\Sigma_{j=1}^{\mathfrak{N}}(1-t_j)|y_j|\\
&+\frac{q(2-\iota)t}{p\mathcal{N}(\iota-1)}|y(t_k)|+\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\int_{t_k}^{1}(1-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}\\
&+\frac{q(\iota-1)t}{p\mathcal{N}(\iota-1)\Gamma(\iota-1)}\Sigma_{j=1}^{\mathfrak{N}}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\iota-2}|y(\mathfrak{z})|d\mathfrak{z}+\frac{qt}{p}\Sigma_{j=1}^{n}|y_j|.
\end{aligned}
$$

$$
\begin{aligned}
&\|\chi(t)-\rho(t)\|\le\Big[\frac{2L_g}{(1+N_g)}\Big(\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathcal{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathcal{N}(\iota-1)\Gamma(\iota+1)}\\
&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathcal{N}(\iota-1)\Gamma(\iota)}\Big)+\frac{C_1}{p}(p(1+\mathfrak{N})+q\mathfrak{N})\\
&+\frac{C_2}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_3}{p^2}(2p+q)+\frac{C_4}{p^2}(q+p)\Big]\|\chi-\rho\|_{PC}\\
&+\Big[\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathcal{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathcal{N}(\iota-1)\Gamma(\iota+1)}\\
&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathcal{N}(\iota-1)\Gamma(\iota)}\\
&+\frac{1}{p^2}(2p^2(1+\mathfrak{N})+3pq\mathfrak{N}+q^2\mathfrak{N})\Big]\varepsilon
\end{aligned}
$$

$$\Rightarrow\|\chi-\rho\|_{PC}\le C_g\varepsilon$$

$$
\begin{aligned}
C_g=\frac{1}{1-K}\Big[&\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathcal{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathcal{N}(\iota-1)\Gamma(\iota+1)}\\
&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathcal{N}(\iota-1)\Gamma(\iota)}\\
&+\frac{1}{p^2}(2p^2(1+\mathfrak{N})+3pq\mathfrak{N}+q^2\mathfrak{N})\Big]
\end{aligned}
$$

$\therefore$ (1.1) is UH stable.

Corollary 2. *Suppose in Theorem 4, define* $x(\varepsilon)=C_g(\varepsilon)$ *s. t.* $x(0)=0$, *so that* (1.1) *represents generalized UH stable.*

A7Consider, x be a non decreasing fn belonging to $C(I<R), \exists\ \mu_x>0, \varepsilon>0$ s.t.,

$$_0I_t^\iota x(t) \leq \mu_x x(t)$$

Theorem 5. *If* (A_1) & (A_3), (A_7) *& the inequality* (3.7) *holds, then* (1.1) *is UHR stability pertaining to* (τ, x).

*Proof.*Suppose $\chi \in \mathfrak{W}$ is a soln of (6) in [6], ρ be a unique solution (1.1) by theorem 4, & Remark 3 in [6] we get the results

$$\begin{aligned}
\|\chi(t)-\rho(t)\| \leq & \Big[\frac{2L_g}{(1+N_g)}\Big(\frac{(2-\iota)(2p^2+2pq+q^2)}{p^2\mathscr{N}(\iota-1)}+\frac{(\iota-1)(2p+q(1+\mathfrak{N})+p(1+2\mathfrak{N}))}{p\mathscr{N}(\iota-1)\Gamma(\iota+1)} \\
&+\frac{(\iota-1)(p(p+q\mathfrak{N})+q^2(1+\mathfrak{N})+pq(1+\mathfrak{N}))}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big)+\frac{C_1}{p}(p(1+\mathfrak{N})+q\mathfrak{N}) \\
&+\frac{C_2}{p^2}(p^2(1+\mathfrak{N})+2pq\mathfrak{N}+q^2\mathfrak{N})+\frac{C_3}{p^2}(2p+q)+\frac{C_4}{p^2}(q+p)\Big]\|\chi-\rho\|_{PC} \\
&+x(t)\mu_x\varepsilon\Big[\frac{2(2-\iota)(p+q)}{p\mathscr{N}(\iota-1)}+\frac{(\iota-1)(3+\mathfrak{N})}{\mathscr{N}(\iota-1)\Gamma(\iota+1)} \\
&+\frac{(\iota-1)(p^2(1+\frac{q}{p})(1+\mathfrak{N})+q^2(1+\mathfrak{N})+2pq\mathfrak{N})}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big] \\
&+\tau\varepsilon\Big[\frac{1}{p^2}(2p^2(1+\mathfrak{N})+3pq\mathfrak{N}+q^2\mathfrak{N}+\frac{q^2(2-\iota)}{\mathscr{N}(\iota)})\Big]
\end{aligned}$$

$$\begin{aligned}
\|\chi-\rho\| \leq & \frac{\Big(x(t)\mu_x\Big[\frac{2(2-\iota)(p+q)}{p\mathscr{N}(\iota-1)}+\frac{(\iota-1)(3+\mathfrak{N})}{\mathscr{N}(\iota-1)\Gamma(\iota+1)}+\frac{(\iota-1)(p^2(1+\frac{q}{p})(1+\mathfrak{N})+q^2(1+\mathfrak{N})+2pq\mathfrak{N})}{p^2\mathscr{N}(\iota-1)\Gamma(\iota)}\Big]}{(1-K)} \\
&+\frac{\tau\Big[\frac{1}{p^2}(2p^2(1+\mathfrak{N})+3pq\mathfrak{N}+q^2\mathfrak{N}+\frac{q^2(2-\iota)}{\mathscr{N}(\iota)})\Big]\Big)\varepsilon}{(1-K)}
\end{aligned}$$

where $K<1$. $\therefore$ (1.1) is UHR stable.

Conclusion

Our research delves into existence, uniqueness, and stability of fractional impulsive differential equations through the Atangana-Baleanu derivative, by leveraging the application of Schafer's and the Banach fixed-point theorem. Also, the stability results are thoroughly analyzed, with integral boundary conditions and mixed delays enriching the depth and rigor of the investigation.

Acknowledgments

The authors extend their appreciation to Northern Border University, Saudi Arabia, for supporting this work through project number (NBU-CRP-2025-3021). This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445). The authors are thankful to the Deanship of Graduate Studies and Scientific Research at University of Bisha for supporting this work through the Fast-Track Research Support Program.

Conflict of interest statement

Authors has no conflict of interest.

References

[1] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their solution and Some of Their Applications*, San Diego: Academic Press (1999), 1-340.

[2] K. S. Nisar, M. Farman, M. Abdel-Aty and C. Ravichandran, *A review of fractional order epidemic models for life sciences problems: Past, present and future*, Alex. Eng. J. **95**(2024), 283-305.

[3] K. S. Nisar, M. Farman, M. Abdel-Aty and C. Ravichandran, *A review of fractional-order models for plant epidemiology*, Prog. Fract. Differ. Appl. **10**(3)(2024), 489-521.

[4] B. Ghanbari and A. Atangana, *A new application of fractional Atangana-Baleanu derivatives: Designing ABC-fractional masks in image processing*, Physica A, (2019), 123516.

[5] I. A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space*, Carpath J. Math. **26**(2010), 103-107.

[6] A. Ali, K. Shah, T. Abdejawad, I. Mahariq and M. Rashdan, *Mathematical analysis of nonlinear boundary value problem of proportional delay implict fractional differential equations with impulsive conditions*, Boundary Value Problem. **2021**(2021):7.

[7] A. Atangana and D. Baleanu, *New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model*, Therm. Sci. **20**(2)(2016): 763-769.

[8] A. Morsy, C. Anusha, K. S. Nisar and C. Ravichandran, *Results on generalized neutral fractional impulsive dynamic equation over time scales using nonlocal initial condition*, AIMS Math. **9**(4)(2024), 8292-8310.

[9] Kamran, M. Asif, K. Shah, B. Abdalla and T. Abdeljawad, *Numerical soln of Bagley-Torvik equation including Atangana-Baleanu derivative arising in fluid mechanics*, Results Phys. **49**(2023), 106468.

[10] M. Partohaghighi, M. Mortezaee, A, Akgül and S.M. Eldin, *Numerical estimation of the fractional advection-dispersion equation under the modified Atangana-Baleanu-Caputo derivative*, Results Phys. **49**(2023), 106451.

[11] F. Jarad, T. Abdeljawad and Z. Hammouch, *On a class of ordinary differential equations in the frame of Atangana-Baleanu derivative*, Chaos. Soliton. Fract. **117**(2018), 16-20.

[12] C. Ravichandran, K. Logeswari and F. Jarad, *New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations*, Chaos. Soliton. Fract. **125**(2019):194-200.

[13] K. Logeswari and C. Ravichandran, *A new exploration on existence of fractional neutral integro-differential equations in the concept of Atangana-Baleanu derivative*, Physica A, 544(2020):1-10.

[14] C. Ravichandran, K. Logeswari, S.K. Panda and K.S. Nisar, *On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions*, Chaos. Soliton. Fract. **139**(2020), 1-9.

[15] E. Bonyah, J.F. Gómez-Aguilar and A. Adu, *Stability analysis and optimal control of a fractional human African trypanosomiasis model*, Chaos. Soliton. Fract. **117**(2018):150-160.

[16] E.O. Alzahrani and M.A Khan, *Modeling the dynamics of Hepatitis E with optimal control*, Chaos. Soliton. Fract. **116**(2018), 287-301.

[17] A. Tassaddiq, I. Khan and K. S. Nisar, *Heat transfer analysis in sodium alginate based nanofluid using MoS2 nanoparticles: Atangana-Baleanu fractional model*, Chaos. Soliton. Fract. **130**(2020), 1-12.

[18] Z. Korpinar, M. Inc and M. Bayram, *Theory and application for the system of fractional Burger equations with Mittag leffler kernel*, Appl. Math. Comput. **367**(2020), 1-11.

[19] D. Filali, A. Ali, Z. Ali, M. Akram and M. Dilshad, *Atangana-Baleanu-Caputo differential equations with mixed delay terms and integral boundary conditions*, Math. Meth. Appl. Sci. **46**(2023):10435-10449.

[20] T. Abdeljawad and D. Baleanu, *Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels*, Adv. Differ. Equ. **2016**(1)(2016):1-18.

[21] T. Abdelijawad and D. Baleanu, *Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel*, J. Nonlinear Sci. App. **10**(3)(2017):1098-1107.

[22] S.T.M. Thabet, M. Abdo, K. Shah and T. Abdeljawad, *Study of tranmission dynamics of COVID-19 mathematical model under ABC fractional order derivative*, Results Phys. **19**(2020), 1-10.