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# Local Fractional Variational Iteration Algorithms for the Parabolic Fokker-Planck Equation Defined on Cantor Sets

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**Abstract:** In this article, we apply the local fractional variational iteration algorithms for solving the parabolic Fokker-Planck equation which is defined on Cantor sets. It is shown by comparing with the three LFVIAs that the LFVIA-II is the easiest to obtain the non-differentiable solutions for linear local fractional partial differential equations. Several other related recent works dealing with local fractional derivative operators on Cantor sets are also indicated.

**Keywords:** Approximate solution, parabolic Fokker-Planck equation, local fractional derivative operators, Cantor sets.

## 1 Introduction, Motivation and Preliminaries

Fractional calculus [1,2,3,4,5,6,7,8,9] has found successful applications in science and engineering, such as fractional-order signals, physics, bioengineering, and dynamics of particles. As an important fractional-order PDEs arising in mathematical physics, the space-and time-fractional Fokker-Planck equation was first derived from the generalised master equation [10] with the time-fractional derivative and after that it ws developed in [11]. Based on it, Barkai [12] successfully found the solutions of the continuous time random walk, whistle Odibat and Momani [13] suggested VIM and ADM to solve it. Meanwhile, Den presented the FEM to obtain the numerical solution [14]. A new version of the time-fractional FP equation was suggested by using dynamical systems of FBM [15] and the fractal time evolution with a critical exponent [16]. Another version of the space-fractional FP equation from the fractional Liouville equation via fractional-power systems [17,18] and the fractional-order governing equation of Lévy motion [19] was reported. Based on this, the upwind difference method was considered by Liu [20] to deal with it. Meanwhile, the finite difference method was proposed by Chen et al. [21] to solve it.

The various versions of local fractional calculus theory [22,23,24,25,26,27,28,29,30,31,32,33] were considered to describe the non-differentiable problems from local fractional PDEs in physics and science due to the surface and structure of materials, which are so-called fractals [34]. Local fractional VIM [35,36] was one of usual methods for finding non-differentiable solutions for the local fractional PDEs. The FP equation defined on Cantor sets was presented as follows (see [37,38]):

$$\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi(\zeta,\eta)=-\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi(\zeta,\eta)+\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi(\zeta,\eta). \tag{1}$$

The parabolic equation defined on Cantor sets can be presented as follows:

$$\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi(\zeta,\eta)=\chi(\zeta,\eta)\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi(\zeta,\eta)+\kappa(\zeta,\eta)\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi(\zeta,\eta), \tag{2}$$

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where $\chi(\zeta,\eta)$ and $\kappa(\zeta,\eta)$ are parameters.

The goal of this article is to suggest the local fractional variational iteration algorithms (LFVIAs) [39,40] to the parabolic equation defined above on Cantor set.

This article is organized as follows. In Section 2, we review the local fractional calculus theory. In Section 3, the local fractional variational iteration algorithms are analyzed. In Section 4, the non-differentiable solution for the parabolic Fokker-Planck equation defined on Cantor sets is obtained. Finally, the conclusions are provided in Section 5.

## 2 Local Fractional Calculus Theory

In this section, we first introduce the concepts and notations of the theory of the local fractional calculus. To begin with, we recall the result for the local fractional continuity.

Let $\wp$ be a subset of the real line and be a fractal. If $g:(\wp,\rho)\to(\Xi,\rho)$ is a bi-Lipschitz mapping, then (for constants $\mu,\eta>0$ and $\wp\subset\mathbb{R}$) we have

$$\mu^s H^s(\wp)\leqq H^s(g(\wp))\leqq\eta^s H^s(\wp) \tag{3}$$

such that, for all $\zeta_1,\zeta_2\in\wp$,

$$\mu^\alpha|\zeta_1-\zeta_2|^\alpha\leqq|g(\zeta_1)-g(\zeta_2)|\leqq\eta^\alpha|\zeta_1-\zeta_2|^\alpha. \tag{4}$$

Following (4), we have

$$|g(\zeta_1)-g(\zeta_2)|\leqq\eta^\alpha|\zeta_1-\zeta_2|^\alpha, \tag{5}$$

which yields

$$|g(\zeta_1)-g(\zeta_2)|\leq\varepsilon^\alpha, \tag{6}$$

where $|\zeta_2-\zeta_1|<\kappa$, $\varepsilon,\kappa>0$, and $H^\alpha$ is an $\alpha$-dimensional Hausdorff measure.

Using (6), one leads to the following relation:

$$\lim_{\zeta\to\zeta_1} g(\zeta)=g(\zeta_1), \tag{7}$$

which exhibits the fact that $g(\zeta)$ is a so-called local fractional continuous function at $\zeta=\zeta_1$.

We write

$$g(\zeta)\in C_\alpha(\sigma,\upsilon), \tag{8}$$

if $g(\zeta)$is only a local fractional continuous function on the interval $(\sigma,\upsilon)$.

In order to make very fine distinction from the classical continuous function theory, we consider the new form (6) as the local fractional continuous condition. We notice that the constant is the special function because it belongs to either Lipschitz or bi-Lipschitz.

The $\alpha$-dimensional Hausdorff measure given by (see [41])

$$H^\alpha(\wp\cap(\sigma,\upsilon))=(\upsilon-\sigma)^\alpha, \tag{9}$$

shows that $\alpha$ $(0<\alpha<1)$ is a fractal dimension value, and it is the graph of Cantor function presented in [42] from $\sigma=0$ to $\upsilon=1$. Namely, it is directly deduced from fractal geometry.

There also are several other useful functions as follows [41]:

$$E_\alpha(\zeta^\alpha)=\sum_{i=0}^{\infty}\frac{\zeta^{i\alpha}}{\Gamma(1+i\alpha)}, \tag{10}$$

$$\sin_\alpha\zeta^\alpha=\sum_{i=0}^{\infty}\frac{(-1)^i\zeta^{(2i+1)\alpha}}{\Gamma[1+(2i+1)\alpha]} \tag{11}$$

and

$$\cos_\alpha\zeta^\alpha=\sum_{i=0}^{\infty}\frac{(-1)^i\zeta^{2\alpha i}}{\Gamma(1+2\alpha i)}. \tag{12}$$

For $0<\alpha<1$ and $g(\zeta)\in C_\alpha(\sigma,\upsilon)$, we define the local fractional derivative of $g(\zeta)$ of order $\alpha$ as follows (see [22,23,35,36,37,38,39,40,41]):

$$g^{(\alpha)}(\zeta_0)=\left.\frac{d^\alpha g(\zeta)}{d\zeta^\alpha}\right|_{\zeta=\zeta_0}=\lim_{\zeta\to\zeta_0}\frac{\Delta^\alpha(g(\zeta)-g(\zeta_0))}{(\zeta-\zeta_0)^\alpha}, \tag{13}$$

where

$$\Delta^{\alpha}\left(g(\zeta)-g(\zeta_0)\right)\cong\Gamma(1+\alpha)\Delta\left(g(\zeta)-g(\zeta_0)\right).$$

Let $0<\alpha<1$, $g(\zeta)\in C_{\alpha}(\sigma,\upsilon)$, $\Delta s_j=s_{j+1}-s_j$, $\Delta s=\max\left\{\Delta s_1,\Delta s_2,\Delta s_j,\cdots\right\}$ and $\left[s_j,s_{j+1}\right]$ $(j=0,\cdots,N-1)$ (with $s_0=\sigma$ and $s_N=\upsilon$) be a partition of the interval $(\sigma,\upsilon)$. Define the local fractional integral of $g(\zeta)$ of $\alpha$ order [22, 23, 35, 36, 37, 38, 39, 40, 41]

$$\begin{aligned}{}_{\sigma}I_{\upsilon}^{(\alpha)}g(s)&=\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\upsilon}g(s)(ds)^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta s\to0}\sum_{j=0}^{j=N-1}f(s_j)(\Delta s_j)^{\alpha}.\end{aligned}\tag{14}$$

In view of (14), we have the following relations:

$${}_{\sigma}I_{\upsilon}^{(\alpha)}g(s)=0\qquad(\sigma=\upsilon)\tag{15}$$

and

$${}_{\sigma}I_{\upsilon}^{(\alpha)}g(s)=-\,{}_{\upsilon}I_{\sigma}^{(\alpha)}g(s)\qquad(\sigma<\upsilon).\tag{16}$$

Basic formulas for the local fractional derivatives (LFDs) and the local fractional integrals (LFIs) of the functions are listed in Table 1.

**Table 1. List of local fractional derivatives and integrals of some functions**

| LFDs | LFIs |
|---|---|
| $\frac{d^{\alpha}}{d\zeta^{\alpha}}g(\zeta)=0$, where $g(\zeta)$ is a given differentiable function | ${}_0I_{\zeta}^{(\alpha)}g(\zeta)$ does not exist where $g(\zeta)$ is a given differentiable function |
| $\frac{d^{\alpha}}{d\zeta^{\alpha}}E_{\alpha}(\zeta^{\alpha})=E_{\alpha}(\zeta^{\alpha})$ | ${}_0I_{\zeta}^{(\alpha)}E_{\alpha}(\zeta^{\alpha})=E_{\alpha}(\zeta^{\alpha})-1$ |
| $\frac{d^{\alpha}}{d\zeta^{\alpha}}\left[\frac{\zeta^{\alpha}}{\Gamma(1+2\alpha)}\right]=\frac{\zeta^{2\alpha}}{\Gamma(1+\alpha)}$ | ${}_0I_{\zeta}^{(\alpha)}\frac{\zeta^{\alpha}}{\Gamma(1+\alpha)}=\frac{\zeta^{2\alpha}}{\Gamma(1+2\alpha)}$ |
| $\frac{d^{\alpha}}{d\zeta^{\alpha}}\cos_{\alpha}(\zeta^{\alpha})=-\sin_{\alpha}(\zeta^{\alpha})$ | ${}_0I_{\zeta}^{(\alpha)}\sin_{\alpha}(\zeta^{\alpha})=1-\cos_{\alpha}(\zeta^{\alpha})$ |
| $\frac{d^{\alpha}}{d\zeta^{\alpha}}\sin_{\alpha}(\zeta^{\alpha})=\cos_{\alpha}(\zeta^{\alpha})$ | ${}_0I_{\zeta}^{(\alpha)}\cos_{\alpha}(\zeta^{\alpha})=\sin_{\alpha}(\zeta^{\alpha})$ |

In Table 1, $g(\zeta)$ is a given differentiable function. For more details of the fractional calculus and the local fractional calculus theory, see [22, 23, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

## 3 The Local Fractional Variational Iteration Algorithms

Let $\varphi^{(\alpha)}(\zeta)$ take on the local fractional differential operator and $\sigma\leqq x\leqq\upsilon$. In view of a local fractional variational principle (see, for example, [23, 43]), the local fractional function reads as follows:

$$\Pi(\varphi)={}_{\sigma}I_{\upsilon}^{(\alpha)}g\left(\zeta,\varphi(\zeta),\varphi^{(\alpha)}(\zeta)\right).\tag{17}$$

The stationary condition of Eq. (17) is given by

$$\frac{\partial g}{\partial\phi}-\frac{d^{\alpha}}{d\zeta^{\alpha}}\left(\frac{\partial g}{\partial\phi^{(\alpha)}}\right)=0.\tag{18}$$

Let $L_{\alpha}$ and $N_{\alpha}$be linear and nonlinear local fractional operators respectively. A correction local fractional functional can be structured as follows (see [39, 40]):

$$\varphi_{n+1}(\zeta)=\varphi_n(\zeta)+{}_{\zeta_0}I_{\zeta}^{(\alpha)}\left\{\vartheta\left[L_{\alpha}\varphi_n(\tau)+N_{\alpha}\widetilde{\varphi}_n(\tau)\right]\right\},\tag{19}$$

where a general local fractional differential equation is presented as follows:

$$L_\alpha \varphi + N_\alpha \varphi = 0, \tag{20}$$

with a restricted local fractional variation $\widetilde{\varphi}_n$ and a fractal Lagrange multiplier $\vartheta$, which is given by (18). After completing the identification of the fractional Lagrange multiplier of (18), the LFVIAs of three types have the following forms: The LFVIA-I:

$$\varphi_{n+1}(\zeta) = \varphi_n(\zeta) + {}_{\zeta_0}I_\zeta^{(\alpha)} \{\vartheta [L_\alpha \varphi_n(\tau) + N_\alpha \varphi_n(\tau)]\}. \tag{21}$$

The LFVIA-II:

$$\varphi_{n+1}(\zeta) = \varphi_0(\zeta) + {}_{\zeta_0}I_\zeta^{(\alpha)} \{\vartheta N_\alpha \varphi_n(\tau)\}. \tag{22}$$

The LFVIA-III:

$$\varphi_{n+1}(\zeta) = \varphi_n(\zeta) + {}_{\zeta_0}I_\zeta^{(\alpha)} \{\vartheta [N_\alpha \varphi_n(\tau) - N_\alpha \varphi_{n-1}(\tau)]\}. \tag{23}$$

Making use of the LFVIAs, the non-differentiable solution of (20) reads as follows:

$$\varphi = \lim_{n\to\infty} \varphi_n. \tag{24}$$

The present method is also utilized to discuss the wave phenomena [44,45].

## 4 Non-Differentiable Solution

Let

$$\chi(\zeta,\eta) = -\frac{2\zeta^\alpha}{\Gamma(1+\alpha)} \quad \text{and} \quad \kappa(\zeta,\eta) = \frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}.$$

Then the parabolic FK equation defined on Cantor sets with local fractional derivative is given as follows:

$$\frac{\partial^\alpha}{\partial \zeta^\alpha}\varphi(\zeta,\eta) = -\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial \eta^\alpha}\varphi(\zeta,\eta) + \frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial \eta^{2\alpha}}\varphi(\zeta,\eta), \tag{25}$$

and the initial condition is given by

$$\varphi(0,\eta) = \frac{\eta^\alpha}{\Gamma(1+\alpha)}. \tag{26}$$

A correction local fractional functional can be structured as follows:

$$\begin{aligned} &\varphi_{n+1}(\zeta,\eta) = \varphi_n(\zeta,\eta) \\ &+ {}_0I_\zeta^{(\alpha)} \left\{ \vartheta \left[ \frac{\partial^\alpha}{\partial \zeta^\alpha}\varphi_n(\zeta,\eta) + \frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial \eta^\alpha}\varphi_n(\zeta,\eta) - \frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial \eta^{2\alpha}}\varphi_n(\zeta,\eta)\right]\right\}. \end{aligned} \tag{27}$$

The stationary conditions of Eq. (27) are presented as follows:

$$\vartheta^{(\alpha)} = 0 \tag{28}$$

and

$$1 + \vartheta\big|_{\tau=\zeta} = 0. \tag{29}$$

Hence, clearly, the fractal Lagrange multiplier is simply identified as follows:

$$\vartheta(\tau) = -1. \tag{30}$$

From (25) and (30), the three LFVIAs for parabolic FP equation defined on Cantor sets are structured as follows:

**The LFVIA-I:**

$$\begin{aligned} &\varphi_{n+1}(\zeta,\eta) = \varphi_n(\zeta,\eta) \\ &- {}_0I_\zeta^{(\alpha)} \left\{ \frac{\partial^\alpha}{\partial \zeta^\alpha}\varphi_n(\zeta,\eta) + \frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial \eta^\alpha}\varphi_n(\zeta,\eta) - \frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial \eta^{2\alpha}}\varphi_n(\zeta,\eta)\right\}. \end{aligned} \tag{31}$$

**The LFVIA-II:**

$$\varphi_{n+1}(\zeta,\eta)=\varphi_0(\zeta,\eta) - {}_0I_{\zeta}^{(\alpha)}\left\{\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_n(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_n(\zeta,\eta)\right\}. \tag{32}$$

**The LFVIA-III:**

$$\varphi_{n+1}(\zeta,\eta)=\varphi_n(\zeta,\eta)-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_n(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_n(\zeta,\eta)\right\} + {}_0I_{\zeta}^{(\alpha)}\left\{\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_{n-1}(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_{n-1}(\zeta,\eta)\right\}. \tag{33}$$

### 4.1 LFVIA-I *for the* FP *Equation Defined on Cantor Sets*

Following (26) and (31), the formulas with non-differentiable terms are presented as follows:

$$\begin{aligned}
&\varphi_1(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi_0(\zeta,\eta)+\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_0(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_0(\zeta,\eta)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}+\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\varphi_2(\zeta,\eta)\\
&=\varphi_1(\zeta,\eta)-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi_1(\zeta,\eta)+\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_1(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_1(\zeta,\eta)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad+{}_0I_{\zeta}^{(\alpha)}\left\{\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\varphi_3(\zeta,\eta)\\
&=\varphi_2(\zeta,\eta)-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi_2(\zeta,\eta)+\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_2(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_2(\zeta,\eta)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad+{}_0I_{\zeta}^{(\alpha)}\left\{\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\varphi_4(\zeta,\eta)\\
&=\varphi_3(\zeta,\eta)-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\varphi_3(\zeta,\eta)+\frac{2\zeta^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\varphi_3(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_3(\zeta,\eta)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}-{}_0I_{\zeta}^{(\alpha)}\left\{\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad-{}_0I_{\zeta}^{(\alpha)}\left\{2\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad+{}_0I_{\zeta}^{(\alpha)}\left\{\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&=\frac{\eta^{\alpha}}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned} \tag{37}$$

$$\vdots$$

$$\begin{aligned}
&\varphi_n(\zeta,\eta)\\
&=\varphi_{n-1}(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{\partial^\alpha}{\partial\zeta^\alpha}\varphi_{n-1}(\zeta,\eta)+\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_{n-1}(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_{n-1}(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\frac{\partial^\alpha}{\partial\zeta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}.
\end{aligned}\tag{38}$$

Therefore, the non-differentiable solution of (25) using (32) is reported to be as follows:

$$\varphi(\zeta,\eta)=\lim_{n\to\infty}\varphi_n(\zeta,\eta)=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}.\tag{39}$$

The corresponding graph is illustrated in Figure 1 when $\alpha=\frac{\ln 2}{\ln 3}$.

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**Figure 1. The non-differentiable solution of (25) when $\alpha=\frac{\ln 2}{\ln 3}$.**

### 4.2 LFVIA-II *for the* FP *Equation Defined on Cantor Sets*

In view of (26) and (32), we have the following formulas with non-differentiable terms:

$$\begin{aligned}
&\varphi_1(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_0(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_0(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\frac{\eta^\alpha}{\Gamma(1+\alpha)}\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{40}$$

$$\begin{aligned}
&\varphi_2(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\varphi_1(\zeta,\eta)-\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_1(\zeta,\eta)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\varphi_3(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\varphi_2(\zeta,\eta)-\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_2(\zeta,\eta)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{42}$$

$$\begin{aligned}
&\varphi_4(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\varphi_3(\zeta,\eta)-\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_3(\zeta,\eta)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{43}$$

$$\vdots$$

$$\begin{aligned}
&\varphi_n(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\varphi_{n-1}(\zeta,\eta)-\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_{n-1}(\zeta,\eta)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial\eta^\alpha}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\tfrac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\tfrac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\tfrac{\eta^\alpha}{\Gamma(1+\alpha)}-\tfrac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}.
\end{aligned}\tag{44}$$

Therefore, the non-differentiable solution of (25) using (33) is reported to be as given below:

$$\varphi(\zeta,\eta)=\lim_{n\to\infty}\varphi_n(\zeta,\eta)=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},\tag{45}$$

which matches with the result from the LFVIA-I in Subsection 4.1.

### 4.3 LFVIA-III *for the* FP *Equation Defined on Cantor Sets*

In connection with (26) and (33), we set up the following formulas in the non-differentiable forms:

$$\begin{aligned}
&\varphi_1(\zeta,\eta)\\
&=\varphi_0(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_0(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_0(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\frac{\eta^\alpha}{\Gamma(1+\alpha)}\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{46}$$

$$\begin{aligned}
&\varphi_2(\zeta,\eta)\\
&=\varphi_1(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_1(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_1(\zeta,\eta)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_0(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_0(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\\
&\quad-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\frac{\eta^\alpha}{\Gamma(1+\alpha)}\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{47}$$

$$\begin{aligned}
&\varphi_3(\zeta,\eta)\\
&=\varphi_2(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_2(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_2(\zeta,\eta)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_1(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_1(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\\
&\quad-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{48}$$

$$\begin{aligned}
&\varphi_4(\zeta,\eta)\\
&=\varphi_3(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_3(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_3(\zeta,\eta)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_2(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_2(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)},
\end{aligned}\tag{49}$$

$$\vdots$$

$$\begin{aligned}
&\varphi_n(\zeta,\eta)\\
&=\varphi_{n-1}(\zeta,\eta)-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_{n-1}(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_{n-1}(\zeta,\eta)\right\}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\varphi_{n-2}(\zeta,\eta)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\varphi_{n-2}(\zeta,\eta)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\\
&\quad+{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&\quad-{}_0I_\zeta^{(\alpha)}\left\{\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha}{\partial\eta^\alpha}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)-\frac{3\zeta^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial\eta^{2\alpha}}\left(\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^\alpha}{\Gamma(1+\alpha)}\right)\right\}\\
&=\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}.
\end{aligned}\tag{50}$$

Therefore, the non-differentiable solution of (23) using (28) is also reported to be as follows:

$$\varphi(\zeta,\eta) = \lim_{n\to\infty} \varphi_n(\zeta,\eta) = \frac{\eta^{\alpha}}{\Gamma(1+\alpha)} - \frac{2\zeta^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{51}$$

which is in conformity with the results from the LFVIA-I and LFVIA-II in Subsections 4.1 and 4.2, respectively.

## 5 Conclusions

In the present work, the linear FP equation defined on Cantor sets was solved by using the LFVIAs and its closed solution is obtained. By comparing with the LFVIA-I and the LFVIA-III, we conclude that the case of the LFVIA-II is the best way to obtain the non-differentiable solutions of linear local fractional partial differential equations.

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