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New Perspectives of the Lambert-Widder Transform: *Singular Non-Local Operators with Exponential Memory*⋆

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Abstract: The Lambert-Widder transform has been seen from a different angle while formulating novel non-local singular operators that possess exponential memory. An extensive examination of the basic singular version of the Lambert-Widder kernel by a transformation toward a memory function in a convolution integral controlled by a fractional parameter has been thoroughly carried out. The kernel monotonicity analysis, its Mellin transform, and consequent formulations of novel fractional integral and fractional derivative have been carried out. Examples of formal differential and integral equations involving the novel operators possessing the Mellin transform have been formulated. The application of the fading memory approach demonstrates a build-up of a heat conduction model with a memory integral involving the Lambert-Widder kernel. Some extensions of the new operator based on the Mittag-Leffler function are suggested.

Keywords: Lambert-Widder transform, singular kernels, exponential memory, fractional operators, Mellin transform,constitutive modeling .

1 Introduction

In the last decade, there has an intensive research on fractional operators with kernels departing from the classical one based on the singular power-law kernel (known also as Abel's kernel) [1] with many new results and hot discussions (here we skip analyses of such conflicting studies but refer to details in [2] and [3]). Fascinated by the potential of the new trends in fractional calculus based on non-singular kernels [4,5] the efforts of many researchers were focused on their properties with results providing features that do not match what was known from the operators with singular kernels but extending their applications to model more complex real-world problems.

There are many approaches to constructing fractional operators with new kernels (some of them are analyzed in [2]) but the dominating style is to constitute them in either Riemann-Liouville or Caputo sense without preliminary analyses (modeling approaches) allowing us to see where they appear as terms of the model equations.

Here we do not focus on the discussion of what is new and what is wrong with this new trend in fractional calculus (see analyses in [2] and [3]). However, the efforts are oriented toward new convolution operators implementing a kernel with properties matching, to some extent, the properties of the operators from the old and the new trends of fractional calculus.

This study is devoted to the Lambert transform [6,7,9], its properties and relations to the fractional (non-local) calculus, and some contemporary important results thereof. The analysis that follows uses a programming (instructive) approach that highlights the key concept concerning the shift from Lambert-Widder original transform [6],to a new kernel and shows how it is applied to new convolution operators controlled by a fractional-order parameter.

In the remainder of the introduction section we refer to the classical formulation of the Lambert transform (Section 1.1) with two principal definitions (Sections 1.1.1 and 1.1.2), its relation to power series (Section 1.1.3) and Laplace transform (Section 1.1.4), asymptotic sums (Sections 1.1.5 and 1.1.6) and its classical inversion (Section 1.1.10) thus forming a solid background of the envisaged kernel used in the new defined non-local operators as claimed in Section 1.2.

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1.1 The Lambert transform in its classical sense: Genealogy, definitions and properties

We start with the origins of the Lambert transform and its interpretations by Widder [6] and consequent versions [7,9] thus allowing the readers to understand the origin and the properties inspiring this study toward formulations (definitions) of new fractional operators.

1.1.1 Widder's definition

In [6] Widder defined the Lambert transform ($\mathscr{W}_{\mathscr{L}}[f(t)]$) (in the original Widder's notations)

$$\mathscr{W}_{\mathscr{L}\mathscr{S}}[f(t)] \Rightarrow F(x) = \int_{0}^{\infty} f(t)\frac{1}{e^{xt}-1}dt, \quad x>0 \tag{1}$$

for some suitable functions $f(t)$

Note 1: We introduce the notation $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ of the Lambert-Widder transform, with a symbol $\mathscr{W}$ as a tribute to David V. Widder, and a subscript $\mathscr{L}\mathscr{S}$ (*Lambert Singular*). As will be seen in the sequel, there are two versions of the Lambert kernel considered by Widder [6]: a singular version (see the kernels in (1) and (4)) and a bounded version (8), at t_{0^+}. After the preliminary analysis of the properties of the discussed versions, speaking in advance, the singular version is the basis of the non-local operators proposed here.

1.1.2 Goldberg's definition

In 1958, Goldberg [7] considered transforms with a kernel expressed as (see also [9])

$$\sum_{k=1}^{\infty} a_k e^{-kxt}dt, \quad x>0 \tag{2}$$

where a_k is a fairly general class of real numbers. That is, the Goldberg version of the transform is (LGT)

$$F(x) = \int_{0}^{\infty} f(t)\sum_{k=1}^{\infty} a_k e^{-kxt}dt, \quad x>0 \tag{3}$$

For $a_k = 1$ and $k = 1,2,\ldots$, the result is

$$\sum_{k=1}^{\infty} a_k e^{-kxt}dt = \frac{1}{e^{xt}-1} \tag{4}$$

The right-hand side of the expression (4) is the integral representation of the Dirichlet series - a very useful relationship that will be explored further in this study (For more details and further development of this point of view see section 1.1.4 in the sequel).

1.1.3 Lambert kernel relationship to series

Following Widder [6], the series introduced by Lambert [10] is defined as

$$\sum_{k=1}^{\infty} \frac{a_k x^k}{1-x^k} \tag{5}$$

Widder introduced its integral analog as [6]

$$F(x) = \int_{0}^{\infty} \frac{a(t)}{e^{xt}-1}dt \tag{6}$$

and its Stieltjes version [6],

$$F(x) = \int_0^\infty \frac{1}{e^{xt}-1} d\alpha(t) \tag{7}$$

Following Widder [6] since the integrand is singular for $t = 0$, the change of variables $xt\alpha(t) = a(t)$ and $d\alpha(t) = xt\alpha(t)$ yields a continuous (non-singular) kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$, namely

$$\frac{xt}{e^{xt}-1} \Rightarrow \mathscr{W}_{\mathscr{L}\mathscr{N}} = \int_0^\infty \frac{xt}{e^{xt}-1} f(t)dt \tag{8}$$

This is a bounded kernel because (see its graphical presentation in Figure 1-panel d)) for any $x > 0$

$$\lim_{t\to 0^+}\left(\frac{xt}{e^{xt}-1}\right) = 1 \tag{9}$$

Further, if x is replaced by e^{-x} and t by e^t, the transform (6) can be expressed as

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt \tag{10}$$

Therefore, the Lambert transform in Widder's definition is a special case of a convolution transform, where [6]

$$G(x) = \left(e^{e-x}-1\right)^{-1}, \quad f(x) = F\left(e^{-x}\right), \quad \varphi(x) = e^x a(e^x) \tag{11}$$

The bilateral Laplace transform of the kernel defined by (11) is [6]

$$\int_{-\infty}^{\infty} e^{-st} G(t)dt = \int_0^\infty \frac{t^{s-1}}{e^t-1} dt = \zeta(s)\,\Gamma(s) \tag{12}$$

*Remark.*In fact, (12) is the Mellin transform (see Appendix D) of the kernel $(e^t-1)^{-1}$ (a good example concerning details of the integration procedure is provided at page 45 of [11]). Here, we have to recall that $\int_0^\infty t^{z-1}e^{-t}dt = \Gamma(z)$ (see page 55 of [11]).

Referring to (7) and shifting to Widder's kernel (8) we get the *Lambert-Stieltjes transform* (LST) [6] (see also [13])

$$\mathscr{L}_{ST} \Rightarrow F(x) = \int_0^\infty \frac{t\,d\alpha(t)}{e^{xt}-1} = \frac{1}{x}\int_0^\infty H(xt)\,d\alpha(t), \quad H(t) = \frac{t}{e^t-1}, \quad 0 < t < \infty, \quad H(0) = 1 \tag{13}$$

In the formulation (13) the assumption is that $\alpha(t)$ is a function of bounded variation in $0 < t < R$, for every positive R and $\alpha(0) = 0$. For example, if $a(t)$ is a function integrable in the sense of Lebesque, and

$$\alpha(t) = \int_0^t a(u)\,du \tag{14}$$

then, we get the *Lambert-Lebesque transform* (LLT) [6].

$$\mathscr{L}_L \Rightarrow F(x) = \frac{1}{x}\int_0^\infty H(x,t)\,a(t)\,dt \tag{15}$$

Further, since for large t the kernel $H(xt)/x$ differs very little from te^{-xt} [6], the result, as Widder suggested, is that the convergence of (13) will be the same as the Laplace integral

$$\int_0^\infty e^{-xt}t\,d\alpha(t) \tag{16}$$

In addition, (13) and (16) converge for the same value x if the abscissa of convergence σ_c of (16) is positive [6]. In conclusion of this point, any Lambert series can be expanded as a unique power series and *vice versa* [8] such as the Dirichlet series, R-series, etc. (see Appendix A)

*Remark.*Regarding the Widder's two definitions it is worth noting that more efforts in [6] were stressed on the transformation with the bounded kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$ (eq. (8)) rather than to the original singular version $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ (eq. (1)) widely analyzed in [6,7,9,12,13]. However, in the light of the present application of the Lambert transform and the consequent interpretations as a kernel of a convolution operator controlled by a fractional parameter, the singular kernel is of primary interest since it, as we will see, exhibits very useful properties and asymptotics.

1.1.4 Relation of the Lambert transform to the Laplace transform

The Laplace transform could be considered as a continuous analog of power series [7]. Precisely, in the case of the Lambert series (see additional information in Appendix B)

$$F(z) = \sum_{k=1}^{\infty} f_k \frac{z^k}{1-z^k} \tag{17}$$

the continuous analog is the Lambert transform (1).

Following Goldberg [7] and considering transforms with kernels (i.e. a Dirichlet series-see Appendix A)

$$K(xt) = \sum_{k=1}^{\infty} a_k e^{-kxt} \tag{18}$$

and when $a_k = 1$ for $k = 1,2,\ldots$ we have the kernel of the Lambert transform (see (1) and (4) defined as

$$K(xt) = \frac{1}{e^{xt}-1} \tag{19}$$

If the transform is expressed as Stieltjes integrals

$$F(x) = \int_{0+}^{\infty} K(xt) d\alpha(t) dt = \int_{0+}^{\infty} \sum_{k=1}^{\infty} a_k e^{-kxt} d\alpha(t) dt \tag{20}$$

then, a special case can be developed as

$$F(x) = \int_{0+}^{\infty} K(xt) \varphi(t) dt = \int_{0+}^{\infty} \sum_{k=1}^{\infty} a_k e^{-kxt} \varphi(t) dt \tag{21}$$

Following Goldberg [7] the relations (20) and (21) can be considered as generalized Lambert transforms (termed here as *Lambert-Transform*). The right-hand expression in (21) is very convenient when we need to interpret the kernel as a function approximating relaxation behavior (see Section 2.3 and Section 2.4) .

1.1.5 Asymptotic estimations of the series

Here we refer to the asymptotic behavior of the non-singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$ as interpreted by Ferreira et al.[12]

If $r > -1$, then (Theorem 1.1. in [7])

$$\sum_{k=1}^{\infty} k^r e^{-kxt} = O\left(t^{-r-1}\right), \quad t \to 0^+ \tag{22}$$

$$\sum_{k=1}^{\infty} k^r e^{-kxt} \sim e^{-t}, \quad t \to \infty \tag{23}$$

Further, $r > -1$, then (Theorem 1.2. in [7])

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^r n^r e^{-knt} = O\left(t^{-r-1}\right) |\log t|, \quad t \to 0^+ \tag{24}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^r n^r e^{-knt} \sim e^{-t}, \quad t \to \infty \tag{25}$$

1.1.6 The Lambert transform at asymptotic values of the transformation parameter

Now, let us consider the asymptotic expansions of the Lambert transform [12]

$$\mathscr{L}_{\mathscr{W}\mathscr{N}}(x)=\int_0^\infty \frac{xt}{e^{xt}-1}f(t)\,dt,\quad x\in C \tag{26}$$

where $f(t)$ is a locally integrable function [12].

1.1.7 Case for $t\to 0^+$

For $t\to 0^+$, if $f(t)$ has an asymptotic expansion at $t=0$ we have [12]

$$f(t)=\sum_{k=0}^{n-1}a_k t^{k+\alpha}+r_n(t),\quad r_n(t)=O\left(t^{n+\alpha}\right),\quad t\to 0^+,\quad \alpha>-1,\quad a_k\in C \tag{27}$$

Then, (Theorem 2.1. in [12])

$$\int_0^\infty \frac{xt}{e^{xt}-1}f(t)\,dt=\sum_{k=0}^{n-1}\Gamma(k+\alpha+2)\,\zeta(k+\alpha+2)\frac{a_k}{x^{k+\alpha+1}}+R_n(x) \tag{28}$$

with a reminder

$$R_n(x)\equiv\int_0^\infty \frac{xt}{e^{xt}-1}r_n(t)dt \tag{29}$$

with $R_n(x)=O^{(x-n-1-\alpha)}$ when $|x|\to\infty$.

1.1.8 Case for $t\to\infty$

If $f(t)$ satisfies $t^n f(t)\in L^1[0,\infty)$ for $n=0,1,2,3\ldots$, then for $\Re x>0$ (Theorem 2.2. in [12])

$$\int_0^\infty \frac{xt}{e^{xt}-1}f(t)\,dt=\sum_{k=0}^{n-1}\frac{B_k}{k!}\mathscr{M}[f;k+1]x^k+R_n(x),\quad n=1,2,3\ldots \tag{30}$$

In (30) are the Bernoulli number since $\frac{xt}{e^{xt}-1}$ is a generating function of

$$\frac{xt}{e^{xt}-1}=\sum_{k=0}^{n-1}\frac{B_k}{k!}(xt)^k+r_n(xt),\quad n=1,2,3,5,7,9,\ldots \tag{31}$$

and $\mathscr{M}[f;k]f(t)=\int_0^\infty t^{k-1}f(t)dt$ is the Mellin transform of $f(t)$. To get this result we need $f(t)$ to have an asymptotic expansion at infinity [12] , namely

$$f(t)=\sum_{k=K}^{n-1}\frac{a_k}{t^{k+\alpha}}+f_n(t),\quad f_n(t)=O\left(t^{-n-\alpha}\right),\quad t\to\infty \tag{32}$$

where $0<\alpha\le 1$, $K\in Z$, $a_k\in C$.

Thus, following Theorem 2.3 of [12] we have

$$\begin{aligned}\int_0^\infty \frac{xt}{e^{xt}-1}f(t)\,dt=&\sum_{k=K}^{n-1}a_k\Gamma(2-\alpha-k)\,\zeta(2-\alpha-k)\,x^{k+\alpha+1}+\\&+\sum_{k=K}^{n-1}\frac{B_k}{k!}\mathscr{M}[f;k+1]x^k+R_n(x)\end{aligned} \tag{33}$$

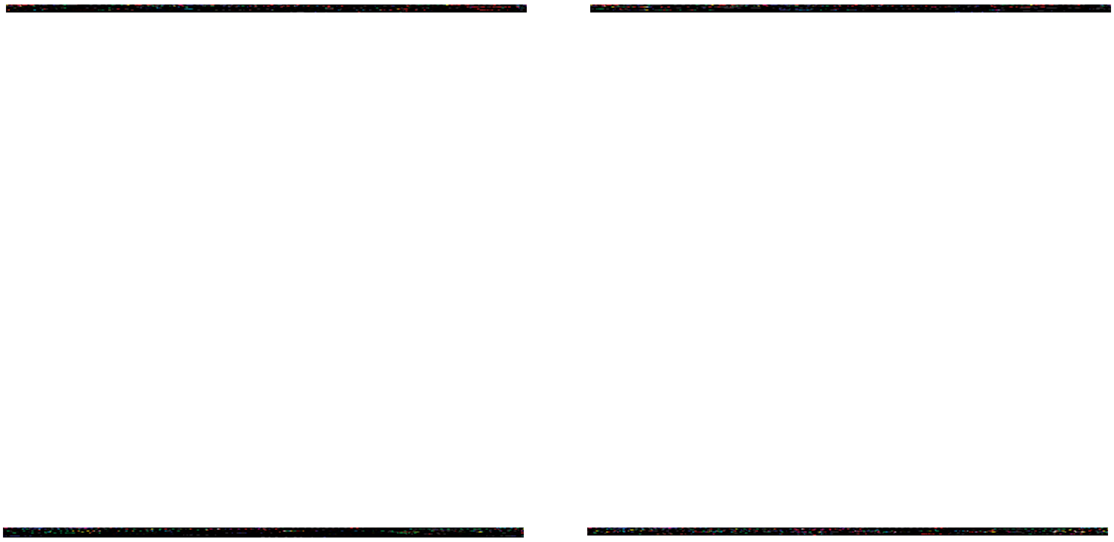

Fig. 1: Graphical presentations of the kernel behaviors: a) The singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ for small values of the controlling parameter x and large time span; b) The singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ for large values of the controlling parameter and short time span; c) The singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ and its asymptotic exponential behavior for large time span with $x = 1$ d) The singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ (and its asymptotic exponential behavior) and the non-singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$ with $x = 1$

With $R_n(x) \equiv (-1)^n \int_0^\infty f_{n,n}(t) \frac{d^n}{dt^n}\left(\frac{xt}{e^{xt}-1}\right) dt$, $f_{n,n} \equiv \frac{(-1)^n}{(n-1)!} \int_t^\infty (u-t)^{n-1} f_n(u)\, du$ and $\mathscr{M}[f;k+1] = \int_0^\infty t^{z-1} f(t) dt$ is the Mellin transform of $f(t)$ with $z = k+1$.

1.1.9 Graphical presentations and comparisons of the kernels

After all these analytical steps in the kernel's analysis it is quite instructive to see their graphical behaviors thus allowing really to appreciate the fact that they are able to match both the singular and non-singular areas. The plots in Figure 1 show how the singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ behaves when the controlling parameter x varies: a) for small values of x, and b) for large values of x). Furthermore, we can see the indistinguishable plots of $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ and e^{-xt} in c) . The common plots of $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ and $\mathscr{W}_{\mathscr{L}\mathscr{N}}$ visually demonstrate the singular behavior of the former and the second one as a bounded kernel.

1.1.10 Inversion of the Lambert-Widder transform

The inversion of the transform $\mathscr{W}_{\mathscr{L}\mathscr{S}}[f(t)]$ defined by (1) is [6]

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} \sum_{n=1}^{\infty} \mu(n) n^k F^{(k)}\left(\frac{kn}{t}\right) \tag{34}$$

where $\mu(n)$ is the Mobious function.

For example, when $f(t) = t$, it follows that $F(x) = \zeta\left(2/x^2\right)$ [6] and therefore the right-hand side of (1) becomes [6]

$$\zeta\left(\frac{2}{x^2}\right) \lim_{k\to\infty} \frac{k+1}{k} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} t \tag{35}$$

Furthermore , since [6,11]

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \mu(n) e^{-s\log n} \tag{36}$$

It follows that the inverse transform yields t [6].

1.2 Motivation and aim of this study

1.2.1 Motivation

Looking for opportunities in non-local modeling toward definitions of relaxation functions (memory kernels) allowing to cover a broad range of real-world non-local (causal) processes is always a challenging task. The inspiring motivation for this study comes from the performance of the Lambert-Widder kernel. Precisely, its monotonicity, as we will see in the sequel, its singularity at t_{0^+}, a behavior close to that of the power law, which is common for the classical branch of the fractional calculus, as well as the ability to perform as an exponential decaying causal function, typical for bounded (regular) kernels. These features are attractive to be investigated since these two aspects give and take a common ground of the two different trends in the contemporary non-local (fractional) calculus and modeling.

1.2.2 Aim

The direct aim is the formulation of new non-local operators based on the Lambert-Widder function (kernel), analysis of its properties and applicability to model non-local problems.The Mellin transform properties relevant to this kernel are also part of the study program as formulated next.

1.3 Study program and further text organization

1.3.1 The study program

In what follows we formulate new applications of the Lambert-Widder kernels in convolution operators. The main idea comes from the relationships with the Dirichlet series and their asymptotic behaviors. Now, will develop this idea by passing through the following steps:

– A check for monotonicity of the Lambert-Widder kernels
– Reformulation of the controlling parameter x as a function of fractional parameter $0 < \alpha < 1$ (Section 2.2 and Section 2.4).
– Construction of a convolution operator with fractional order controlling parameter
– Estimations of the asymptotics of the constructed convolution operators .
– Analysis of the convolution operator far from the extremes (far from asymptotic behaviors).
– Mellin convolution transforms of the novel operators.
– Formal differential and integral equations with the new operators.
– Modeling approaches with the new operators.

1.3.2 Further text organization

In what follows, the sequel is organized as: The task formulated above concerning the kernel properties are analyzed in Section 2 (kernel monotonicity in Section 2.1 and reformulation of the controlling parameter through a fractional order: in Section 2.2 and Section 2.4). A new non-local operator based on the Lambert-Widder singular kernel controlled by a fractional order $0<\alpha<1$ is conceived in Section 3 (concerning its asymptotic properties in Section 3.1 and relation to the Dirichlet series (Section 3.1.1), its normalization function (Section 3.1.3) as well as its behavior far from extremes (Section 3.2)). The integral transform, precisely the Mellin convolution is considered in Section 4. Tests of the index law validity and semi-group properties are performed in Section 5. Formal differential and integral equations with the new operators are considered in Section 6. Modeling approaches with the operators are discussed in Section 7. Ideas to generalize the proposed transform by application of the Mittag-Leffler function and its version are are conceived in Section 7.3. Outcomes of the study and some envisaged problems to be resolved are analyzed in Section 8.

*Remark.*Making the presentation slender and concise many properties of the functions used and auxiliary information are summarized in the Appendices 10.

2 The Lambert-Widder kernel properties reconsidered

2.1 Monotonicity the kernels

2.1.1 A simple test

It is easy to check that the general requirement for the monotonicity of the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$

$$(-1)^n \frac{\partial^n}{\partial t^n}\left[\mathscr{W}_{\mathscr{L}\mathscr{S}}\right] \geq 0, \quad n=0,1,2,\ldots \tag{37}$$

is obeyed, because, for example, through straightforward computations, we have

$$\begin{gathered}\frac{d}{dt}\left[\mathscr{W}_{\mathscr{L}\mathscr{S}}\right]=-\frac{xe^{xt}}{\left(e^{xt}-1\right)^2}, \quad \frac{d^2}{dt^2}\left[\mathscr{W}_{\mathscr{L}\mathscr{S}}\right]=\frac{x^2 e^{xt}\left(e^{xt}+1\right)}{\left(e^{xt}-1\right)^3}, \\ \frac{d^3}{dt^3}\left[\mathscr{W}_{\mathscr{L}\mathscr{S}}\right]=-\frac{x^3 e^{xt}\left(\left(e^{xt}\right)^2+4e^{xt}+1\right)}{\left(e^{xt}-1\right)^4}, \quad x>0, \quad t>0\end{gathered} \tag{38}$$

Similarly, for the non-singular kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$ we have

$$\frac{d}{dt}\left[\mathscr{W}_{\mathscr{L}\mathscr{N}}\right]=-\frac{x\left(e^{xt}(xt-1)+1\right)}{\left(e^{xt}-1\right)^2}, \quad \frac{d^2}{dt^2}\left[\mathscr{W}_{\mathscr{L}\mathscr{N}}\right]=\frac{x^2 e^{xt}\left(e^{xt}(xt+1)-2e^{xt}+2\right)}{\left(e^{xt}-1\right)^3}, \quad x>0, \quad t>0 \tag{39}$$

Hence, both kernels of interest are completely monotonous (CM) since, obviously it satisfies the condition (37). We applied straightforward calculations but proofs can be found in [15,16] as byproducts of other problems involving exponential function and particularly the kernel $\mathscr{W}_{\mathscr{L}\mathscr{N}}$.

2.1.2 Monotonicity estimation via the Mellin transform

Now, to complete the case with the kernel monotonicity we will apply a different approach developed in [17]. Following [17], if a function $f(x)$, $x \in (0,\infty) \to \mathbb{R}$ is completely monotonous if it is of class C^∞ and satisfies the condition (37) for all $n \in \mathbb{N}$ and $x>0$. However, concerning the integral transforms relevant to the function at issue, a concept for complete monotonicity was formulated in [17] as follows. Since the Laplace transform is Mellin convolution type transform [17], the Mellin transform can be used to define the completely monotone functions. Following this concept, the presentation

$$f(x)=\int_0^\infty e^{xt}F(t)\,dt, \quad x>0 \tag{40}$$

holds for non-negative function F with known Mellin transform [17].

Then the function f is completely monotone and its Mellin transform is given by the formula [17]

$$f_M(z) = \Gamma(z) F_M(1-z), \quad f_M = \mathscr{M}[f(x)], \quad F_M = \mathscr{M}[F(t)] \tag{41}$$

and therefore [17]

$$F_M(z) = \frac{f_M(1-s)}{\Gamma(1-s)} \tag{42}$$

Further, if $F(t)$, $t > 0$ is non-negative, then the function $G(t) = t^{\gamma} F\left(t^{-\beta}\right)$ is non-negative for any $\gamma, \beta \in \mathbb{R}$. As a consequence, the function $g(x)$ presented as [17]

$$g(x) = \int_0^{\infty} e^{-xt} G(t)\, dt, \quad x > 0 \tag{43}$$

is completely monotone. Hence, it follows from the relation (41) that [17]

$$g_M(z) = \Gamma(z) G_M(1-z) \tag{44}$$

In the case of the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}} = \frac{1}{e^{xt}-1}$ with a Mellin transform $\mathscr{M}[\mathscr{W}_{\mathscr{L}\mathscr{S}}] = \Gamma(z)\zeta(z)$, the function F_M is

$$F_M = \frac{\Gamma(1-z)\zeta(1-z)}{\Gamma(1-z)} = \zeta(1-z) \tag{45}$$

Hence, we get the Zeta function (see more details in Appendix C), and then the function $f(x)$ is completely monotone. With this, we confirm the complete monotonicity of the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$. However, we have to stress the attention on a practical rule provided by [18] : *If the function $f(x)$ is completely monotone, then the function $F_M(z)$ with Mellin integral transform (42) is non-negative, and vise versa, that is, if the function $F_M(z)$ is non-negative, then the function $f(x)$ with a Mellin transform (41) is completely monotone.*

2.1.3 Monotonicity: A test to a Bernstein function

Now, we start from the results presented in the preceding point 2.1.2. Following the definition of Bernstein function (BF) it first derivative should be a completely monotone function. That is, denoting $\frac{d}{dt}[W_{LS}] = F_W(xt)$ we need the following conditions to be satisfied

$$(-1)^m \frac{\partial^m F_W(x,t)}{\partial t^m} \geq 0, \quad m = 1, 2, \ldots \tag{46}$$

or $\mathscr{W}_{\mathscr{L}\mathscr{S}}$, if and only if, can be represented by a Riemann-Stieltjes integral

$$\mathscr{W}_{\mathscr{L}\mathscr{S}}(s) = \int_0^{\infty} e^{-\lambda s} d\alpha(\lambda) \tag{47}$$

where $\alpha(\lambda)$ is bounded and non-decreasing, and the integral converges for $0 < \lambda < \infty$.

We will skip the treatment of the problem in terms of the requirement (47) since this is clear from Widder's analysis in Section 1.1, but it is easy to demonstrate directly that (46), precisely the reformulated condition (48), is satisfied by $\mathscr{W}_{\mathscr{L}\mathscr{S}}(xt)$. Thus, in terms of the derivatives used to estimate the complete monotonicity, we have that $m = n+1$. Therefore, if we like to work as in the point 2.1.2 we may recast the Bernstein condition as

$$(-1)^{n-1} \frac{\partial^n}{\partial t^n}[\mathscr{W}_{\mathscr{L}\mathscr{S}}(x,t)] \leq 0, \quad t > 0, \quad n = 2, 3, \ldots \tag{48}$$

Then, the straightforward computations provide the following results

$$(-1)\frac{\partial F_W(x,t)}{\partial t} = (-1)\left(-\frac{xe^{xt}}{(e^{xt}-1)^2}\right) \geq 0 \tag{49}$$

$$(-1)^{n-1}\frac{\partial^n}{\partial t^n}[\mathscr{W}_{\mathscr{L}\mathscr{S}}(xt)] = (-1)^{(2-1)}\frac{\partial^2}{\partial t^2}[\mathscr{W}_{\mathscr{L}\mathscr{S}}(xt)] = (-1)\frac{x^2 e^{xt}(e^{xt}+1)}{(e^{xt}-1)^3} \leq 0 \tag{50}$$

$$(-1)^{(3-1)}\frac{\partial^3}{\partial t^3}[\mathscr{W}_{\mathscr{L}\mathscr{S}}(x,t)] = (-1)^2\left(-\frac{x^3 e^{xt}\left((e^{xt})^2 + 4e^{xt} + 1\right)}{(e^{xt}-1)4}\right) \leq 0 \tag{51}$$

Therefore, we demonstrated simply that condition (48) is satisfied and the Lambert-Widder kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ is a Bernstein function.

2.2 Reformulation of the controlling parameter x

From the definition (1) we have $x > 0$ and we may reformulate it such that

$$x = \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1), \quad x \in (0,\infty) \tag{52}$$

where for $\alpha \to 0$ we get $x \to 0$, while $\alpha \to 1$ corresponds to $x \to \infty$, and therefore we get a modified kernel

$$\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t) = \frac{1}{e^{\frac{\alpha}{1-\alpha}t} - 1}, \quad 0 < \alpha < 1, \quad t > 0 \tag{53}$$

controlled by a fractional parameter α.

2.3 Fractional order parameter definition: A physically related approach

From a mathematical point of view, the dimension of the controlling parameter x and its substitution $x = \alpha/(1-\alpha)$ is not so important because all issues considered so far and some in the sequel are not related to real-world physical problems. However, there is a reasonable question of what should be defined and how they could be determined if the relaxation function, i.e. the kernel should be fitted to real relaxation data. Both x and $\alpha/(1-\alpha)$ have dimensions inverse to that of t. When t is assumed as time in relaxation experiments, then the dimension of $\alpha/(1-\alpha)$ should be inverse of time. Here, we face a problem, since α, by definition is dimensionless. The problem was resolved in [19] by the introduction of the process time scale t_0 (the duration of the experiment) allowing to perform a non-dimensalization of the time as $0 \leq \bar{t} = t/t_0 \leq 1$. Then, considering the original formulation of the exponent in the kernel as $xt = 1/(1/x)$ we can see that $1/x = u$ can be considered as a specific relaxation time. This allows a non-dimensalization such that $xt \to (t/t_0)/(u/t_0) = \bar{t}/\bar{u}$. Now, looking at $x = \alpha/(1-\alpha)$ we can see that $\bar{u} = (1-\alpha)/\alpha$. Therefore the problem with the correct dimensions is resolved and we can calculate α from real data if both the time constant x and the time scale t_0 are defined by a successful data fitting. Precisely, from

$$\bar{u} = \frac{\alpha}{1-\alpha} \Rightarrow \alpha = \frac{1}{1+\bar{u}} \tag{54}$$

Therefore, from this definition, we always have $0 < \alpha < 1$; $\alpha \to 0$ means infinite relaxation time ($\bar{u} \to \infty$), whereas $\alpha = 1$ corresponds to $\bar{u} = 0$ (no relaxation exists, that is there is an instantaneous process). As a result of this definition, we can see that the behavior of the kernel considered further as a practically oriented function is restricted in the square $0 \leq \alpha \leq 1$, $0 \leq \bar{u} \leq 1$. However, when the problems are purely mathematical this is not so important, but for the real-world application of the kernel, this definition is crucial.

2.4 Behavior of the singular kernel with fractional controlling parameter α

Now, we can see in Figure 2 what the behavior of the new kernel is when the fraction parameter α varies in the range $\alpha \in (0,1)$. We can see that with the increase in α the decaying of the kernel is faster (see Figure 2-panels a) and b)) that is equivalent to the case of high values of the controlling parameter x since $\alpha \to 1 \Rightarrow x \to \infty$. We can see the kernel singularity for $\alpha \to 0$ and its complete vanishing for $\alpha \to 1$ (Figure 2-panel c))

In terms of the dimensionless time $\bar{u}$ we can observe the same effects of the fractional order (see Figure 3). However, we can see another effect physically relevant to the relaxation processes that could be approximated by this kernel. Precisely, there are two types of profiles in Figure 3: profiles completely vanishing for $u \leq 1$ and profiles vanishing for $u > 1$. This can be simply explained by the fact that the considered kernel is an integral analog of the Dirichlet series. The latter could be to some extent considered as a decomposition (or a superposition) as a series of simple exponential relaxations. The components of such a series with short relaxation times (corresponding to high values of α see (54)) vanish faster, since these times are shorter than the time scale. Otherwise, when the relaxation times are higher than the time scale, then the kernel does not decay completely for $u < 1$. To explain this effect in terms of the fractional order and referring to (54) we can see that $\alpha = 0.5$ corresponds to the case $u = 1$, that is the relaxation time matches the time scale t_0. For larger relaxation times we have $u > 1$ and therefore from $u = (1-\alpha)/\alpha$ it follows $\alpha < 0.5$ approximately, whilst $u < 1$ means a fractional order in the range $0.5 < \alpha < 1.0$ (i.e. relaxation times shorter than the time scale).

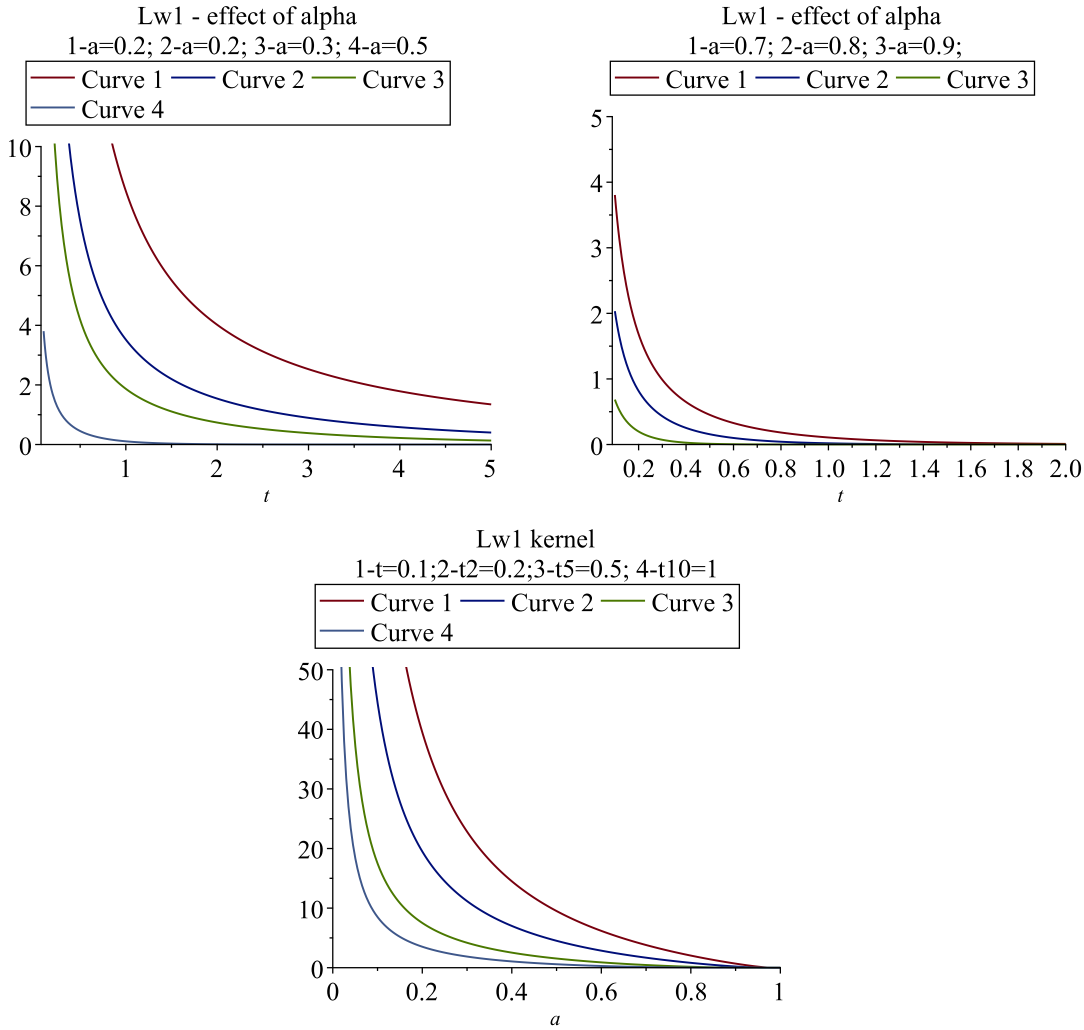

Fig. 2: The kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$: a) Relaxations for $0 < \alpha < 0.5$ and large span of time ; b) Relaxations for $\alpha > 0.5$ and short span of time ; c) The kernel as a function of the fractional order α and short times.

3 Convolution operator with the singular Lambert-Widder kernel

Further, setting the upper terminal to t and the dummy variable τ we formulate the following convolution integral (introducing the notation $J_{WS}(\alpha,t)[f(t)]$)

$$J_{WS}(\alpha,t)[f(t)] = \frac{1}{N(\alpha)} \int_{0}^{t} f(\tau) \frac{1}{\left[e^{\frac{\alpha}{1-\alpha}(t-\tau)} - 1\right]} d\tau \tag{55}$$

The denominator $N(\alpha)$ should ensure the convergence of the integral and its functional form should be formulated as a consequence of the asymptotic behavior of the kernels, as will be demonstrated in the sequel.

Now, we can see that we may have two asymptotic behaviors of the kernel, namely:

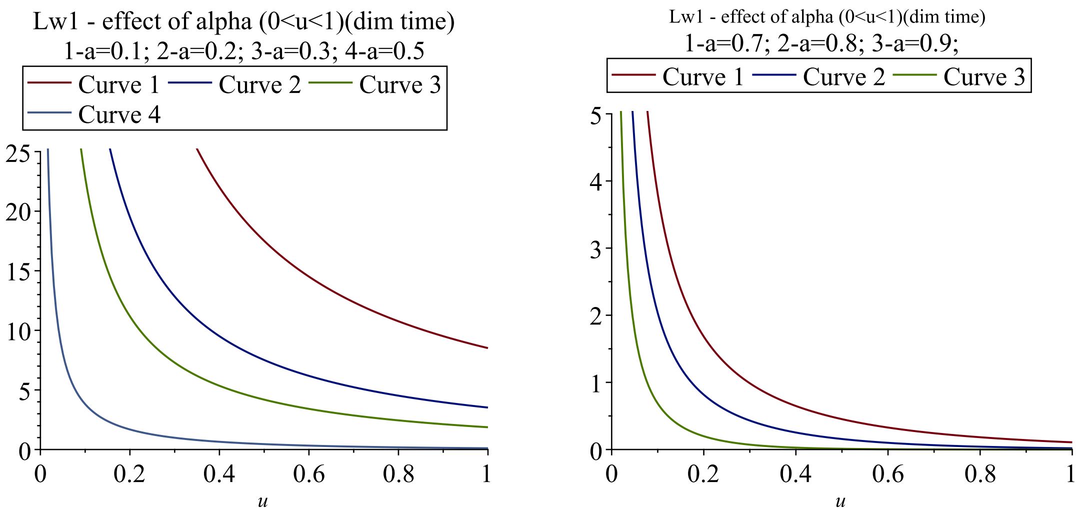

Fig. 3: The kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ as a function of the dimensionless time $u = t/t_0$: a) Cases for $0.1 < \alpha < 0.5$; b) Cases for $\alpha > 0.5$. Note: As $\alpha \to 1$, then $u = t/t_0 \to 0$ (approaching zero relaxation times) and *vice versa*.

For $e^{xt} = e^{\frac{\alpha}{1-\alpha}t} \gg 1$ we get a convolution integral with an exponential memory

$$J_{WS}(\alpha,t)\,[f(t)] \sim \frac{1}{N(\alpha)} \int_0^t f(\tau)\, e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau \tag{56}$$

This is in agreement with the asymptotic behavior (see (25)) for $t \to \infty$.

For $e^{xt} = e^{\frac{\alpha}{1-\alpha}t} \leq 1$ and expressing the exponential function as a series $e^{xt} \approx 1 + xt + (xt)^2/2! + \ldots (xt)^n/n!$, through a truncation omitting the high powers we get

$$J_{WS}(\alpha,t)\,[f(t)] \sim \frac{1}{N(\alpha)} \int_0^t f(\tau)\, \frac{1}{x(t-\tau)^{\gamma}} d\tau \tag{57}$$

For more details concerning this approximation see the next point and the related remark.

3.1 Asymptotics of the new formulated operator

We especially introduced the exponent γ since referring to the asymptotic behaviors (see Section 1.1.5) for $t \to 0^+$ the sum of exponents $\sum_{k=1}^{\infty} k^r e^{-kxt} = O\left(t^{-r-1}\right) \sim t^{-(1+r)} \sim t^{-\gamma}$ (see (22)).

Hence, we have two asymptotic behaviors of the kernel (see the preliminary analyzes in Section 1.1.5 and Section 1.1.6) :

–Decaying Power law for short times which is unbounded at $t \to 0^+$

–Exponential decaying kernel for large times $t \to \infty$

However, for intermediate times, this is a singular kernel since for $e^{xt} = e^{\frac{\alpha}{1-\alpha}t} = 1$ for either $t = \tau$ (no relaxation) or for $\alpha = 0$ we have $1/\left(e^{\frac{\alpha}{1-\alpha}(t-\tau)} - 1\right) \to \infty$.

*Remark.*It is worth noting to stress the attention on the power-law approximation of the exponential sums when $t \to 0^+$ which was successfully used in the dynamic of viscoelastic fluids [20] and allowed to see how a series of terms with exponential memories may be approximated, thus explain a relationship between operators with singular and non-singular memories [19].

3.1.1 Relations to the Dirichlet series and results thereof

Furthermore, we have to mention an important issue related to the fact that this kernel is an integral form of the Dirichlet series. Precisely, from (2) we have

$$\sum_{k=1}^{\infty} a_k e^{-kxt} = \frac{1}{e^{kxt}-1} \tag{58}$$

Then, constructing the convolution integrals we obtain (without normalization functions, for the sake of simplicity of presentation)

$$\int_0^t \sum_{k=1}^{\infty} a_k e^{-kx(t-\tau)} f(\tau)\, d\tau \equiv \int_0^t \frac{1}{e^{kxt(t-\tau)}-1} f(\tau)\, d\tau \tag{59}$$

Now, with $x = \frac{\alpha}{1-\alpha}$ and interchanging the order of the summation and integration in the left-hand side of (59) we get

$$\sum_{k=1}^{\infty} \int_0^t a_k e^{-k\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau \equiv \int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1} f(\tau)\, d\tau \tag{60}$$

Bearing in mind that $\int_0^t a_k e^{-k\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)$ is the principle construction of the constitutive integral of the Caputo-Fabrizio operator [4], and we have

$$\sum_{k=1}^{\infty} {}^{CF}I_t^{k\alpha}[f(t)] \equiv \int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1} f(\tau)\, d\tau, \quad 0<\alpha<1, \quad t>0 \tag{61}$$

where [2]

$${}^{CF}I_t^{k\alpha}[f(t)] \equiv \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau \tag{62}$$

is a constitutive integral of the Caputo-Fabrizio operator as formulated in [2] and [3].

3.1.2 Stieltjes version of the new operator

As an example, to be coherent with the initial analysis of the Lambert-Widder kernel, if $f(t) = da(t)$ (see section 1.1.3, then we get the Stiltjes version of the operator in the right-hand side of (61), that is

$$\int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1} f(\tau)\, d\tau \Rightarrow \int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1} da(\tau) \tag{63}$$

3.1.3 Normalization of the operator

Now, we stress the attention on the definition of the normalization function $N(\alpha)$, mainly when there are asymptotic behavior of operators. For this determination, we initially will use the asymptotic formulations of the operator (see above Section sec:Asymptotics-Operator), namely

For short times when $t \to 0^+$

In such a case, from (57) we may suggest, without loss of generality, that $\gamma = 1-\alpha$ and then an asymptotic version of the operator can be formulated in the sense of a Riemann-Liouville integral, suggesting that $N(\alpha) = \Gamma(\alpha)$. The result is a weighted Riemann-Liouville integral, where the weighting function is $m(\alpha) = (1-\alpha)/\alpha$, namely

$${}^{RL}I_{WS}(\alpha,t)[f(t)] \sim \frac{m(\alpha)}{\Gamma(\alpha)} \int_0^t f(\tau) \frac{1}{(t-\tau)^{1-\alpha}} d\tau, \quad m(\alpha) = (1-\alpha)/\alpha \tag{64}$$

That is, the relation of the newly formulated convolution integral to the classical Riemann-Liouville integral is

$${}^{WS}I_t^{\alpha}[f(t)] = m(\alpha)\, {}^{RL}I_{WS}(\alpha,t) \tag{65}$$

Consequently, integrating in (64), concerning τ , we can define

–A derivative in the Riemann-Liouville sense

$$ {}^{WSR}D_t^{\alpha}\left[f(t)\right] \sim \frac{m(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t f(\tau)\frac{1}{(t-\tau)^{\alpha}}d\tau = m(\alpha)\,{}^{RL}D_t^{\alpha}\left[f(t)\right] \tag{66}$$

as well as

–A derivative in the Caputo sense, such as

$$ {}^{WSC}D_t^{\alpha}\left[f(t)\right] \sim \frac{m(\alpha)}{\Gamma(1-\alpha)}\int_0^t \frac{df(\tau)}{d\tau}\frac{1}{(t-\tau)^{\alpha}}d\tau = m(\alpha)\,{}^{C}D_t^{\alpha}\left[f(t)\right] \tag{67}$$

*Remark.*It is important to stress the attention on the fact that the weighting function $m(\alpha)$ is not a normalization function as such appearing in the formulation of the constitutive integrals of operators with non-singular kernels (see detailed analysis in [3]).

For large times when $t \to \infty$

In such cases, we have $\frac{\alpha}{1-\alpha}t \to \infty$ and therefore $e^{\frac{\alpha}{1-\alpha}(t-\tau)} \gg 1$. Then, the second asymptotic formulation is in the sense of the Caputo-Fabrizio constitutive integral [3]

$$ {}^{WS-CF}I_t^{\alpha} = \frac{1}{P(\alpha)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f(\tau)\,d\tau \tag{68}$$

Looking for convenience in the integral formulation we suggest $N(\alpha) \to P(\alpha) = 1-\alpha$, so that (68) becomes

$$ {}^{WS-CF}I_t^{\alpha} = \frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f(\tau)\,d\tau \tag{69}$$

If $f(t) = dg(t)/dt$, we can formulate the Caputo-Fabrizio operator

$$ {}^{WS-CF}D_t^{\alpha}\left[g(t)\right] = \frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\frac{dg(\tau)}{d\tau}d\tau \tag{70}$$

Note: The superscript and subscript WS mean Widder-singular as related to the kernel $\mathscr{WLS}$.

*Remark.*Therefore, depending on the asymptotic behavior of the kernel we have two different versions of the normalization function $N(\alpha)$. This is not strange since for $t \to t_{0+}$ the singularity dominates and $N(\alpha) = \Gamma(\alpha)$ is the adequate solution. Far from the zone of dominating singularity $N(\alpha) = 1-\alpha$ is a function already well working with the exponential memory of the Caputo-Fabrizio operator. We we briefly comment this problem again when the behavior of the kernel far from extremes will be discussed (see the remarks at the end of the next section).

3.2 The behavior of the new operators far from the extremes

Let us now consider the new operators far from the extremes ($t \to 0^{+}$ and $t \to \infty$) where the more interesting question is : what the behavior is of either the integral $J_{WS}(\alpha,t)$ (71) or the derivative ${}^{WS}D_t^{\alpha}f(t)$ (72) when $\alpha \to 1$ and $\alpha \to 0$? Hence, we have three formulations needing such an analysis, namely

$$ J_{WS}(\alpha,t) = \frac{1}{N(\alpha)}\int_0^t \frac{f(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{71}$$

$$ {}^{WS}D_t^{\alpha}f(t) = \frac{1}{N(\alpha)}\int_a^t \frac{f^{(1)}(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{72}$$

alternatively in the Riemann-Liouville sence

$$ {}^{WSR}D_t^{\alpha}f(t) = \frac{d}{dt}J_{WS}(\alpha,t) = \frac{1}{N(\alpha)}\int_0^t \frac{f(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{73}$$

3.2.1 The integral $J_{WS}(\alpha,t)f(t)$

Now, let us assume, for better analysis, that the lower terminal of the convolution integral is set to $a \neq 0$. Then, we have

$$J_{WS}(\alpha,t) = \frac{1}{N(\alpha)}\int_a^t \frac{f(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \frac{1}{N(\beta)}\int_a^t \frac{f(\tau)}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau, \quad \beta = \frac{1-\alpha}{\alpha} \tag{74}$$

For $\alpha \to 1$, we have $\beta \to 0$ and therefore

$$\lim_{\alpha\to 1}\frac{1}{N(\alpha)}\int_a^t f(\tau)\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \lim_{\beta\to 0}\frac{N(\beta)}{\beta}\int_a^t f(\tau)\frac{1}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau = f(t) \tag{75}$$

bearing in mind that

$$\lim_{\beta\to 0}\left[\frac{1}{\beta}\frac{1}{e^{\frac{(t-\tau)}{\beta}}}\right] = \delta(t-\tau) \tag{76}$$

For $\alpha \to 0$ we have $\beta \to +\infty$ and the result is

$$\lim_{\alpha\to 0}\frac{1}{N(\alpha)}\int_a^t f(\tau)\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \lim_{\beta\to +\infty}\frac{N(\beta)}{\beta}\int_a^t f(\tau)\frac{1}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau = \int_a^t f(t)\,dt \tag{77}$$

3.2.2 The derivative ${}^{WS}D^t_\alpha f(t)$

$${}^{WS}D_t^\alpha f(t) = \frac{1}{N(\alpha)}\int_a^t \frac{f^{(1)}(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \frac{1}{N(\beta)}\int_a^t \frac{f^{(1)}(\tau)}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau, \quad \beta = \frac{1-\alpha}{\alpha} \tag{78}$$

$$\begin{aligned}\lim_{\alpha\to 1}\left[{}^{WS}D_t^\alpha f(t)\right] &= \lim_{\alpha\to 1}\frac{1}{N(\alpha)}\int_a^t f^{(1)}(\tau)\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \\ &= \lim_{\beta\to 0}\frac{N(\beta)}{\beta}\int_a^t f^{(1)}(\tau)\frac{1}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau = f^{(1)}(t)\end{aligned} \tag{79}$$

$$\begin{aligned}\lim_{\alpha\to 0}\left[{}^{WS}D_t^\alpha f(t)\right] &= \lim_{\alpha\to 0}\frac{1}{N(\alpha)}\int_a^t f^{(1)}(\tau)\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau = \\ &= \lim_{\beta\to +\infty}\frac{N(\beta)}{\beta}\int_a^t f^{(1)}(\tau)\frac{1}{e^{\frac{(t-\tau)}{\beta}}-1}d\tau = f(t)-f(a)\end{aligned} \tag{80}$$

*Remark.*From the analysis above it follows that the adequate version of the normalization function, far from the cases $t \to t_{0^+}$ and $t \to \infty$, that is when $exp\left(-\frac{\alpha}{1-\alpha}\right)t >> 1$, is $N(\alpha) = 1-\alpha$.

*Remark.*It is worth noting that the estimation of the above properties when $0 \leq \alpha \leq 1 \to 0 < \beta < \infty$ resembles the analysis in [4] if we may approximate to some extent that $\alpha \to 1\,(\beta \to 0) \Rightarrow \lim_{\beta\to 0}\left[\exp\left(\frac{t-\tau}{\beta}\right)\right] \to \infty$ and then the denominator of the kernel can be approximated as $\left(e^{\frac{(t-\tau)}{\beta}}-1\right)^{-1} \approx e^{\frac{-(t-\tau)}{\beta}}$, as it was demonstrated by the behavior of the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$ (see Figure 1, Figure 2 as well as Figure 3). With this approximation the common kernel of both $J_{WS}(\alpha,t)$ and ${}^{WS}D_t^\alpha f(t)$ approaches the memory function of the Caputo-Fabrizio operator, with only, and important exception concerning its singularity for $t \to t_{0^+}$.

4 Integral transforms of the new operators

4.1 Mellin transform and fractional (non-local) calculus

The Mellin transform is strongly related to the development of fractional calculus [14,17,21,22,23,24]. The main efforts and significant results are on operators with singular power-law kernels (such as Riemann-Liouville and Erdelyi-Kober derivatives) representing them as Mellin convolution-type transforms [17] starting from the Erdelyi-Kober fractional integrals [21,22]. Many results in this direction are available in [24]. With this short remark, we close the comments on the Mellin transforms to fractional operators of the classical type (with singular kernels), but interested readers may find comprehensive information in [14,17,24,25,26,27,28] (for concise and clear presentation of the Mellin transform properties and its applications see [25,29]). In what follows in this section, we will consider the Mellin transforms oriented to the convolution operator with the Lambert-Widder kernel as formulated in the preceding sections.

4.2 Preliminary notes on applicable integral transforms

Following the analysis in [30] (based on the work of Bharatya [31]) the Laplace transform method applies successfully if the integral equation can be presented as (in the original notations- here the variables and do not match the variables used in the preceding sections)

$$\int_0^x k(x-t)\, f(t)\, dt = g(x) \tag{81}$$

If the kernel's Laplace transform can be presented as

$$L[k(x);p] = K(p) = p(p-\alpha)^m (p-\beta)^{-n} \tag{82}$$

As to the Mellin transform, it is especially suitable for cases where the integral equation can be presented as [30]

$$\int_0^x k(x/t\,)\, f(t)\, dt = g(x) \tag{83}$$

and that the kernel has a Mellin transform involving only Gamma $\Gamma(\cdot)$ functions. The substitution $u = t/x$ yields [30]

$$\int_0^x k(u)\, f(ux)\, x du = g(x) \tag{84}$$

It is worth noting that the Mellin transform (this is not related to Mellin's convolution-see below) does not possess the associative and commutative properties [30] but it is a linear transform since

$$\mathscr{M}[af(x)+bg(x)] = af_M(x)+bg_M(x) \tag{85}$$

and the scaling property $\mathscr{M}[f(at)] = a^{-z} f_M(z)$ holds [17,25]. Many properties and related formulas are summarized in Appendix D.

4.3 Mellin transform to multiplicative convolution

Here we stress the attention on the Mellin transform to multiplicative convolution which is directly related to the problems developed in this study. That is, the general formulations are [14,17,25,26,28] (for concise and clear presentation of the Mellin transform properties and its applications see [29]).

$$\mathscr{M}[f*g] = M\left[\int_0^\infty f\left(\frac{t}{u}\right) g(u)\frac{du}{u};z\right] = f_M(z)\, g_M(z) \tag{86}$$

$$\mathscr{M}^{-1}[f_M(z)\, g_M(z)] = \int_0^\infty f\left(\frac{t}{u}\right) g(u)\frac{du}{u} \tag{87}$$

taking into account the commutative and associative properties of the multiplicative convolution [26] (p.339) : $f*g = g*f$ and $(f*g)*h = f*(g*f)$, respectively.

*Remark.*It is natural to raise the question: why the Mellin transform is widely applied in this study in contrast to the common application of the Laplace transform in classical fractional calculus? The answer is simple: the function used as a kernel of *the Lambert-Widder transform does not posses Laplace transformation* (image). This is easy to check: the only close functions, but not the same, with Laplace transforms are :

(see [38] (Table 2.3-formula 19)

$$\mathscr{L}\left[\frac{1}{a}\left(1-e^{-at}\right)\right] \Rightarrow \frac{1}{s\left(s+a\right)} \tag{88}$$

and

(see [54] (page 318-formula 9)

$$\mathscr{L}\left[\frac{1}{a}\left(e^{at}-1\right)\right] \Rightarrow \frac{1}{s\left(s-a\right)} \tag{89}$$

Thus, the most suitable transform to the kernel analyzed here is the Mellin transform and we will demonstrate this next.

4.4 Mellin transforms to multiplicative convolutions with the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$

Now we focus our attention on the Mellin transforms with the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$ where Laplace transforms do not exist.

The Mellin transform of the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$ is (see 216) in Appendix D.1) (this result was mentioned earlier as an outcome of a bilateral Laplace transform)

$$\mathscr{M}\left[\left(e^{ax}-1\right)^{-1}\right] \Rightarrow a^{-z}\Gamma\left(z\right)\zeta\left(z\right), \quad \operatorname{Re}\left(z\right)>1 \tag{90}$$

Then, the Mellin transform of the $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$, following the properties of the convolution presented in Appendix D.6, is

$$\mathscr{M}\left\{\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)*f(t)\right\}=\mathscr{M}\left[\frac{f(t)}{e^{\frac{\alpha}{1-\alpha}t}-1}\right]=\left[\left(\frac{1-\alpha}{\alpha}\right)^{z}\Gamma\left(z\right)\zeta\left(z\right)\right]\mathscr{M}\left[f\left(t\right)\right] \tag{91}$$

Hence, for example, in two cases :

With a power function $f(t)=t^{p}$ (see (206))

$$\mathscr{M}\left\{\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)*t^{p}\right\}=\mathscr{M}\left[\frac{t^{p}}{e^{\frac{\alpha}{1-\alpha}t}-1}\right]=\left[\left(\frac{1-\alpha}{\alpha}\right)^{z}\Gamma\left(z\right)\zeta\left(z\right)\right]\left[\frac{1}{p}\left(\frac{z}{p}\right)\right] \tag{92}$$

With $f(t)=\sin(at)$ (see (219)) we have

$$\mathscr{M}\left\{\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)*\sin(at)\right\}=\left[\left(\frac{1-\alpha}{\alpha}\right)^{z}\Gamma\left(z\right)\zeta\left(z\right)\right]\left[a^{-z}\Gamma\left(z\right)\sin\left(\frac{\pi z}{2}\right)\right], \quad -1<\operatorname{Re}\left(z\right)<1 \tag{93}$$

More examples can be constructed if the functions $f(t)$ have Mellin transforms [32,33,35,36,37].

4.5 Mellin transform to the operator $J_{WS}(\alpha,t)$

Now, we have to evaluate the Mellin transform of the new integral operator, namely

$$\mathscr{M}\left[J_{WS}(\alpha,x);z\right]=\mathscr{M}\left[\frac{1}{N(\alpha)}\int_{0}^{t}\frac{f(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau\right] \tag{94}$$

or equivalently

$$\mathscr{M}\left[J_{WS}(\alpha,x);z\right]=\int_{0}^{\infty}x^{z-1}\left[\frac{1}{N(\alpha)}\int_{0}^{t}\frac{f(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau\right]dx \tag{95}$$

Note: For the sake of better clarity of calculations we use instead in the following calculations that are coherent of the symbols used in most of the articles devoted to the Mellin transforms of convolution operators.

As a first step, applying the Fubini's theorem to (94) we get

$$\mathscr{M}\left[J_{WS}(\alpha,x);z\right]=\frac{1}{N(\alpha)}\int_0^{\infty}x^{z-1}f(\tau)\left[\int_0^{x}\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}dx\right]d\tau \tag{96}$$

Then, changing the variable to $u=\tau/x$ (with $\tau=x\rightarrow u=1$ and $\tau=0\rightarrow u=0$) we may rearrange the inner integral in (96) as (for the sake of simplicity and clarity of the transform we replaced $\alpha/(1-\alpha)$ by λ)

$$\left[\int_0^{x}\frac{1}{e^{\lambda(t-\tau)}-1}dx\right]\Rightarrow\frac{1}{\lambda}\int_0^{\infty}\frac{1}{e^{y}-1}dy,\quad y=\lambda x(1-u) \tag{97}$$

Then, by the change of variable $u=\tau/x$, we get that x^{z-1} can be presented as

$$x^{z-1}=\frac{\tau^{z-1}}{u^{z-1}}=\tau^{z-1}\frac{y^{z-1}}{\lambda^{z-1}} \tag{98}$$

Therefore, after these changes of the variables and some algebra, the general expression (96) can be presented as

$$\mathscr{M}\left[J_{WS}(\alpha,x);z\right]=\frac{1}{N(\alpha)}\int_0^{\infty}\tau^{z-1}f(\tau)\frac{1}{\lambda^{z-1}}\left[\frac{1}{\lambda}\int_0^{\infty}\frac{y^{z-1}}{e^{y}-1}dy\right]d\tau \tag{99}$$

With $\int_0^{\infty}\frac{y^{z-1}}{e^{y}-1}dy=\Gamma(z)\zeta(z)$ (see (12) and Appendix D), we get

$$\mathscr{M}\left[J_{WS}(\alpha,x);z\right]=\frac{1}{N(\alpha)}\frac{\Gamma(z)\zeta(z)}{\lambda^{z}}\int_0^{\infty}\tau^{z-1}f(\tau)d\tau=\frac{1}{N(\alpha)}\frac{\Gamma(z)\zeta(z)}{\lambda^{z}}f_M(z),\quad \lambda=\frac{\alpha}{1-\alpha} \tag{100}$$

*Remark.*We can see that the integral transform of the new integral operator works like a multiplicative convolution, as demonstrated in Section 4.3, since all changes of variables (required to represent the convoluted functions) hold and assure that, as demonstrated above. Hence, we may use this feature of the new integral operator (and of the derivative), as it will be demonstrated in Section 4.6, as a rule allowing easy performance of further calculations.

4.5.1 Some examples with $J_{WS}(\alpha,t)f(t)$

Power function $f(t)=t^{p}$

With $f(t)=t^{p}$ and its Mellin's transform $\mathscr{M}\left[t^{p}\right]=\frac{1}{p}\left(f_M\left(\frac{z}{p}\right)\right)$ we have

$$\mathscr{M}\left[J_{WS}(\alpha,t);z;(t^{p})\right]=\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\frac{1}{z}\left[\Gamma(z)\zeta(z)\right]\left(\frac{1}{p}\left(\frac{z+1}{p}\right)\right) \tag{101}$$

With $f(t)=sin(at)$ (see $\mathscr{M}\left[sin(at)\right]$ from (93)), we have

$$\mathscr{M}\left[J_{WS}(\alpha,t)(\sin(at))\right]=\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[a^{-(z+1)}\Gamma(z+1)\frac{\sin(\pi(z+1))}{2}\right]\right\} \tag{102}$$

With $f(t)=e^{-at}$

Taking into account that $\mathscr{M}\left[e^{-at}\right]=a^{-z}\Gamma(z)$ we have

$$\mathscr{M}\left[{}^{WS}J_{WS}(\alpha,t)\left(e^{-at}\right)\right]=\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[a^{-(z+1)}\Gamma(z+1)\right]\right\} \tag{103}$$

4.6 Mellin transform of the operator ${}^{WS}D_t^{\alpha} f(t)$

Now, looking at the definition of the derivative ${}^{WS}D_t^{\alpha}\ f(t)$ (70) we will use the results from the preceding point and the definitions (208) and (209) in Appendix D.4.1, we may write $\mathscr{M}\left[f^{(1)}(t);z;t\right] = -(z-1)\,f_M(z-1)$. Then, from the general formulation

$$\mathscr{M}\left[{}^{WS}D_t^{\alpha}(f(t);z;t)\right] = M\left[\frac{1}{N(\alpha)}\int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}\frac{df(\tau)}{d\tau}d\tau\right] = \tag{104}$$
$$= \int_0^{\infty} t^{z-1}\left[\frac{1}{N(\alpha)}\int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}\frac{df(\tau)}{d\tau}d\tau\right]dt$$

and passing all the steps, as in the evaluation of $M[J_{WS}(\alpha,t);z]$, we get

$$\mathscr{M}\left[{}^{WS}D_t^{\alpha}(f(t);z;t)\right] = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[-(z-1)\,f_M(z-1)\right]\right\} \tag{105}$$

4.6.1 Some examples with ${}^{WS}D_t^{\alpha} f(t)$

Power function $f(t) = t^p$

With $f(t) = t^p$ and its Mellin's transform $\mathscr{M}[t^p] = \frac{1}{p}\left(f_M\left(\frac{z}{p}\right)\right)$ we have

$$\mathscr{M}\left[{}^{WS}D_t^{\alpha}(t^p)\right] = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[(1-z)\frac{(z-1)^{p+1}}{p}\right]\right\} \tag{106}$$

With $f(t) = sin(at)$

Applying $\mathscr{M}[sin(at)]$ from (93), we have

$$\mathscr{M}\left[{}^{WS}D_t^{\alpha}(\sin(at))\right] = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[a^{-(z-1)}\Gamma(z-1)(1-z)\sin\left(\frac{\pi(z-1)}{2}\right)\right]\right\} \tag{107}$$

With $f(t) = e^{-at}$

Taking into account $\mathscr{M}[e^{-at}] = a^{-z}\Gamma(z)$, we have

$$\mathscr{M}\left[{}^{WS}D_t^{\alpha}(e^{-at})\right] = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\frac{1}{z}\left\{\left[\Gamma(z)\zeta(z)\right]\left[a^{-(z-1)}(1-z)\Gamma(z-1)\right]\right\} \tag{108}$$

5 Tests for index laws and semi-group properties

After all these demonstrations of the new operator features it is mandatory to perform tests answering two principal questions: Do the index law and the semi-group property, known from the classical fractional calculus, hold? Now, we can do this by applying simple tests through the Mellin transforms.

5.1 Does the index law hold?

Concerning the integral operator $J_{WS}(\alpha,t)$ let us assume that the following rule is valid

$$J_{WS}(\alpha,t)\,J_{WS}(\beta,t) = J_{WS}(\alpha+\beta,t), \quad 0<\alpha<1, \quad 0<\beta<1 \tag{109}$$

That is, with $\alpha+\beta=\gamma$ we should have

$$J_{WS}(\alpha,t)\,J_{WS}(\beta,t) = \int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}}-1}\left[\int_0^t \frac{1}{e^{\frac{\beta}{1-\beta}}-1}f(\tau)\,d\tau\right]d\tau = \int_0^t \frac{1}{e^{\frac{\gamma}{1-\gamma}}-1}f(\tau)\,d\tau \tag{110}$$

Applying the Mellin transform to both sides of (110) we get

$$\mathscr{M}\left[J_{WS}(\alpha,t)\,J_{WS}(\beta,t)\,;z\right]=\frac{1}{A^{z}}\left[\Gamma(z)\,\zeta(z)\right]\frac{1}{B^{z}}\left[\Gamma(z)\,\zeta(z)\right]f_{M}(z)=\frac{1}{C^{z}}\left[\Gamma(z)\,\zeta(z)\right]f_{M}(z)$$
$$A=\frac{\alpha}{1-\alpha},\quad B=\frac{\beta}{1-\beta},\quad C=\frac{\gamma}{1-\gamma} \tag{111}$$

That is, because both sides of (111) differ , we have

$$\mathscr{M}\left[J_{WS}(\alpha,t)\,J_{WS}(\beta,t)\,;z\right]=\frac{1}{A^{z}B^{z}}\left[\Gamma(z)\,\zeta(z)\right]^{2}f_{M}(z)\neq\frac{1}{C^{z}}\left[\Gamma(z)\,\zeta(z)\right]f_{M}(z) \tag{112}$$

Therefore, since the Mellin transforms are not equal, then the index law does not hold.

5.2 Do semi-groups exist?

Now, we have to see does the rule

$$J_{WS}(\alpha,t)\,{}^{WS}D(\alpha,t)\,f(t)=f(t)\,,\quad 0<\alpha<1,\quad 0<\beta<1 \tag{113}$$

is valid? Explicitly, we have to see does the following is true

$$J_{WS}(\alpha,t)\,{}^{WS}D_{t}^{\alpha}(\alpha,t)=\int_{0}^{t}\frac{1}{e^{\frac{\alpha}{1-\alpha}}-1}\left[\int_{0}^{t}\frac{1}{e^{\frac{\alpha}{1-\alpha}}-1}f^{(1)}(\tau)\,d\tau\right]d\tau=f(t) \tag{114}$$

Applying to (114) the Mellin transform we get

$$\mathscr{M}\left[J_{WS}(\alpha,t)\,{}^{WS}D_{t}^{\alpha}(\alpha,t)\,;z\right]=\left[\frac{\Gamma(z)\,\zeta(z)}{A^{z}}\right]^{2}\left[-(z-1)\,f_{M}(z-1)\right]\neq f_{M}(z) \tag{115}$$

Therefore, the rule $J_{WS}(\alpha,t)\,{}^{WS}D(\alpha,t)\,f(t)=f(t)$ does not hold. It is easy to check, also that ${}^{WS}D(\alpha,t)\,J_{WS}(\alpha,t)\,f(t)\neq f(t)$.

*Remark.*It was easy to predict the missing index law and the semi-group properties bearing in mind that starting the departure of t from limit point $t_{0^{+}}$ (where the kernel approaches asymptotically the power-law) the kernel behavior tends to be close to the exponential one, where the desired properties do not hold, facts known from the studies on the Caputo-Fabrizio kernel' properties [4]. The intended, but nonexistent, features would not make computations easier, but if the modeling using these operators is done correctly, this is not as significant of an issue.

6 Some formal equations

6.1 Differential equation with ${}^{WS}D_{t}^{\alpha}f(t)$

Now, we try to construct the general rule of solutions of fractional differential equations involving the derivative ${}^{WS}D_{t}^{\alpha}$

$${}^{WS}D_{t}^{\alpha}y(t)=q(t) \tag{116}$$

Following [34] (Chapter 5) where similar problems with the Liouville derivative are solved, we can apply the Mellin transforms to both sides of (116), namely

$$\mathscr{M}\left[{}^{WS}D_{t}^{\alpha}y(t)\,;z\right]=\mathscr{M}\left[q(t);z\right]\Rightarrow Y(z)\,y_{M}(z-1)=q_{M}(z)\,,$$
$$-(z-1)y_{M}(z-1)=\mathscr{M}\left[y^{(1)}(t)\right],\quad q_{M}(z)=\mathscr{M}\left[q(t)\right] \tag{117}$$

Rearranging (117), we get (see (105))

$$y_{M}(z-1)=\frac{q_{M}(z)}{Y(z)},\quad Y(z)=\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\frac{1}{z}\left[\Gamma(z)\,\zeta(z)\right]\left[-(z-1)\right] \tag{118}$$

Here, $y_M(z-1) = \frac{q_M(z)}{Y(z)}$ can be considered as a transfer function $H(z)$. Then, applying the inverse transform $\mathscr{M}^{-1}$ to (118), the result is

$$y(t) = \mathscr{M}^{-1}\left[\frac{q_M(z)}{Y(z)}\right] = \mathscr{M}^{-1}[H(z)] \tag{119}$$

The $\mathscr{M}^{-1}$ can be evaluated by either tabulated formulas, if they exist, or through the Mellin-Barnes integral

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(z)t^{-z}dz, \quad H(z) = \frac{q_M(z)}{Y(z)} \tag{120}$$

*Remark.*About the associate integral of ${}^{WS}D_t^{\alpha}y(t)$: The last results indicating the missing index law and semi-group properties directly lead to the conclusion that the quest for an associate integral will be unsuccessful.

6.2 Integral equation with $J_{WS}(\alpha,t)f(t)$

The general form of an integral equation involving the operator $J_{WS}(\alpha,t)$ can be constructed as

Integral equation of 1st kind

$$y(t) = J_{WS}(\alpha,t)[\varphi(t)] = \frac{1}{N(\alpha)}\int_0^t \frac{\varphi(t)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{121}$$

or

Integral equation of 2nd kind

$$\varphi(t) = y(t) + J_{WS}(\alpha,t)[\varphi(t)] = y(t) + \frac{1}{N(\alpha)}\int_0^t \frac{\varphi(t)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{122}$$

imitating the construction of the Volterra integral equations.

These are general constructions that do not exclude other integral equations to be formulated. The constructions (121) and (122) are Volterra's *convolution equations*, in a general formulation

$$y(t) = \varphi(t) + \int K(t-\tau)y(\tau)d\tau, \quad K(t-\tau) = \frac{1}{N(\alpha)}\int_0^t \frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau \tag{123}$$

Their general solutions, by application of the Mellin transforms, can be presented as

Integral equation of 1st kind

$$\begin{aligned}
&\mathscr{M}[y(t)] = \mathscr{M}\{J_{WS}(\alpha,t)[\varphi(t)]\} = \mathscr{M}\left[\frac{1}{N(\alpha)}\int_0^t \frac{\varphi(t)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}d\tau\right] \Rightarrow \\
&\Rightarrow y_M(z) = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\Gamma(z)\zeta(z)\varphi_M(z) \Rightarrow \\
&\Rightarrow \mathscr{M}^{-1}[\varphi_M(z)] = \mathscr{M}^{-1}\left[y_M(z)\left(\frac{\alpha}{1-\alpha}\right)^z \frac{1}{N(\alpha)}\frac{1}{\Gamma(z)\zeta(z)}\right]
\end{aligned} \tag{124}$$

In terms of transfer function $H_1(z)$ we have

$$H_1(z) = \frac{y_M(z)}{\varphi_M(z)} = \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\Gamma(z)\zeta(z) \tag{125}$$

Integral equation of 2nd kind

$$\begin{aligned}
&\varphi_M(z) = y_M(z) + \mathscr{M}[J_{WS}(\alpha,t)[\varphi(t);z]] = y_M(z) + \frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^z}\Gamma(z)\zeta(z)\varphi_M(z) \Rightarrow \\
&\Rightarrow \varphi_M(z) = y_M(z)\left[N(\alpha)\left(\frac{\alpha}{1-\alpha}\right)^z \frac{1}{N(\alpha)\left(\frac{\alpha}{1-\alpha}\right)^z - \Gamma(z)\zeta(z)}\right]
\end{aligned} \tag{126}$$

Applying $\mathscr{M}^{-1}$ to both sides of (126) $\varphi(t)$ could be determined either analytically or numerically. Alternatively, in terms of transfer function, we have

$$H_2(z) = \frac{y_m(z)}{\varphi_M(z)} = \left[N(\alpha) \left(\frac{\alpha}{1-\alpha} \right)^z \frac{1}{N(\alpha)\left(\frac{\alpha}{1-\alpha}\right)^z - \Gamma(z)\zeta(z)} \right]^{-1} \tag{127}$$

*Remark.*To close this section it is important to say that we especially do not provide examples with solutions of fractional or integral equations involving the newly defined operators. Their solutions, especially addressing the procedures of the inverse Mellin transforms need special attention and techniques to be developed, problems beyond the scope of this article.

7 Modeling with $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$ kernel: Some considerations and examples

7.1 Determination of the fractional order

The starting point in the determination of the fractional order α is the approximation of the experimental relaxation curve by the kernel $\mathscr{W}_{\mathscr{L}\mathscr{S}}$. Now we will explain this procedure step by step.

First, let us present the reciprocal of $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ as

$$\frac{1}{\mathscr{W}_{\mathscr{L}\mathscr{S}}} = \mathscr{W}_{recip} = e^{\lambda t} - 1 \Rightarrow \mathscr{W}_{recip} + 1 = e^{\lambda t}, \quad \lambda = \frac{\alpha}{1-\alpha} \tag{128}$$

Then in semi-logarithmic coordinates, we have a linear plot with a slope λ , namely

$$\ln(\mathscr{W}_{recip} + 1) = \lambda t \tag{129}$$

The fitting of the experimental points defines λ, with some error of course, that allows to calculate α from

$$\lambda = \frac{\alpha}{1-\alpha} \Rightarrow \alpha = \frac{\lambda}{1+\lambda} \tag{130}$$

As mentioned earlier in Section 2.3, the fractional order α can be determined by the scaled (dimensionless) relaxation time $\bar{u}$ (see equation (54)), that is $\lambda/t_0 = \bar{u}$, where t_0 is the time scale of the modeled relaxation process.

Therefore, the primary test verifying that the kernel may approximate the behavior of the modeled relaxation process by fitting experimental data is mandatory; this means that equation (128) can fit the experimental points, thus defining λ. Then, the fractional order α can be determined from (130).

*Remark.*It is obvious, but we have to mention that data fitting for determination λ can be carried out at some times afterward the relaxation onset, bearing in mind that close to $t = 0$, i.e., for $t \to t_{0^+}$ the kernel is singular. In fact, we have a very high value that cannot be measured by the data acquisition devices; thus, mathematically, it is assumed to be extremely high, approaching infinity, which explains the use of the singular kernel in relaxation approximation.

7.2 Examples

7.2.1 Kinetic-type non-local equation

Kinetic equations describing various time-dependent phenomena such as chemical reactions, and growth processes having varying production and destruction rates $C(t)$are local where the time balances of the creation and destruction rates are presented as [39,40,41]

$$\frac{dC}{dt} = \pm dC(t,\tau) \mp pC(t,\tau), \quad C(t,\tau) = C(t-\tau), \quad \tau > 0 \tag{131}$$

where $\pm$ indicates a growth in the time process, while $\mp$ relying on a time-decaying process. Looking deeply at the transport process we can see that both the destruction $dC(t,\tau)$ and the production rates $pC(t,\tau)$ are history-dependent over a relaxation time τ. That is, there is no instantaneous process, but a time-sift exists, and therefore the process is non-local in time. Then, the integral version of (131) can be presented as

$$\frac{dC}{dt} = \pm cC(t,\tau) = \pm c\int_0^t R(t-\tau)C(\tau)\,d\tau = \pm cR(\tau)*C(\tau) \tag{132}$$

In the particular case of a Riemann-Liouville integral ${}_0D_t^{-\alpha}C(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{C(\tau)}{(t-\tau)^{1-\alpha}}d\tau$, it is popular as [39,40,41]

$$C(t) = C_0 \pm c^{\alpha}{}_0D_t^{-\alpha}N(t) \tag{133}$$

Now, using the general construction (132) we may use the new kernel and the integral $J_{WS}(\alpha,t)$ to construct the following model

$$C(t) = C_0 \pm c^{\alpha}J_{WS}(\alpha,t)[C(t)] \Rightarrow C(t) - C_0 = \pm c^{\alpha}J_{WS}(\alpha,t)[C(t)] \tag{134}$$

or

$$\Delta C(t) = (C(t) - C_0) = \pm c^{\alpha}\int_0^t \frac{C(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)} - 1}d\tau \tag{135}$$

which is a non-local (convolution) equation of 2nd kind. Applying the Mellin transform to both sides of (135) we get

$$\begin{aligned}&\mathscr{M}[\Delta C(t)] = \mathscr{M}[(C(t) - C_0)] = \pm k^{\alpha}\mathscr{M}[C(z)] \Rightarrow \mathscr{M}[(C(t) - C_0)] = \\ &= \pm k^{\alpha}\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\Gamma(z)\zeta(z)C_M(z)\end{aligned} \tag{136}$$

Now, we have to recall that the constants do not possess the Mellin transform in contrast to the Laplace transform. Therefore, we may apply this kinetic model if $C_0 = C(t=0) = 0$, that is, it is valid for time growing processes, or when , for instance. $C_0 = k_0t^{\beta}$ is time-dependent. In such cases, the non-local kinetic equation, in integral form, can be formulated as

$$C(t) = k_0 f(t) + J_{WS}(\alpha,t)[C(t)] \Rightarrow C(t) = k_0 f(t) + c^{\alpha}\int_0^t \frac{C(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)} - 1}d\tau \tag{137}$$

The term $k_0 f(t)$ is the functional related to long times when the second term in (136) vanishes. It is commonly presented as power-law $f(t) \equiv t^{\beta}$, $\beta > 0$. In such a case, for instance, we have

$$C(t) = k_0t^{\beta} + c^{\alpha}\int_0^t \frac{C(\tau)}{e^{\frac{\alpha}{1-\alpha}(t-\tau)} - 1}d\tau \tag{138}$$

Then, the Mellin transforms yield

$$C_M(z) = k_0\frac{1}{(\beta+z)}\frac{z}{\beta} + c^{\alpha}\frac{1}{N(\alpha)}\frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{z}}\Gamma(z)\zeta(z)C_M(z) \tag{139}$$

and

$$\begin{aligned}&C_M(z) = \Phi(z) \Rightarrow C(t) = \mathscr{M}^{-1}[\Phi(z)] \\ &\Phi(z) = k_0\frac{1}{(\beta+z)}\frac{z}{\beta}\left[1 - c^{\alpha}\frac{1}{N(\alpha)}\left(\frac{1-\alpha}{\alpha}\right)^{z}\Gamma(z)\zeta(z)C_M(z)\right]^{-1}\end{aligned} \tag{140}$$

*Remark.*In this example, we especially choose $C_0 = k_0t^{\beta}$ since it exists in the cases solved in [39,40] thus assuring a ground for comparative analysis.

7.2.2 Example:Constitutive approach in the model build-up with non-local operator

In this section, we address basic constructions of non-local modeling, applying two general principles :

– The Boltzmann superposition principle (BSP)

–The fading memory approach (FMA)

The Boltzmann linear superposition functional [42] for hereditary viscoelasticity with a time-dependent memory kernel (correlation function $R(t,z)$) is

$$\varphi(x,t) = A_1 [v_x(x,t)] + A_2 \int_0^t R(t,z) v_x(z) dz \tag{141}$$

The fading memory concept relating the flux to its gradient, for simple materials [43,44,45], is modeled by the following integrodifferential equation

$$j(x,t) = -A_1 \frac{\partial}{\partial x} C(x,t) - A_2 \int_{-\infty}^{t} R(t-\tau) \frac{\partial}{\partial x} C(x,t) d\tau \tag{142}$$

as a manifestation of (141). In (142) the transport coefficients A_1 and A_2 are, in general, diffusivities.

Since $C(x,t)$ is a causal function (vanishing for $t<0$) and considered only for $0<t<\infty$, then the lower terminal of the convolution integral in (142) should be set to zero, thus matching the construction of (141). Therefore (142) takes the form

$$\frac{\partial}{\partial x} j(x,t) = -A_1 \frac{\partial}{\partial x} C(x,t) - A_2 \int_0^t R(t-\tau) \frac{\partial}{\partial x} C(x,t) d\tau \tag{143}$$

The setting of the lower terminal of the memory integral to zero has a deep physical meaning when applying hereditary integrals. The essence is well expressed by Hilfer in [46] that in fractional operators the time is not the chronological time (*instant time*) but the intrinsic time of the process (the *time of duration*), starting at the point accepted as $t=0$, that is,there is no relaxation process before $t=0$.

*Remark.*The deep thermodynamic sense of the fading memory formulation is that the non-locality represented by the convolution term works for short times, while for long times we get local diffusion flux, the first term in (143)), i.e. the instant reaction of the system. Moreover, models constructed with the fading memory principle obey the *causality principle* (through the convolution term), thermodynamic consistency, and *model observability* (objectivity) [47,48,49]. Detailed presentations of how these two basic principles are successfully applied to fractional modeling are available in [50,51,52].

7.2.3 Example: Heat conduction model with $\mathscr{W}_{\mathscr{L}\mathscr{S}}$ as a memory term

Here we start with a simple heat conduction model where the energy conservation equation is

$$\frac{\partial(\rho C_p T)}{\partial t} = -\frac{\partial q}{\partial x} \tag{144}$$

If the constitutive flux-gradient relation is

$$q(x,t) = -k \frac{\partial T(x,t)}{\partial x} \tag{145}$$

then we get the local in-time heat conduction model (Fourier model).

$$\frac{\partial(\rho C_p T)}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{146}$$

or with density $\rho = const.$ and heat capacity $C_p = const.$ as

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, \quad a = \frac{k}{\rho C_p} \tag{147}$$

where a is the thermal diffusivity $[m^2/s]$.

Note: *We consider this classic model especially because it is memoryless, i.e. with Dirac's delta as a memory kernel and allows easy to see the differences when the non-local approach is applied.*

Now, assuming that our memory kernel is $\mathscr{W}_{\mathscr{L}\mathscr{S}}(\alpha,t)$ we may construct a heat diffusion model in terms of $T(x,t)$ and involving the integral $J_{WS}(\alpha,t)$, namely

$$q=-a_1\frac{\partial T}{\partial x}-a_2J_{WS}(\alpha,t)\left[\frac{\partial T}{\partial x}\right]\Rightarrow -a_1\frac{\partial T(x,t)}{\partial x}-a_2n(\alpha)\frac{1}{N(\alpha)}\int_0^t\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}\left[\frac{\partial T(\tau,x)}{\partial x}\right]d\tau \tag{148}$$

where $n(\alpha)$ is a balancing function that should be zero for $\alpha=1$, and determined through the model development (see the remark at the end of this section).

Then, applying the energy balance (the continuity equation) we get a heat conduction equation where the hereditary term uses the Lambert-Widder kernel, namely

$$\frac{\partial T(x,t)}{\partial t}=a_1\frac{\partial^2 T(x,t)}{\partial x^2}+a_2(1-\alpha)\frac{1}{N(\alpha)}\int_0^t\frac{1}{e^{\frac{\alpha}{1-\alpha}(t-\tau)}-1}\left[\frac{\partial^2 T(\tau,x)}{\partial x^2}\right]d\tau \tag{149}$$

For large times the second term in (149) vanishes. Moreover, the memory function in the second term of (149) for $\alpha=1$ is zero (because $\left(\exp\left(\frac{\alpha}{1-\alpha}t\right)\right|_{\alpha\to 1}\to\infty$, for any t , and taking into account that for $\alpha=1$ the normalization function $N(\alpha)\to 0$ (see the analysis in Section 3.2 and equations (75) and (75)). Then, for $\alpha=1$ the model (149) reduces to the Fourier model.

*Remark.*As to the function $N(\alpha)=1-\alpha$ we refer to the analysis in [53] (see also the same approach in [50,51,52]). The main *trick* is that for large t the Lambert-Widder kernel approaches the behavior of the Caputo-Fabrizio kernel (see the remark after equation (80)).

7.3 A few ideas that go beyond exponential memory

After all these multifaceted studies on the new perspective of the Lambert-Widder transform it is natural to raise the question: Is it possible to extend the idea beyond the exponential memory? Answering this question the first idea that comes to mind is to create a new kernel by a migration of the simple exponential function to its generalization, that is to the Mittag-Leffler function with one parameter. In such a case, we can construct mechanistically, by intuition, the following kernel and a related transform

$$\mathscr{W}_{ML1}=\frac{1}{E_\alpha(t)-1}\Rightarrow\int_0^\infty\frac{f(t)}{E_\alpha(t)-1}dt,\quad t>0,\quad 0<\alpha<1 \tag{150}$$

Looking precisely at the denominator we can see that the kernel can be presented as

$$\frac{1}{E_\alpha(t)-1}=\frac{1}{\sum\limits_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(\alpha k+1)}},\quad t>0,\quad 0<\alpha<1 \tag{151}$$

bearing in mind that for $k=0$ $E_\alpha(t)|_{k=1}=1$;for $t\to t_{0+}$ the kernel defined by (150) is singular because $E_\alpha(t)|_{k\geq 1}(t=0)=0$.

Hence, applying $E_\alpha(t)$ in the kernel, we use all its terms except that for $k=0$. If we like this to be avoided we may suggest that the kernel could be based on the Mittag-Leffler function of two parameters, namely

$$\mathscr{W}_{ML2}=\frac{1}{E_{\alpha,\beta}(t)-1}\Rightarrow\int_0^\infty\frac{f(t)}{E_{\alpha,\beta}(t)-1}dt,\quad 0<\alpha<1,\quad \beta\neq 1 \tag{152}$$

Then, the kernel can be expressed as

$$\mathscr{W}_{ML2}=\frac{1}{E_{\alpha,\beta}(t)-1}=\frac{1}{\sum\limits_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}-1}=\frac{1}{\left(\frac{1}{\Gamma(\beta)}-1\right)+\sum\limits_{k=1}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}} \tag{153}$$

For $t \to t_{0+}$ the kernel defined by (153) tends to $\frac{1}{\Gamma(\beta)} - 1$, and therefore it is bounded. The condition $\frac{1}{\Gamma(\beta)} - 1 > 0$ needs $\frac{1}{\Gamma(\beta)} - 1 > 0$ and $\frac{1}{\Gamma(\beta)} > 1 \to \Gamma(\beta) < 1$. From the properties of the Gamma function, this requirement is valid if $\beta \in (1,2)$; for $\beta = 1$ and $\beta = 2$ the kernel becomes singular for $t \to t_{0+}$. Outside this range $\Gamma(\beta) > 1$.

The step toward the formulation of the convolution operators on the ideas just conceived is simple, namely

$$\mathscr{W}_{ML1}(\alpha,t) = \frac{1}{E_\alpha(t-\tau)-1} \Rightarrow \int_0^t \frac{f(\tau)}{E_\alpha(t-\tau)-1}d\tau = \int_0^t \frac{f(\tau)}{\sum_{k=0}^{\infty}\frac{(t-\tau)^{\alpha k}}{\Gamma(\alpha k+1)}-1}d\tau = \int_0^t \frac{f(\tau)}{\sum_{k=1}^{\infty}\frac{(t-\tau)^{\alpha k}}{\Gamma(\alpha k+1)}}d\tau \tag{154}$$

$$\mathscr{W}_{ML2}(\alpha,t) = \frac{1}{E_{\alpha,\beta}(t-\tau)-1} \Rightarrow \int_0^t \frac{f(\tau)}{E_{\alpha,\beta}(t-\tau)-1}d\tau = \int_0^t \frac{f(\tau)}{\sum_{k=0}^{\infty}\frac{(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\beta)}-1}d\tau \tag{155}$$

Plots of the generating functions (Mittag-Leffler functions $E_\alpha(t)$ and $E_{\alpha,\beta}(t)$) and the kernels ($\mathscr{W}_{ML1}$ and $\mathscr{W}_{ML2}$) based on them are shown in Figures 4 and 5, respectively.

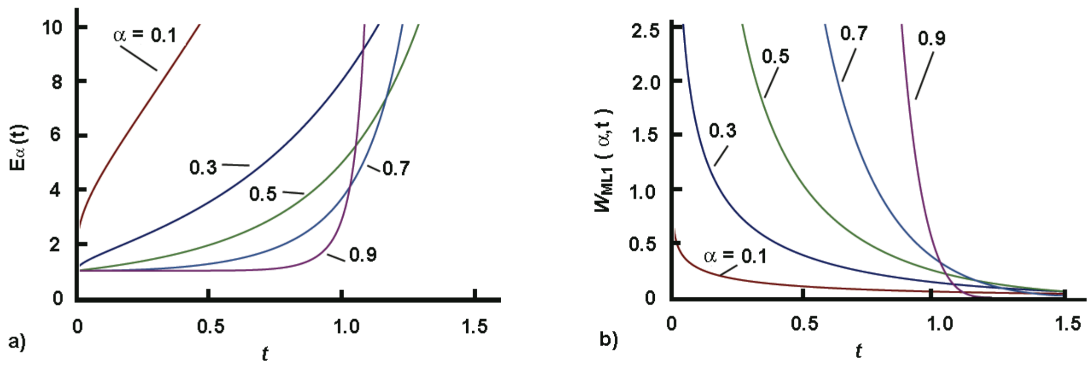

Fig. 4: Mittag-Leffler function of one parameter $E_\alpha(t)$ (panel a) and a singular kernel $\mathscr{W}_{ML1}$ based on it (panel b) for some values of the fractional parameter α against the independent variable t.

The same idea can be extended by increasing the parameters of the Mittag-Leffler function, toward the Prabhakar function $E^{\gamma}_{\alpha,\beta}(+\lambda t^{\alpha})$, for example, as

$$\mathscr{W}_{ML3} = \frac{1}{E^{\gamma}_{\alpha,\beta}(+\lambda t^{\alpha})-1} \Rightarrow \int_0^1 \frac{f(\tau)}{E^{\gamma}_{\alpha,\beta}(+\lambda \tau^{\alpha})-1}d\tau \tag{156}$$

with well-defined ranges of variations of the parameters, thus creating either singular or non-singular versions of the kernels. However, from a personal point of view, too much does not mean better, since the kernel control becomes more complicated. But, the idea was already conceived and new experiments applying it are expected to be carried out.

Here we show the main transformation path, from the kernel constitution toward definitions of new operators, concerning only the inner constructions of the convolution integrals; precisely the kernel constructions. In these constructions $f(\tau)$ can be a certain function or its derivative (first derivative as in the operators in Caputo sense).

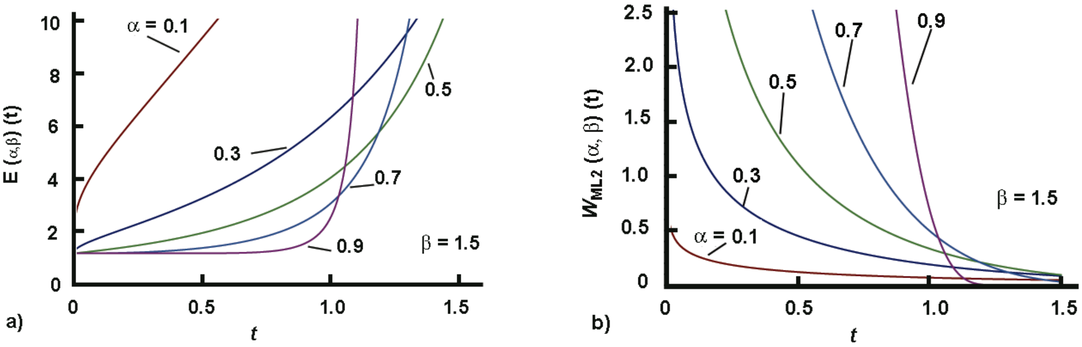

Fig. 5: Mittag-Leffler function of two parameter $E_{\alpha,\beta}(t)$ (panel a) and a singular kernel $\mathscr{W}_{ML2}$ based on it (panel b), with $\beta = 1.5$, for some values of the fractional parameter α against the independent variable t.

8 Outcomes and envisaged problems to be resolved

8.1 Outcomes

Since this is just the beginning, it is difficult to account for all the achievements and evaluate the new steps taken in the creation of new operators. We might, however, mention a few moments that could be outlined from our point of view..

First, it was demonstrated that it is possible to create a singular kernel based on the exponential function. This was possibly staying on the shoulders of many previous studies, mainly on the thorough works of Widder and some new forcing thoughts coming from new trends in the modern fractional calculus.

The new kernel is completely monotonic, singular at the origin t_{0^+}, and loses this property far from it, thus combining features of operators emerging in both the classical and new trends in fractional calculus.

Two non-local operators based on the convolution integral involving the Lambert-Widder kernel are proposed: The fractional integral $J(\alpha,t)[f(t)]$ (71) and two derivatives based on it: ${}^{WSR}D^t_\alpha[f(t)]$ (73) and ${}^{WS}D^t_\alpha[f(t)]$ (71) of Riemann-Liouville and Caputo sense, correspondingly.

The new operators do not obey the index law and semi-group properties, known from the classical fractional calculus based on the power-law memory, but this is not new in the light of some new trends in fractional calculus where similar in nature kernels are applied.

It was successfully demonstrated that the new kernel possessing the Mellin transform can be applied to some functions frequently emerging in applied models when new non-local operators based on it appear.

Formal differential and integral equations with new non-local integral and derivative are considered with two examples:

A kinetic equation with time-dependent initial conditions involving the derivative ${}^{WS}D^t_\alpha[f(t)]$, and

A heat conduction model, developed through the fading memory approach, with a damping term based on the integral $J(\alpha,t)[f(t)]$.

Further, it was suggested, but not yet studied, extensions of the conceived here new non-local operators by replacement of the exponential function by the Mittag-Leffler function, as its generalization, and more complex versions such as the two-parameter Mittag-Leffler function and the Prabhakar kernel.

8.2 Envisaged problems to be solved

Finally, we would like to pose some inquiries that can be considered as possible fields of research and investigation.

First of all, the new kernel proposed needs a more detailed investigation concerning its functional spaces of applications. In the context of this suggestion, a good basis for future studies exists in the works referred to in the introduction section.

A second and quite important issue is the development of analytical or numerical approaches to solve convolution integrals as an alternative to the application of the Mellin transform.

In contrast to the Laplace transform method where the images are algebraic functions that are easy to handle in the problem solutions, the Mellin transform yields images expressed in terms of special functions, precisely the Gamma and Zeta functions. Not all functions have images when we try to apply the Mellin transforms, and therefore, to a greater extent, this limits their application. As to the emerging problems with the solutions of equations where the non-local operators involve the Lambert-Widder kernel, we may envisage resolving problems through efficient analytical or numerical techniques in the inverse steps when the Mellin-Barnes integral is applied.

Tests with experimental data that allow successful fitting of relaxation processes with the new kernel are highly required. In this context, we demonstrated here how the fractional order is to be determined if experimental data exist.

As a consequence of the work done, it would be interesting to extend the study toward a deeper analysis of a non-local operator based on the non-singular Widder's kernel mentioned in the introduction section.

Last but not least, there are very encouraging insights. We believe that the proposed, but not yet completely investigated, singular operators based on the Mittag-Leffler function and its versions may exhibit new features and undiscovered areas of application.

9 Conclusion

The main points of this study can be briefly outlined as:

The Lambert-Widder transform was considered from a different point of view while formulating novel non-local singular operators possessing exponential memory.

An extensive examination of the basic singular version of the Lambert-Widder kernel through a transformation toward a memory function in a convolution integral controlled by a fractional parameter was systematically carried out.

The new kernel is completely monotonic, singular at the origin, and loses this property far from it, thus combining features of operators emerging in both the classical and new trends in fractional calculus.

Non-local operators based on the convolution integral involving the Lambert-Widder kernel are proposed: a fractional integral and two derivatives based on it (in the Riemann-Liouville and Caputo senses, respectively).

It was successfully demonstrated that the new kernel possessing the Mellin transform can be applied to some functions frequently emerging in applied models when new non-local operators based on it are applied. Formal differential and integral equations involving the novel operators possessing the Mellin transform were expressed with basic steps in their solutions through the Mellin transform.

Examples of a heat conduction model and a fractional kinetic equation involving the new non-local operators are developed.

10 Appendices

A Dirichlet Series

Following Widder [55] (pp.19-20) there are commonly applied three integral transforms, namely (in the original notations) :

$$f(s) = \int_0^\infty G(s,t)\,\varphi(t)\,dt \qquad (157)$$

$$f_n = \sum_{k=0}^{\infty} G(n,k)\varphi_k \tag{158}$$

$$f(s) = \sum_{k=0}^{\infty} G(s,t)\,\varphi_k \tag{159}$$

Assuming $s = \sigma + i\tau$ as a complex variable. These operations convert one function or a sequence into another.

The main properties of the aforementioned transforms, defining the kernels (transforming functions) are:

Laplace transform: with $G(s,t) = e^{-st}$

Cesaro summability: with $G(n,k) = \frac{1}{n}$, $k \leq \frac{1}{n}$

Power series : with $G(s,k) = s^k$

If $G(s,k) = \sum_{k=0}^{\infty} \exp(-\lambda_k s)$, where $0 \leq \lambda_1 < \lambda_2 < \ldots \lambda_{k-1}$, and $\lambda_k \to \infty$ for $k \to \infty$ we define a Dirichlet series of type λ_k [55].

$$\sum_{k=1}^{\infty} a_k \exp(-\lambda_k s) \tag{160}$$

When $\lambda_k = k$, then the region of convergence is the right-hand half-plane, namely: $|e^{-s}| < \rho$ or $\sigma > \log(1/\rho\ )$ [55]. In addition, a general Dirichlet series *needs to have no singularity on the axis of convergence* [55].

If $\lambda_k = \log k$ we get *an ordinary series* [55].

$$\sum_{k=1}^{\infty} \frac{a_k}{k^s} \tag{161}$$

According to Widder [55] this type of series was used by Dirichlet in his number studies including the Riemann *zeta-function* (see Appendix C)

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{162}$$

Further, if $\lambda_k = k$, as mentioned above, we obtain a power series. Moreover, the series (161) can be reduced to a power series, as well as if $\lambda_k = k\log 2$ the same result can be obtained from (160). The Dirichlet series have absolute convergences and are uniformly bounded in any horizontal strip of finite width inside the region of convergence

$$\sigma \geq \sigma_c + \delta, \quad |\tau| \leq R, \quad \delta > 0, \quad R > 0 \tag{163}$$

where σ_c is the abscissa of convergence and the following theorem holds [55].

Note: *All theorems in the sequel are provided without proofs. Detailed analyzes and proofs are available in* [55]).

Theorem 1. Convergence of a Dirichlet series

If the series

$$\sum_{k=1}^{\infty} a_k \exp(-\lambda_k s) \tag{164}$$

converges at s_0 , then it converges in the half-plane $\sigma > \sigma_0$.

and it's absolute convergence follows simply from the following theorem

Theorem 2. Absolute convergence of a Dirichlet series

The series

$$\sum_{k=1}^{\infty} a_k \exp(-\lambda_k s) \tag{165}$$

converges absolutely for $\sigma > \sigma_0$ as a consequence of the relation [55]

$$\sum_{k=1}^{\infty} a_k \exp(-\lambda_k s) \ll \sum_{k=1}^{\infty} |a_k| \exp(-\lambda_k \sigma_0), \quad \sigma > \sigma_0 \tag{166}$$

The Dirichlet series represents an analytic function in its region of convergence, as it is proved by the following theorem [55]

Theorem 3. Analycity of a Dirichlet series

If

$$f(s)=\sum_{k=1}^{\infty}a_k\exp(-\lambda_k s),\quad \sigma_c<+\infty \tag{167}$$

Then

$$f^{(p)}(s)=\sum_{k=1}^{\infty}(-\lambda_k)^p a_k\exp(-\lambda_k s),\quad p=1,2,3,\ldots,\sigma>\sigma_c \tag{168}$$

To close this point, a Dirichlet series converges uniformly in any compact region of its half-plane of convergence (it also converges uniformly in certain regions that extend to infinity) and the correct formula for σ_c is due to Cahen (1894) [56] (see also pp.29-33 in [55]).

$$\sigma_c=\lim_{n\to\infty}\frac{\log|U_n|}{\lambda_n},\quad U_n=\sum_{k=0}^{n}a_k,\quad \sigma_c>0 \tag{169}$$

It is worth noting that a given function $f(s)$ cannot be the sum of two different Dirichlet series that refer to uniqueness (see [55]-pp.34-36).

In the end, there are two important points: A function $f(s)$ defined by a Dirichlet series is [55]-p.48):

– Never analytic at $s=\infty$ (unless it is constant)

– Need have no singularity on the axis of convergence

B Some additional information on the exponential analogs of the Lambert series

The series (5) is closely related to the so-called associated power series [8]

$$\sum_{0}^{\infty}a_k z^k \tag{170}$$

Replacing x^k by $e^{-\lambda_k x}$ (where λ_k is any real sequence such that $\lambda_k<\lambda_{k+1}\to\infty$) in (5) and (170) the results are [8]

General R-series

$$\sum_{k=1}^{\infty}a_k\frac{e^{-\lambda_k x}}{1-e^{-\lambda_k x}} \tag{171}$$

Associated Dirichlet series

$$\sum_{n=1}^{\infty}a_k e^{-\lambda_k x} \tag{172}$$

A special case of (171), when $\lambda_k=\log k$ is the *ordinary R-series*

$$\sum_{k=2}^{\infty}a_k\frac{k^{-x}}{1-n^{-x}} \tag{173}$$

In addition, the *R-series* is a special case of a more general case , when $\alpha=1$ [8]

$$\sum_{k=1}^{\infty}a_k\frac{e^{-\lambda_k x}}{1-e^{-\alpha\lambda_k x}},\quad \alpha>1 \tag{174}$$

C The Zeta function

The zeta-function (ZF), defined by the Dirichlet series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{175}$$

converges for $\sigma > 1$ and is an analytic function [55].

It can be extended analytically for $\sigma > 0$ except for a pole of order 1 and residue 1 at $s = 1$ (Theorem 2, p.52 in [55]).

A very important property of the ZF is defined by a series involving all positive integers [55]). That is, it can be presented as an infinite product involving all primes. Denoting the k^{th} prime as [55], in increasing order, by p_k

$$p_1 = 2\ , p_2 = 3\ , p_3 = 5\ , p_4 = 7\ ,\ p_7 = 11 \tag{176}$$

and by the Euler product for $\zeta(s)$ we get (Theorem 3, p.53 in [55])

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}}], \quad \sigma > 1 \tag{177}$$

The product (177) converges absolutely if [55]

$$\sum_{k=1}^{\infty} \frac{1}{|p_k^s|} < \infty \tag{178}$$

From the Euler's product, we have that $\zeta(s) \neq 0$ for $\sigma > 1$ [55].

The zeta function (see section 3.7 in [55], p.60) can be derived from a functional equation but before its formulation, we have mentioned some identities

For $\sigma > 1$ it follows that

$$\int_0^{\infty} \frac{x^{\sigma-1}}{e^x - 1} dx = \zeta(\sigma)\Gamma(\sigma) \tag{179}$$

As a consequence of integration term by term using the approximation of e^x as a power series (Lemma 7.2 in [55]), namely

$$\int_0^{\infty} \frac{x^{\sigma-1}}{e^x - 1} dx = \sum_{k=1}^{\infty} \int_0^{\infty} x^{\sigma-1} e^{-kx} dx = \sum_{k=1}^{\infty} \frac{\Gamma(\sigma)}{k^{\sigma}} = \zeta(\sigma)\Gamma(\sigma) \tag{180}$$

Furthermore, for $x \neq 2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$ it follows that (Lemma 7.3 in [55])

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2x}{x^2 + 4k^2\pi^2} \tag{181}$$

This is the Mittag-Leffler development of the meromorphic function on the left. After these preliminaries, we go to the definition of Riemann's functional equation (Theorem 7.1 in [55])

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \quad s \neq 1 \tag{182}$$

Bearing in mind that the rational function $1/s$ has two different integral representations in the left and right half-planes [55]

$$\frac{1}{s} = \int_0^1 x^{s-1} dx, \quad \sigma > 0 \tag{183}$$

$$\frac{1}{s} = -\int_1^{\infty} x^{s-1} dx, \quad \sigma < 0 \tag{184}$$

Then from (180) and using (183) we have (replacing s by $\sigma - 1$)

$$\zeta(\sigma)\Gamma(\sigma)=\int_0^1\left(\frac{1}{e^x-1}-\frac{1}{x}\right)x^{\sigma-1}dx+\frac{1}{\sigma-1}+\int_1^\infty\frac{x^{\sigma-1}}{e^x-1}dx \tag{185}$$

The first integrand in (185) approaches $-1/2$ for $x\to 0_+$, then the first integral converges for $\sigma>0$, followed by the restriction to the real axis as an analytic function in the half-plane $\sigma>0$ [55]. Further, the last integral in (185) defines an entire function of s (when σ is replaced by s). Therefore, the right-hand of (185) is analytic for $\sigma>0$, except the pole at $s=1$.

Equation (185) coincides with $\zeta(s)\Gamma(s)$ on a part of the real axis and this can be used to extend analytically again for $\sigma>0$ through the second integral (184), with s replaced by $\sigma-1$, namely:

$$\zeta(\sigma)\Gamma(\sigma)=\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}\right)x^{\sigma-1}dx,\quad 0<\sigma<1 \tag{186}$$

The integration with $\sigma>0$ (see details in [55]) and using again (184) yields

$$\zeta(\sigma)\Gamma(\sigma)=\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}\right)x^{\sigma-1}dx,\quad -1<\sigma<0 \tag{187}$$

Further, applying (182) and (181) we get [55]

$$\zeta(\sigma)\Gamma(\sigma)=\sum_{k=1}^{\infty}\int_0^\infty\frac{2x^\sigma}{x^2+4k^2\pi^2}dx=\frac{\pi}{\cos\frac{\pi\sigma}{2}}\sum_{k=1}^{\infty}(2k\pi)^{\sigma-1} \tag{188}$$

This is the functional equation defining $\zeta(s)$ for $\sigma<-1$, by a simple replacement of σ by $1-s$ .The result is valid because all terms of the series are positive and since the resulting series converges for $-1<\sigma<0$ [55].

Further, bearing in mind the definition of the zeta-function (175) as a continuable meromorphic function over the entire s plane, having unique pole of order 1 and residue 1 at $s=1$ (the only real zeros at points $s=-2,-4,-6,\dots$), we may define the function $\eta(s)$

$$\eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^s},\quad \sigma>0 \tag{189}$$

The function $\eta(s)$ is continuable over the whole plane and therefore it is an entire function.

At the end of this section, we have to mention some properties [55]

$$\frac{\zeta'(s)}{\zeta(s)}\sim\frac{-1}{s-1},\quad s\to 1 \tag{190}$$

The function $\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-\frac{s}{2}}$ satisfies the functional equation [55]

$$f(s)=f(1-s) \tag{191}$$

and

$$\int_0^\infty\frac{x^{\sigma-1}}{e^x+1}dx=\Gamma(\sigma)\eta(\sigma),\quad \sigma>0 \tag{192}$$

as well as [55]

$$\int_0^\infty\frac{x^{\sigma-1}}{e^x+1}dx=\left(1-2^{-\sigma}\right)\int_0^\infty\frac{x^{\sigma-1}}{e^x-1}dx,\quad \sigma>0 \tag{193}$$

D Mellin's transform

D.1 Definition of the Mellin transform

The integral [35,36,37]

$$\mathscr{M}\left[f\left(x\right),z\right] = f\left(z\right) = \int_{0}^{\infty} x^{z-1} f\left(x\right) dx, \quad x > 0 \tag{194}$$

is the Mellin transform of the function $f(x)$ with respect to the complex parameter $z = \sigma + i\tau$.

The substitution $x = e^{-t}$ transforms (194) into a two-sided (bilateral) Laplace

$$f\left(z\right) = \int_{-\infty}^{\infty} f\left(e^{-t}\right) e^{-tz} dt \tag{195}$$

or as a sum of two one-sided Laplace integrals of parameters z and $-z$ [35], namely

$$f\left(z\right) = \int_{0}^{\infty} f\left(e^{-t}\right) e^{-tz} dt + \int_{0}^{\infty} f\left(e^{-t}\right) e^{-t(-z)} dt \tag{196}$$

The domain of absolute and ordinary convergence of the integral (194) consist of the respective strips [35]

$$\beta < \mathrm{Re}\left(z\right) < -\beta', \quad \alpha < \mathrm{Re}\left(z\right) < -\alpha' \tag{197}$$

where β and β' denotes the abscissa of absolute and ordinary convergence, respectively, in (196), while α and α' for the second integral in (196) [35].

D.2 Inversion

The inversion of the integral (194) is defined as [35,37]

$$f\left(x\right) = \mathscr{M}^{-1}\left[f\left(z\right);x\right] \tag{198}$$

D.3 Relation of the Mellin transform to other integral transforms

Laplace transform

$$\mathscr{L}\left[f\left(t\right);x\right] = \int_{0}^{\infty} f\left(t\right) e^{-xt} dt \tag{199}$$

Then [35],

$$\mathscr{M}\left(L\left[f\left(t\right);x\right];z\right) = \Gamma\left(z\right) \mathscr{M}\left[f\left(x\right);z-1\right] \tag{200}$$

Fourier sine transform

$$F_s\left[f\left(t\right);x\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f\left(t\right) \sin\left(xt\right) dt \tag{201}$$

Then [35]

$$\mathscr{M}\left(F_s\left[f\left(t\right);x\right];z\right) = \sqrt{\frac{2}{\pi}} \Gamma\left(z\right) \sin\left(\frac{\pi z}{2}\right) \mathscr{M}\left[f\left(x\right);1-z\right] \tag{202}$$

Finite Mellin transform

The substitution of $t = -\log x$ transforms (199) into a finite Mellin transform [35]

$$f\left(z\right) = \int_{0}^{1} x^{z-1} f\left(x\right) dx, \quad f\left(x\right) = f\left(\log\frac{1}{x}\right) \tag{203}$$

D.4 Mellin transform: Some formulas

Some general formulas related to the present study are summarized next. More detailed tables are available in [35]

D.4.1 General formulas

$$\mathscr{M}\left[af_1(x)+bf_2(x)\right]=af_{1M}(z)+bf_{2M}(z) \tag{204}$$

$$\mathscr{M}\left[f(ax)\right]\Rightarrow a^{-z}f_M(z),\quad a>0 \tag{205}$$

$$\mathscr{M}\left[f(x^p)\right]\Rightarrow p^{-1}f_M\left(\frac{z}{p}\right),\quad \mathscr{M}\left[f\left(x^{-p}\right)\right]\Rightarrow p^{-1}f_M\left(-\frac{z}{p}\right)\quad p>0 \tag{206}$$

$$\mathscr{M}\left[\int_0^x f(t)\,dt\right]\Rightarrow -z^{-1}f_M(z+1),\quad \mathscr{M}\left[\int_x^\infty f(t)\,dt\right]\Rightarrow z^{-1}f_M(z+1) \tag{207}$$

$$\mathscr{M}\left[f_x'(x)\right]=-(z-1)f_M(z-1) \tag{208}$$

$$\mathscr{M}\left[f_x^{(n)}(x)\right]=(-1)^n)\frac{\Gamma(z)}{\Gamma(z-n)}f_M(z-n) \tag{209}$$

$$\mathscr{M}\left[xf_x'(x)\right]=-zf_M(z) \tag{210}$$

Here $f_M(\cdot)$ denotes the Mellin transform of the function $f(x)$.

D.4.2 Powers of arbitrary order [35,37]

$$\mathscr{M}\left[x^\nu\right],x<a\Rightarrow(\nu+z)^{-1}a^{\nu+z},\quad x<a,\quad \mathrm{Re}(z)>-\mathrm{Re}(\nu) \tag{211}$$

$$\mathscr{M}\left[(a+x)^{-1}\right]\Rightarrow \pi a^{z-1}\csc(\pi z),\quad 0<\mathrm{Re}(z)<1 \tag{212}$$

$$\mathscr{M}\left[(a+x)^{-\frac{1}{2}}\right]\Rightarrow(\pi a)^{\frac{1}{2}}a^z\Gamma(z)\Gamma\left(\frac{1}{2}-z\right),\quad 0<\mathrm{Re}(z)<\frac{1}{2} \tag{213}$$

$$\mathscr{M}\left[(a+x)^{-n}\right]\Rightarrow(-1)^{n+1}\frac{\pi}{(n-1)!}(z-1)(z-2)\dots(z-n+1)a^{z-n}\csc(\pi z),\quad n=2,3,4,\dots,\quad 0<\mathrm{Re}(z)<n \tag{214}$$

$$\mathscr{M}\left[(a-x)^{-1}\right]\Rightarrow \pi a^{z-1}\cot(\pi z),\quad 0<\mathrm{Re}(z)<1 \tag{215}$$

D.4.3 Exponential functions [35,37]

$$\mathscr{M}\left[(e^{ax}-1)^{-1}\right]\Rightarrow a^{-z}\Gamma(z)\zeta(z),\quad \mathrm{Re}(z)>1 \tag{216}$$

$$\mathscr{M}\left[(e^{ax}-1)^{-1}e^{-bx}\right]\Rightarrow a^{-z}\Gamma(z)\zeta\left(z,\frac{1+b}{a}\right),\mathrm{Re}(z)>1 \tag{217}$$

$$\mathscr{M}\left[(e^{ax}+1)^{-1}\right]\Rightarrow a^{-z}\Gamma(z)\zeta\left(1-2^{2-z}\right)\zeta(z),\quad \mathrm{Re}(z)>1 \tag{218}$$

D.4.4 Trigonometric functions [35,37]

$$\mathscr{M}\left[\sin(ax)\right]\Rightarrow a^{-z}\Gamma(z)\sin\left(\frac{\pi z}{2}\right),\quad -1<\mathrm{Re}(z)<1 \tag{219}$$

$$\mathscr{M}\left[\cos(ax)\right]\Rightarrow a^{-z}\Gamma(z)\cos\left(\frac{\pi z}{2}\right),\quad -1<\mathrm{Re}(z)<1 \tag{220}$$

D.5 Inverse Mellin transforms: Some formulae

D.5.1 General relationships

$$\mathscr{M}^{-1}\left[a^{-z}f_M(z)\right] \Rightarrow f(ax), \quad a>0 \tag{221}$$

$$\mathscr{M}^{-1}[z+a] \Rightarrow x^a f(x) \tag{222}$$

$$\mathscr{M}^{-1}[f_M(pz)] \Rightarrow p^{-1}f\left(x^{1/p}\right), \quad \mathscr{M}^{-1}[f_M(-pz)] \Rightarrow p^{-1}f\left(x^{-1/p}\right) \tag{223}$$

$$\mathscr{M}^{-1}\left[f_M{}^{(n)}(z)\right] \Rightarrow f(x)(\log x)^n \tag{224}$$

D.5.2 Powers of arbitrary orders

$$\mathscr{M}^{-1}\left[z^{-1}a^z\right] \Rightarrow 1, \quad x<a, \quad \mathscr{M}^{-1}\left[z^{-1}a^z\right] \Rightarrow 0, \quad x>a, \quad \operatorname{Re}(z)>0 \tag{225}$$

$$\mathscr{M}^{-1}\left[z^{-1}a^z\right] \Rightarrow 0, \quad x<a, \mathscr{M}^{-1}\left[z^{-1}a^z\right] \Rightarrow -1, \quad x>a, \quad \operatorname{Re}(z)<0 \tag{226}$$

$$\mathscr{M}^{-1}\left[(\nu+z)^{-1}a^z\right] \Rightarrow \left(\frac{x}{a}\right)^{\nu}, \quad x<a, \quad \mathscr{M}^{-1}\left[(\nu+z)^{-1}a^z\right] \Rightarrow 0, \quad x>a, \quad \operatorname{Re}(z)>-\operatorname{Re}(\nu) \tag{227}$$

$$\mathscr{M}^{-1}\left[z^{-\nu}a^z\right] \Rightarrow \frac{1}{\Gamma(\nu)}\left[\log\left(\frac{a}{x}\right)\right]^{\nu-1}, \quad x<a, \quad \mathscr{M}^{-1}\left[z^{-\nu}a^z\right] \Rightarrow 0, \quad x>a, \quad \operatorname{Re}(z)>0, \quad \operatorname{Re}(\nu)>0 \tag{228}$$

D.5.3 Gamma function and related functions

$$\mathscr{M}^{-1}[\Gamma(z)] \Rightarrow e^{-x}, \quad \operatorname{Re}(z)>0 \tag{229}$$

$$\mathscr{M}^{-1}[\Gamma(z)] \Rightarrow e^{-x}-1, \quad -1<\operatorname{Re}(z)<0 \tag{230}$$

$$\mathscr{M}^{-1}[\sec(\pi z)\Gamma(z)] \Rightarrow e^x Erfc\left(\sqrt{x}\right) \tag{231}$$

D.6 The Mellin convolution transform

The Mellin transform (194) by the convolution of the function $A(x)$ and $B(x)$, defined for $x>0$ we have [36]

$$(A*B)(x) = \int_0^\infty A(t)B\left(\frac{x}{t}\right)\frac{dt}{t} = \int_0^\infty A\left(\frac{x}{t}\right)B(t)\frac{dt}{t} = \mathscr{M}^*_{AB}(x) \tag{232}$$

Further (Theorem 2.3 in [36]), if the functions $x^{j-1}A_j(x)$ (or $x^{j-1}B_j(x)$), $j=1,2$, as well as the function $(A*B)(x)$ are absolutely integrable on the axis $(0,\infty)$, for each z such that $\operatorname{Re}(z)=\gamma$ then the convolution (232) exists and it is equal to the product of the images of the convoluted functions, namely

$$\mathscr{M}[(A*B)(x);z] = \mathscr{M}(A;z)\mathscr{M}(B;z) = \mathscr{M}(z), \quad \operatorname{Re}(z)=\gamma \tag{233}$$

Moreover, following [36], the conditions imposed on the functions $x^{j-1}A_j(x)$ as sufficient. Then, it follows from (232) that (the Parseval's formula) that

$$\mathscr{M}^*_{AB}(x) = \int_0^\infty A(t)B\left(\frac{x}{t}\right)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} A^*(z)B^*(z)x^{-z}dz \tag{234}$$

In addition (from Theorem 2.4 in [36]), if the function $A(x)$ is continuous for $x>0$ and decreases faster at $t\to\infty$ than any power of x, but is possible in the neighborhood of $x=0$ to be expanded as a convergent series $A(x)=\sum_{k=0}^{\infty} a_k x^{\lambda_k}$ ($x\leq\varepsilon$, $0\leq\lambda_0<\lambda_1<\ldots\lambda_k\to\infty$), then the Mellin transform $\mathscr{M}[A(x);z]=A^*(z)$ is equal to the Gama function $\Gamma(z)$. Further, if $a_k=(-1)^k/k!$ and $\lambda_k=k$, then $A(x)=e^{-x}$ with a Mellin transform equal to $\Gamma(z)$. Therefore, $\Gamma(z)$ can be continued analytically into the plane z , except the points $z=0,-1,2,\ldots$where it has simple poles. It this context, it is well-known that the residues of $\Gamma(z)$are $\underset{z=-k}{res}\,\Gamma(z)=\frac{(-1)^k}{k!}$, $k=0,1,2,\ldots$.

D.7 The Mellin transform of fractional operators (with power-law memories)

The Mellin transform of the Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha < 1$ of the function $f(t)$ is [1]

$$\mathscr{M}\left[{}^{RL}D_t^{\alpha}\right] = \mathscr{M}\left[{}^{C}D_t^{\alpha}\right] = \frac{(1-z+\alpha)}{(1-z)}F_M(z+\alpha) \tag{235}$$

Also, the Mellin transform of the Riemann-Liouville integral is [1]

$$\mathscr{M}\left[{}^{RL}D_t^{-\alpha}\right] = \frac{(1-z-\alpha)}{(z-\alpha)}F_M(z+\alpha) \tag{236}$$

D.8 The Mellin transform of some special functions in fractional calculus

Two-parameter Mittag-Leffler function [57]

$$\mathscr{M}\left[E_{\alpha,\beta}(-t);z\right] = \int_0^{\infty} E_{\alpha,\beta}(-t)t^{z-1}dt = \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(\beta-\alpha z)} \tag{237}$$

The Prabhakar function [57]

$$\mathscr{M}\left[E_{\alpha,\beta}^{\gamma}(-t);z\right] = \int_0^{\infty} E_{\alpha,\beta}^{\gamma}(-t)t^{z-1}dt = \frac{\Gamma(z)\Gamma(\gamma-z)}{\Gamma(\gamma)\Gamma(\beta-\alpha z)} \tag{238}$$

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