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Limit-Residual Function Method: A New Approach to Creating a Power Series Solution for Fractional Differential Equations

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Abstract: This study develops and applies a new iterative scheme to generate analytical solutions for linear and nonlinear fractional differential equations. The solution methodology involves producing a fractional power series solution in the form of a rabidly convergent series using the concepts of the limit and the residual function without the need to transform the target equations into other spaces or calculate the derivative to determine the power series solution coefficients, which requires less computational effort than other approaches. This method can be an alternative to the residual power series method, which can be easily and efficiently used to deal with nonlinear fractional differential equations arising in various physical phenomena. To illustrate the procedure and validate the effectiveness of the suggested approach, several applications are studied. This enabled us to demonstrate the potential, accuracy, and the method's versatility in dealing with these equations under restrictive constraints. The complete reliability and efficiency of the proposed algorithm are readily demonstrated by numerical results combined with graphical representations.

Keywords: Fractional derivative and integral; analytic approximation of solutions; power series method.

1 Introduction

Nowadays, fractional differential equations (FDEs) are widely used as models for various biological, engineering, and physical problems [1,2,3,4,5,6,7,8]. Since most dynamical systems involve memory or hereditary effects, the non-local characteristics of fractional derivatives make them a more realistic model than classical local operators. Due to the lack of exact analytical solutions for most nonlinear fractional equations, researchers have proposed and applied analytical and numerical methods to obtain their approximate solutions. Consequently, researchers have been interested in developing effective methods to solve linear and nonlinear problems. These methods include the variational iteration method [9,10], the Adomian decomposition method [11,12], the operational matrix method [13], the Laplace transform method [14], the fractional differential transform method [15], the Homotopy analysis method [16,17], and the Homotopy perturbation method [18,19].

It is known that the power series (PS) method is an effective tool for solving fractional and non-fractional differential equations, where the solution is represented as an infinite series. However, obtaining approximate solutions for nonlinear differential equations and determining the series coefficients presents significant challenges. To address these limitations, several modifications to the PS method have been developed, such as the residual power series (RPS) method [20,21,22,23] and the Laplace residual power series (LRPS) method [24,25,26,27,28,29,30].

The RPS method is an innovative approach for determining the analytical Taylor series solutions of both linear and nonlinear FDEs. By utilizing the residual error concept and calculating derivatives to determine the coefficients of series solutions, the RPS approach provides approximate analytical solutions to the problem in truncated series form. The RPS approach has successfully solved a wide variety of FDEs, including fractional matrix equations [20], Boussinesq–Burgers equations [21], fractional multi-pantograph system [22], neutron diffusion equations in the spherical and hemispherical reactors [23]. On the other hand, the LRPS method combines the RPS methodology with the Laplace transform. The

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effectiveness of the LRPS method has been demonstrated by solving numerous FDEs, such as the hyperbolic system of Caputo-time-fractional partial differential equations with variable coefficients [24], fractional Lane-Emden equation [25], fractional Riccati equation [26], fractional KdV-Burgers equations [27], and others [28,29,30].

In this paper, we propose a novel iterative scheme called the limit residual function (LRF) method [31,32] to create series solutions for fractional ordinary differential equations (FODEs). This method skips the necessity for calculating derivatives or solving algebraic equations in the presence of the Laplace transform and its inverse at every stage of the solution process. The algorithm of the LRF method relies only on the concepts of the residual function and the limit to determine the coefficients of the series solutions. Few calculations are needed to obtain the coefficients compared to the RPS and LRPS methods. The theories governing the new technique are presented with detailed proof. To clarify the mechanism of action of the proposed method and test its efficiency, it was applied to a class of FODEs.

The structure of this paper is as follows: in the next section, we present some concepts and principles of fractional calculus as well as the fractional PS. To build and predict the fractional PS expansion solution, the fundamental principles of the LRF method are illustrated in Section 3. The capability and simplicity of the suggested method are demonstrated in Section 4 by solving several mathematical and physical FODEs of various kinds and orders. Finally, Section 5 provides the conclusion.

2 Essential Preliminaries and Notations

This section introduces fractional calculus and the principles of fractional PS to develop analytical solutions for FODEs using the LRF method.

The mathematical literature has several definitions of fractional derivatives, including those for the Riemann-Lioville derivative, Caputo fractional derivative, and Conformable fractional derivative. This article focuses on the Caputo derivative of order β, which is defined for the function $\omega(\tau)$ as in the next definition.

Definition 2.1. [5,6] The Caputo derivative of order $\beta \in (m-1, m]$, $m \in \mathbb{N}$ of the function $\omega(\tau)$ is defined as follows:

$$\mathscr{D}^{\beta}\omega(\tau) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau - t)^{\beta-1}\, \omega^{(m)}(t)\, dt, & m-1 < \beta < m\,,\ \tau > t \geq t_0 \geq 0, \\ \omega^{(m)}(\tau), & \beta = m. \end{cases} \tag{1}$$

There are many properties of the operator $\mathscr{D}^{\beta}$ that can be found in the Refs. [5,6]. However, a few properties of $\mathscr{D}^{\beta}$ that are critical to our task are as follows: $\mathscr{D}^{\beta}\delta = 0$ and $\mathscr{D}^{\beta}(\tau-\tau_0)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\beta)}(\tau-\tau_0)^{\alpha-\beta}$, $\delta \in \mathbb{R}$, and $\alpha > -1$.

Some definitions and theorems relevant to the classical PS are expanded here to include the fractional case in the Caputo sense. Additionally, novel results about the convergence of the series $\sum_{n=0}^{\infty} a_n (\tau-\tau_0)^{n\beta}$ are also provided. Throughout this work, these definitions and results are required to construct solutions for the FODEs.

Definition 2.2. [20] A PS representation of the form

$$\sum_{n=0}^{\infty} a_n(\tau-\tau_0)^{n\beta} = a_0 + a_1(\tau-\tau_0)^{\beta} + a_2\,(\tau-\tau_0)^{2\beta} + \ldots, \tag{2}$$

where $0 \leq m-1 < \beta \leq m$, $\tau \geq \tau_0$ is called a fractional PS about τ_0 where τ is a variable and a_n's are constants called the coefficients of the series.

Theorem 2.1. [20] The fractional PS $\sum_{n=0}^{\infty} a_n(\tau-\tau_0)^{n\beta}, \tau \geq \tau_0$ has two cases

1. The series converges at all points in the interval $[\tau_0,\ c]$ whenever it converges at the point $c > \tau_0$.
2. The series diverges at all points in the interval $[d, \infty)$ whenever it diverges at the point $d > 0$.

Theorem 2.2. [20] Exactly one of the following statements is true for the fractional PS $\sum_{n=0}^{\infty} a_n(\tau-\tau_0)^{n\beta}$.

1. The series is convergent at the point τ_0.
2. The series is convergent at each real number.
3. The series converges when $|\tau-\tau_0| < r$ and diverges when $|\tau-\tau_0| > r$ for a certain positive real number, r, which is known as the radius of convergence.

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Theorem 2.3. [33] The fractional PS $\sum_{n=0}^{\infty} a_n(\tau-\tau_0)^{n\beta}=0$ for all τ in $|\tau-\tau_0|<r$ iff each coefficient a_n equals zero.

To determine the coefficients of the fractional PS solution, which is considered to be a solution to the FDEs that are discussed in this study, we prove the following significant fact.

Theorem 2.4. Suppose that $\omega(\tau)$ has the fractional PS expansion $\omega(\tau)=\sum_{n=0}^{\infty} a_n(\tau-\tau_0)^{n\beta}$ and $\omega(\tau)=0$, for all τ in some interval I. Then

$$\lim_{\tau\to\tau_0} \frac{\omega_j(\tau)}{(\tau-\tau_0)^{(j-k)\beta}}=0, \quad j,k=1,2,\ldots, \quad \tau\neq\tau_0, \tag{3}$$

where

$$\omega_j(\tau)=\sum_{n=0}^{j} a_n(\tau-\tau_0)^{n\beta}. \tag{4}$$

Proof. Considering that $\omega(\tau)=0$, it follows that $\omega(\tau)/(\tau-\tau_0)^{(j-1)\beta}=0$, for all values of $\tau\neq\tau_0$, and $j=1,2,\ldots$. Thus,

$$\lim_{\tau\to\tau_0} \frac{\omega(\tau)}{(\tau-\tau_0)^{(j-k)\beta}}=0, \quad j,k=1,2,\ldots.$$

Since $\omega(\tau)$ has a fractional PS expansion, the limit can be reformulated as

$$\lim_{\tau\to\tau_0} \sum_{n=0}^{\infty} \frac{a_n\,(\tau-\tau_0)^{n\beta}}{(\tau-\tau_0)^{(j-k)\beta}}=0, \quad j,k=1,2,\ldots,$$

which is equivalent to

$$\lim_{\tau\to\tau_0} \left\{\sum_{n=0}^{j} \frac{a_n(\tau-\tau_0)^{n\beta}}{(\tau-\tau_0)^{(j-k)\beta}}+\sum_{n=j+1}^{\infty} a_n(\tau-\tau_0)^{n\beta-(j-k)\beta}\right\}=0, \quad j,k=0,1,2,\ldots.$$

Of course, the second infinite series is equal to zero. Hence, the first finite series is also zero, i.e.

$$\lim_{\tau\to\tau_0} \sum_{n=0}^{j} \frac{a_n(\tau-\tau_0)^{n\beta}}{(\tau-\tau_0)^{(j-k)\beta}}=\lim_{\tau\to\tau_0} \frac{\omega_j(\tau)}{(\tau-\tau_0)^{(j-k)\beta}}=0, \quad j,k=0,1,2,\ldots\,. \tag{5}$$

Thus, the proof of the theorem is complete.

Unsurprisingly, the coefficients, a_n, $n<j-k$, in series (5) are equal to zero, ensuring the limit equals zero. Therefore, Equation (5) is equivalent to

$$\lim_{\tau\to\tau_0} \sum_{n=j-k}^{j} \frac{a_n(\tau-\tau_0)^{n\beta}}{(\tau-\tau_0)^{(j-k)\beta}}=0, \quad j,k=1,2,\ldots\,. \tag{6}$$

3 Procedures of the Limit-Residual Function Method for Solving FODEs

This section illustrates the LRF method strategy for creating solutions for FODEs. This method is based on the ideas of the residual function and the limit at zero. To demonstrate the procedures of LRF method, we study the following class of FODEs :

$$\mathscr{D}^{k\beta}\omega(\tau)=H\left(\tau^{\beta},\omega,\ \mathscr{D}^{\beta}\omega,\mathscr{D}^{2\beta}\omega,\ldots,\mathscr{D}^{(k-1)\beta}\omega\right), \quad \tau\geq\tau_0,\ 0<\beta\leq 1, \tag{7}$$

with the initial conditions

$$\omega(\tau_0)=b_0, \quad \mathscr{D}^{\beta}\omega(\tau_0)=b_1,\ \ldots,\ \mathscr{D}^{(k-1)\beta}\omega(\tau_0)=b_{k-1}, \tag{8}$$

where β is the order of the Caputo fractional differential operator $\mathscr{D}^{\beta}$, k is positive integers, and H is an analytic function.

LRF method assumes writing the function $\omega(\tau)$ on the following fractional PS:

$$\omega(\tau)=\sum_{n=0}^{\infty} a_n\,(\tau-\tau_0)^{n\beta}, \quad 0\leq\tau-\tau_0<\delta, \tag{9}$$

for some constant δ.

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Consequently, by truncating the series (9), we get the jth approximate solution of Problem (7)-(8) as follows:

$$\omega_j(\tau) = \sum_{n=0}^{j} a_n (\tau - \tau_0)^{n\beta}. \tag{10}$$

Using the initial conditions (8), the $(k-1)$th approximation of $\omega(\tau)$ is given as:

$$\omega_{k-1}(\tau) = \sum_{n=0}^{k-1} \frac{b_n}{\Gamma(n\beta+1)} (\tau - \tau_0)^{n\beta}. \tag{11}$$

Define a zeroth function known as the residual function for Equation (7) as the following:

$$\mathscr{R}f(\tau) = \mathscr{D}^{k\beta}\omega(\tau) - H\left(\tau^{\beta}, \omega, \mathscr{D}^{\beta}\omega, \mathscr{D}^{2\beta}\omega, \ldots, \mathscr{D}^{(k-1)\beta}\omega\right) = 0, \quad \tau \geq \tau_0. \tag{12}$$

So, the jth residual function can be given as:

$$\mathscr{R}f_j(\tau) = \mathscr{D}^{k\beta}\omega_j(\tau) - H\left(\tau^{\beta}, \omega_j, \mathscr{D}^{\beta}\omega_j, \mathscr{D}^{2\beta}\omega_j, \ldots, \mathscr{D}^{(k-1)\beta}\omega_j\right). \tag{13}$$

Since $\omega(\tau)$ has a fractional PS expansion and H is analytic, $\mathscr{R}f(\tau)$ and $\mathscr{R}f_j(\tau)$ also have fractional PS expansions in the following expressions:

$$\mathscr{R}f(\tau) = \sum_{n=0}^{\infty} A_n (\tau - \tau_0)^{n\beta}, \quad 0 \leq \tau - \tau_0 < \delta, \tag{14}$$

and

$$\mathscr{R}f_j(\tau) = \sum_{n=0}^{j} A_n (\tau - \tau_0)^{n\beta}, \tag{15}$$

where A_n is a combination of $a_0, a_1, a_2, \ldots, a_{n-1}$.

The essential tool of the LRF method, which effectively identifies the unknown coefficients in Equation (10) is built by adapting Theorem 2.4 to correspond with our structure so that it becomes as follows:

$$\lim_{\tau \to \tau_0} \frac{\mathscr{R}f_j(\tau)}{(\tau - \tau_0)^{(j-k)\beta}} = 0. \tag{16}$$

To find the kth approximate solution of Problem (7)-(8), we need to determine the value of the coefficient a_n in the series (10). We can obtain our desired result by substituting $\omega_k(\tau) = \omega_{k-1}(\tau) + a_k(\tau - \tau_0)^{k\beta}$ into $\mathscr{R}f_k(\tau)$ and solving the algebraic equation $\lim_{\tau \to \tau_0} \mathscr{R}f_k(\tau) = 0$ for a_k.

To obtain the $(k+1)$th approximate solution, we replace $\omega_{k+1}(\tau) = \omega_k(\tau) + a_{k+1}(\tau - \tau_0)^{(k+1)\beta}$ in $\mathscr{R}f_{k+1}(\tau)$ and solving the algebraic Equation (16), when $j = k+1$, for a_{k+1}.

Generally, the jth approximate solution of Problem (7)-(8) is determined by substituting $\omega_j(\tau) = \omega_{j-1}(\tau) + a_j(\tau - \tau_0)^{j\beta}$ into $\mathscr{R}f_j(\tau)$ and then solving the algebraic Equation (16) for a_j.

In the next section, we present the proposed method by applying it to a series of well-known physical equations.

4 Illustrative Examples of Physical Equations

In this section, we will showcase the simplicity and effectiveness of our technique by solving five interesting and significant applications of FODE. We will assess the accuracy of the method by comparing it with the exact solutions. To perform our computations, we will be using MATHEMATICA 11.

Example 1. Given the following fractional Riccati equation [26]:

$$\mathscr{D}^{\beta}\omega(\tau) + \omega^2(\tau) = 1, \quad 0 < \beta \leq 1, \quad \tau \geq 0, \tag{17}$$

subject to the initial condition

$$\omega(0) = 0. \tag{18}$$

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The exact solution in the classical case ($\beta = 1$) can be determined analytically and given as follows:

$$\omega(\tau) = \left(e^{2\tau} - 1\right)\left(e^{2\tau} + 1\right)^{-1}. \tag{19}$$

To derive an analytical series solution of the Problem (17)-(18) using LRF method, assume that the solution has the following fractional series expansion:

$$\omega(\tau) = \sum_{n=0}^{\infty} a_n\, \tau^{n\beta}. \tag{20}$$

According to the initial condition specified in Equation (18), we have $a_0 = 0$. Therefore, the jth approximate solution of $\omega(\tau)$ can be expressed as:

$$\omega_j(\tau) = \sum_{n=1}^{j} a_n\, \tau^{n\beta}. \tag{21}$$

To find the other unknown coefficients of the series mentioned in Equation (20), we proceed with the second step of the LRF method procedure by defining the residual function of Equation (17) as follows:

$$\mathscr{R}f(\tau) = \mathscr{D}^{\beta}\omega(\tau) + \omega^2(\tau) - 1, \tag{22}$$

and the jth residual function is expressed as follows:

$$\mathscr{R}f_j(\tau) = \mathscr{D}^{\beta}\omega_j(\tau) + \omega_j^2(\tau) - 1. \tag{23}$$

By substituting the first approximation $\omega_1(\tau) = a_1\tau^{\beta}$ into the first residual function, $\mathscr{R}f_1(\tau)$, we obtain

$$\mathscr{R}f_1(\tau) = a_1\,\Gamma(\beta+1) + a_1^2\,\tau^{2\beta} - 1. \tag{24}$$

Solving the equation $\lim_{\tau\to 0}\mathscr{R}f_1(\tau) = 0$ for a_1, we get

$$a_1 = \frac{1}{\Gamma(\beta+1)}. \tag{25}$$

To set the second coefficient a_2, substitute the second approximation $\omega_2(\tau) = \frac{\tau^{\beta}}{\Gamma(\beta+1)} + a_2\tau^{2\beta}$ into $\mathscr{R}f_2(\tau)$ to have

$$\mathscr{R}f_2(\tau) = a_2\,\frac{\Gamma(2\beta+1)\,\tau^{\beta}}{\Gamma(\beta+1)} + \left(\frac{\tau^{\beta}}{\Gamma(\beta+1)} + a_2\tau^{2\beta}\right)^2. \tag{26}$$

Following simple calculations, we find that $\lim_{\tau\to 0}\frac{\mathscr{R}f_2(\tau)}{\tau^{\beta}} = 0$ yields

$$a_2 = 0. \tag{27}$$

In the same manner, to determine the values of the coefficient $a_3(x)$, we substitute $\omega_3(\tau) = \frac{\tau^{\beta}}{\Gamma(\beta+1)} + a_3\tau^{3\beta}$ into $\mathscr{R}f_3(\tau)$, and then we solve the following equation:

$$\lim_{\tau\to 0}\frac{\mathscr{R}f_3(\tau)}{\tau^{2\beta}} = a_3\,\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} + \frac{1}{\Gamma^2(\beta+1)} = 0. \tag{28}$$

Then we have

$$a_3 = -\frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)\,\Gamma^2(\beta+1)}. \tag{29}$$

For $j = 4, 5, \ldots, 11$, the coefficients of the 11th approximation can be obtained iteratively by solving the algebraic equations $\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-1)\beta}} = 0$, $j = 4, 5, \ldots, 11$. The needed coefficients are summarized in Table 1.

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Table 1. The coefficients of the 11th approximate solution of $\omega(\tau)$ for the Problem (17)-(18).

j	a_j
0,2,4,6,8,10	0
1	$\frac{1}{\Gamma(\beta+1)}$
3	$-\frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)\Gamma^2(\beta+1)}$
5	$\frac{2\Gamma(1+2\beta)\Gamma(1+4\beta)}{\Gamma(1+\beta)^3\Gamma(1+3\beta)\Gamma(1+5\beta)}$
7	$-\frac{\Gamma(1+2\beta)\Gamma(1+6\beta)(4\Gamma(1+3\beta)\Gamma(1+4\beta)+\Gamma(1+2\beta)\Gamma(1+5\beta))}{\Gamma(1+\beta)^4\Gamma(1+3\beta)^2\Gamma(1+5\beta)\Gamma(1+7\beta)}$
9	$\frac{2\Gamma(1+2\beta)^2\Gamma(1+8\beta)(\Gamma(1+5\beta)\Gamma(1+6\beta)+2\Gamma(1+4\beta)\Gamma(1+7\beta))}{\Gamma(1+\beta)^5\Gamma(1+3\beta)^2\Gamma(1+5\beta)\Gamma(1+7\beta)\Gamma(1+9\beta)} + \frac{8\Gamma(1+2\beta)\Gamma(1+4\beta)\Gamma(1+6\beta)\Gamma(1+8\beta)}{\Gamma(1+\beta)^5\Gamma(1+3\beta)\Gamma(1+5\beta)\Gamma(1+7\beta)\Gamma(1+9\beta)}$
11	$-\left(\frac{8\Gamma(1+2\beta)^2\Gamma(1+4\beta)\Gamma(1+10\beta)(\Gamma(1+7\beta)\Gamma(1+8\beta)+\Gamma(1+6\beta)\Gamma(1+9\beta))}{\Gamma(1+\beta)^6\Gamma(1+3\beta)^2\Gamma(1+5\beta)\Gamma(1+7\beta)\Gamma(1+9\beta)\Gamma(1+11\beta)} + \frac{2\Gamma(1+2\beta)^2\Gamma(1+10\beta)(\Gamma(1+2\beta)\Gamma(1+5\beta)^2\Gamma(1+6\beta)+2\Gamma(1+3\beta)\Gamma(1+4\beta)^2\Gamma(1+7\beta))}{\Gamma(1+\beta)^6\Gamma(1+3\beta)^3\Gamma(1+5\beta)^2\Gamma(1+7\beta)\Gamma(1+11\beta)} + \frac{4\Gamma(1+2\beta)\Gamma(1+6\beta)\Gamma(1+8\beta)\Gamma(1+10\beta)(4\Gamma(1+3\beta)\Gamma(1+4\beta)+\Gamma(1+2\beta)\Gamma(1+5\beta))}{\Gamma(1+\beta)^6\Gamma(1+3\beta)^2\Gamma(1+5\beta)\Gamma(1+7\beta)\Gamma(1+9\beta)\Gamma(1+11\beta)}\right)$

So, the 11th approximate solution, $\omega_{11}(\tau) = \sum_{n=0}^{5} a_{2n+1}\ \tau^{(2n+1)\beta}$, of the Problem (17)-(18), at $\beta = 1$, is given by

$$\omega_{11}(\tau) = \tau - \frac{\tau^3}{3} + \frac{2\tau^5}{15} - \frac{17\tau^7}{315} + \frac{62\tau^9}{2835} - \frac{1382\tau^{11}}{155925}, \tag{30}$$

which is the first twelve terms of expanding the exact solution given in (19).

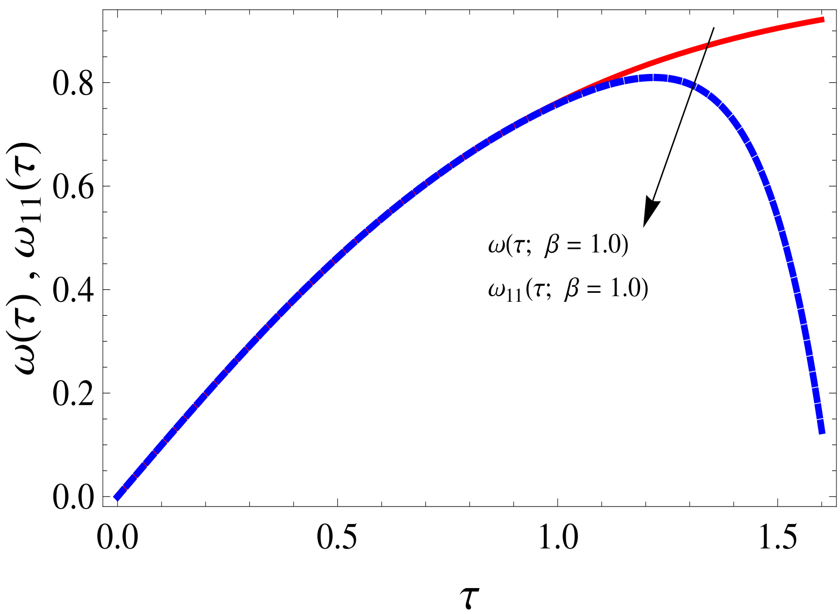

Figure 1. The curves of the exact and the 11th approximate solutions of Problem (17)-(18) at $\beta = 1$.

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Figure 1 shows the graph of the exact and the 11th approximate solutions of Problem (17)-(18) at $\beta = 1$. It illustrates the interval of convergence of the series solution (20) by comparing the 11th approximate solution with the exact solution. Figure 2 shows the 11th approximate solutions of Problem (17)-(18) at various values of β. We can observe in Figure 2 the impact of the derivative order on the behavior of the solution, where the solution curve rises as the order of the fractional derivative decreases, and the convergence interval of the solution shrinks as the order of the derivative increases.

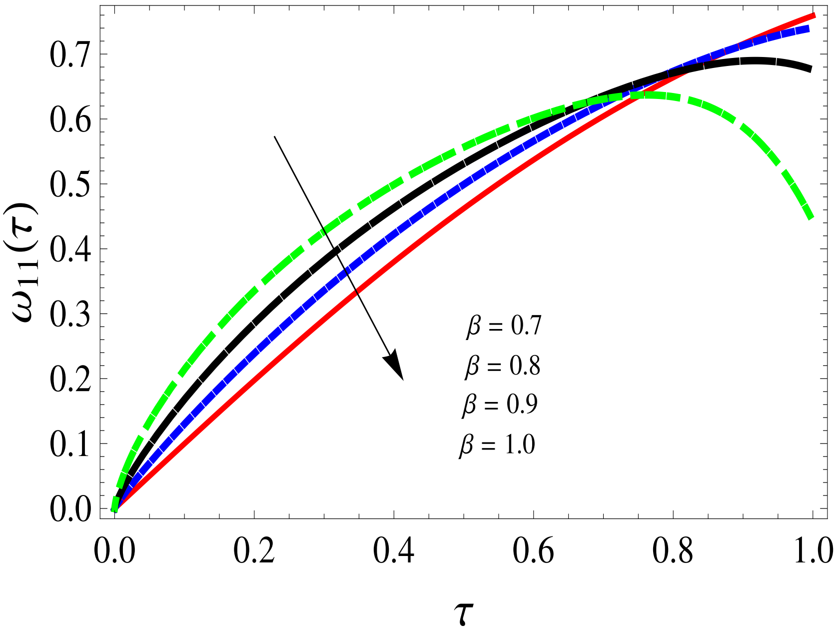

Figure 2. The curves of the 11th approximate solution of Problem (17)-(18) at different values of β.

Example 2. Given the following fractional composite oscillation equation [34]:

$$\mathscr{D}^{2\beta}\omega(\tau) - b\mathscr{D}^{\beta}\omega(\tau) - c\omega(\tau) = 8, \quad 0 < \beta \leq 1, \quad \tau \geq 0, \tag{31}$$

subject to the initial conditions

$$\omega(0) = 0, \ \ \mathscr{D}^{\beta}\omega(0) = 0. \tag{32}$$

The exact solution, when $\beta = 1$ and $b = c = -1$, can be given explicitly as follows:

$$\omega(\tau) = 8 - 8\, e^{-\frac{\tau}{2}} \left(\cos\frac{\sqrt{3}}{2}\,\tau + \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}\,\tau \right). \tag{33}$$

LRF method suggests the solution of the Problem (31)-(32) to be on the following fractional series expansion:

$$\omega(\tau) = \sum_{n=0}^{\infty} a_n\, \tau^{n\beta}. \tag{34}$$

Employing the initial conditions in Equation (32), the jth approximate solution will be as:

$$\omega_j(\tau) = \sum_{n=2}^{j} a_n\, \tau^{n\beta}. \tag{35}$$

To determine the jth approximate solution of the Problem (31)-(32), we define the residual and jth residual functions, respectively, as follows:

$$\mathscr{R}f(\tau) = \mathscr{D}^{2\beta}\omega(\tau) - b\mathscr{D}^{\beta}\omega(\tau) - c\omega(\tau) - 8, \quad \tau \geq 0, \tag{36}$$

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$$\mathscr{R}f_j(\tau) = \mathscr{D}^{2\beta}\omega_j(\tau) - b\mathscr{D}^{\beta}\omega_j(\tau) - c\omega_j(\tau) - 8, \quad \tau \geq 0. \tag{37}$$

After substituting the jth approximation (35) into (37), we obtain the jth residual function formula as follows:

$$\mathscr{R}f_j(\tau) = \sum_{n=2}^{j} a_n \frac{\Gamma(n\beta+1)}{\Gamma((n-2)\beta+1)}\,\tau^{(n-2)\beta} - b\sum_{n=2}^{j} a_n \frac{\Gamma(n\beta+1)}{\Gamma((n-1)\beta+1)}\,\tau^{(n-1)\beta} - c\sum_{n=2}^{j} a_n\,\tau^{n\beta} - 8. \tag{38}$$

By the formulation in the preceding section, The jth approximation can be achieved by obtaining the coefficients a_2, $a_3, \ldots, a_j$ via solving the following algebraic equations iteratively:

$$\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}} = 0 \ . \tag{39}$$

For $j = 2$, the second residual function will be $\mathscr{R}f_2(\tau) = a_2\,\Gamma(2\beta+1) - b\,a_2\,\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}\tau^{\beta} - ca_2\tau^{2\beta} - 8$. Solving the equation $\lim_{\tau\to 0}\mathscr{R}f_2(\tau) = a_2\ \Gamma(2\beta+1) - 8 = 0$, we get $a_2 = \frac{8}{\Gamma(2\beta+1)}$. Thus, the second approximation will be $\omega_2(\tau) = \frac{8}{\Gamma(2\beta+1)}\tau^{2\beta}$.

The coefficients of the 10th approximation, which can be obtained iteratively by solving the algebraic equations (39), are summarized in Table 2.

Table 2. The coefficients of the 10th approximate solution of $\omega(\tau)$ for the Problem (31)-(32).

j	$\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}} = 0$	a_j
2	$-8+\Gamma(1+2\beta)a_2 = 0$	$\frac{8}{\Gamma(1+2\beta)}$
3	$\frac{-8b+\Gamma(1+3\beta)a_3}{\Gamma(1+\beta)} = 0$	$\frac{8b}{\Gamma(1+3\beta)}$
4	$\frac{-8\left(b^2+c\right)+\Gamma(1+4\beta)a_4}{\Gamma(1+2\beta)]} = 0$	$\frac{8(b^2+c)}{\Gamma(1+4\beta)}$
5	$\frac{-8(b^3+2bc)+t\Gamma(1+5\beta)a_5}{\Gamma(1+3\beta)} = 0$	$\frac{8b(b^2+2c)}{\Gamma(1+5\beta)}$
6	$\frac{-8(b^4+3b^2c+c^2)+\Gamma(1+6\beta)a_6}{\Gamma(1+4\beta)} = 0$	$\frac{8(b^4+3b^2c+c^2)}{\Gamma(1+6\beta)}$
7	$\frac{-8(b^5+4b^3c+3bc^2)+\Gamma(1+7\beta)a_7}{\Gamma(1+5\beta)} = 0$	$\frac{8b(b^2+c)(b^2+3c)}{\Gamma(1+7\beta)}$
8	$\frac{-8(b^6+5b^4c+6b^2c^2+c^3)+\Gamma(1+8\beta)a_8}{\Gamma(1+6\beta)} = 0$	$\frac{8(b^6+5b^4c+6b^2c^2+c^3)}{\Gamma(1+8\beta)}$
9	$\frac{-8(b^7+6b^5c+10b^3c^2+4bc^3)+\Gamma(1+9\beta)a_9}{\Gamma(1+7\beta)} = 0$	$\frac{8b(b^2+2c)(b^4+4b^2c+2c^2)}{\Gamma(1+9\beta)}$
10	$\frac{-8(b^8+7b^6c+15b^4c^2+10b^2c^3+c^4)+\Gamma(1+10\beta)a_{10}}{\Gamma(1+8\beta)} = 0$	$\frac{8(b^2+c)(b^6+6b^4c+9b^2c^2+c^3)}{\Gamma(1+10\beta)}$

According to our assumption in Equation (34), the solution of the Problem (31)-(32) has the following expansion:

$$\begin{aligned}\omega(\tau) = {} & \frac{8}{\Gamma(1+2\beta)}\tau^{2\beta} + \frac{8b}{\Gamma(1+3\beta)}\tau^{3\beta} + \frac{8(b^2+c)}{\Gamma(1+4\beta)}\tau^{4\beta} + \frac{8b(b^2+2c)}{\Gamma(1+5\beta)}\tau^{5\beta} + \frac{8(b^4+3b^2c+c^2)}{\Gamma(1+6\beta)}\tau^{6\beta} \\ & + \frac{8b(b^2+c)(b^2+3c)}{\Gamma(1+7\beta)}\tau^{7\beta} + \frac{8(b^6+5b^4c+6b^2c^2+c^3)}{\Gamma(1+8\beta)}\tau^{8\beta} + \frac{8b(b^2+2c)(b^4+4b^2c+2c^2)}{\Gamma(1+9\beta)}\tau^{9\beta} \\ & + \frac{8(b^2+c)(b^6+6b^4c+9b^2c^2+c^3)}{\Gamma(1+10\beta)}\tau^{10\beta} + \ldots .\end{aligned} \tag{40}$$

For $\beta = 1$ and $b = c = -1$, the expansion (40) becomes as follows:

$$\omega(\tau) = 4\tau^2 - \frac{4\tau^3}{3} + \frac{\tau^5}{15} - \frac{\tau^6}{90} + \frac{\tau^8}{5040} - \frac{\tau^9}{45360} + \ldots, \tag{41}$$

which is the expansion of the solution indicated in (33).

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In Figure 3, we can see the graph of the 10th approximate and exact solution of Problem (31) - (32) at $\beta = 1$. It shows that the convergence interval of the series solution (34) is wider than the convergence interval of the solution of the series in Example 1. This can be attributed to the fact that the differential equation in Example 2 is linear while in Example 1 is nonlinear. In Figure 4, as in Figure 2, we can see that the convergence interval of the solution decreases as the order of the derivative increases. In addition, the order of the solution curves changes after some point in the interval, with the higher curve having a lower value of β, and after that point, the order reverses.

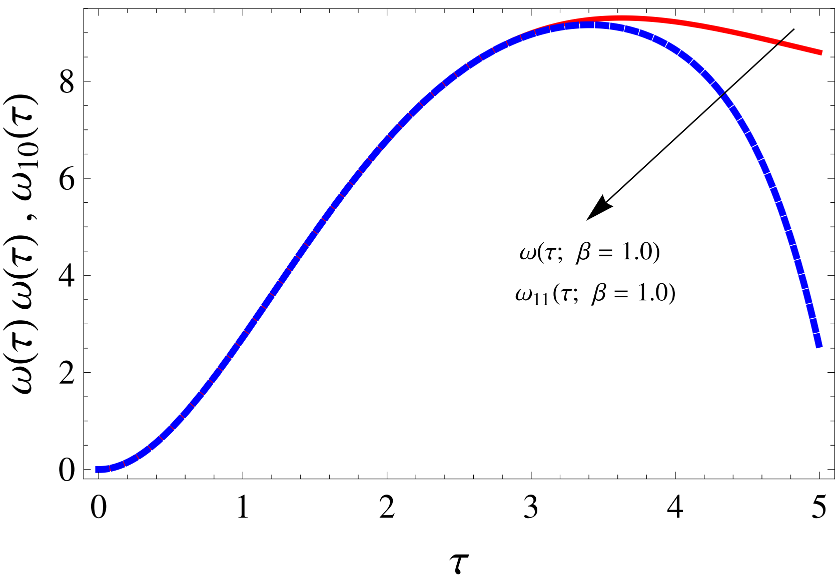

Figure 3. The curves of the exact and the 10th approximate solutions of Problem (31)-(32) at $\beta = 1$.

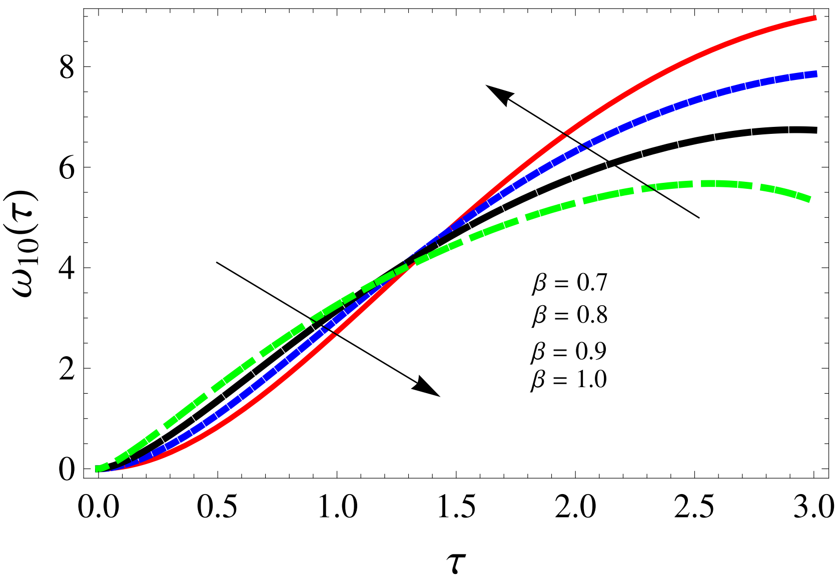

Figure 4. The curves of the 10th approximate solution of Problem (31)-(32) at different values of β.

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Example 3. Given the following fractional Airy differential equations [35]:

$$\mathscr{D}^{2\beta}\omega(\tau)-k^2\,\tau^\beta\,\omega(\tau)\;=\;0,\quad k\in\mathbb{R},\quad 0<\beta\le 1,\quad \tau\ge 0, \tag{42}$$

subject to the initial conditions

$$\omega(0)=a,\;\;\mathscr{D}^\beta\omega(0)=b. \tag{43}$$

Assume the solution of the Problem (42)-(43) has a fractional series expansion as:

$$\omega(\tau)=\sum_{n=0}^{\infty}a_n\,\tau^{n\beta}. \tag{44}$$

So, based on the initial conditions in Equation (43), the jth approximate solution is

$$\omega_j(\tau)=a+\frac{b}{\Gamma(1+\beta)}\tau^\beta+\sum_{n=2}^{j}a_n\,\tau^{n\beta}. \tag{45}$$

The residual and jth residual functions of Equation (42) are respectively given as follows:

$$\mathscr{R}f(\tau)=\mathscr{D}^{2\beta}\omega(\tau)-k^2\,\tau^\beta\,\omega(\tau)\;,\quad \tau\ge 0, \tag{46}$$

$$\mathscr{R}f_j(\tau)=\mathscr{D}^{2\beta}\omega_j(\tau)-k^2\,\tau^\beta\omega_j(\tau),\quad \tau\ge 0. \tag{47}$$

Substituting the jth approximation (45) into the jth residual function (47) gives the series form of the jth residual function as follows:

$$\mathscr{R}f_j(\tau)=\sum_{n=2}^{j}a_n\frac{\Gamma(n\beta+1)}{\Gamma((n-2)\beta+1)}\;\tau^{(n-2)\beta}-ak^2\,\tau^\beta-\frac{k^2\,b}{\Gamma(1+\beta)}\tau^{2\beta}-k^2\sum_{n=2}^{j}a_n\;\;\tau^{(n+1)\beta}. \tag{48}$$

Solving the next algebraic equation for $a_j,\;j=2,3,\ldots$ provides the value of the coefficients of the series solution (44):

$$\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}}=0\;,\quad j=2,3,\ldots. \tag{49}$$

The coefficients of the 10th approximation are summarized in Table 3.

Table 3. The coefficients of the 10th approximate solution of $\omega(\tau)$ for the Problem (42)-(43).

j	$\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}}=0$	a_j
2	$\Gamma(1+2\beta)a_2=0$	0
3	$-ak^2+\frac{\Gamma(1+3\beta)a_3}{\Gamma(1+\beta)}=0$	$\frac{ak^2\Gamma(1+\beta)}{\Gamma(1+3\beta)}$
4	$\frac{-bk^2}{\Gamma(1+\beta)}+\frac{\Gamma(1+\beta)a_4}{\Gamma(1+2\beta)}=0$	$\frac{bk^2\Gamma(1+2\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)}$
5	$\frac{\Gamma(1+5\beta)a_5}{\Gamma(1+3\beta)}=0$	0
6	$-\frac{ak^4\Gamma(1+\beta)}{\Gamma(1+3\beta)}+\frac{\Gamma(1+6\beta)a_6}{\Gamma(1+4\beta)}=0$	$\frac{ak^4\Gamma(1+\beta)\Gamma(1+4\beta)}{\Gamma(1+3\beta)\Gamma(1+6\beta)}$
7	$-\frac{ak^4\Gamma(1+2\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)}+\frac{\Gamma(1+7\beta)a_7}{\Gamma(1+5\beta)}=0$	$\frac{bk^4\Gamma(1+2\beta)\Gamma(1+5\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)}$
8	$\frac{\Gamma(1+8\beta)a_8}{\Gamma(1+6\beta)}=0$	0
9	$-\frac{ak^6\Gamma(1+\beta)\Gamma(1+4\beta)}{\Gamma(1+3\beta)\Gamma(1+6\beta)}+\frac{\Gamma(1+9\beta)a_9}{\Gamma(1+7\beta)}=0$	$\frac{ak^6\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)}{\Gamma(1+3\beta)\Gamma(1+6\beta)\Gamma(1+9\beta)}$
10	$-\frac{bk^6\Gamma(1+2\beta)\Gamma(1+5\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)}+\frac{\Gamma(1+10\beta)a_{10}}{\Gamma(1+8\beta)}=0$	$\frac{bk^6\Gamma(1+2\beta)\Gamma(1+5\beta)\Gamma(1+8\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)\Gamma(1+10\beta)}$

So, the fractional series expansion (44) becomes as:

$$\omega(\tau)=a\left(1+\frac{k^2\Gamma(1+\beta)}{\Gamma(1+3\beta)}\tau^{3\beta}+\frac{k^4\Gamma(1+\beta)\Gamma(1+4\beta)}{\Gamma(1+3\beta)\Gamma(1+6\beta)}\tau^{6\beta}+\frac{k^6\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)}{\Gamma(1+3\beta)\Gamma(1+6\beta)\Gamma(1+9\beta)}\tau^{9\beta}+\ldots\right)$$

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$$b\left(\frac{1}{\Gamma(1+\beta)}\tau^{\beta}+\frac{k^{2}\Gamma(1+2\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)}\tau^{4\beta}+\frac{bk^{4}\Gamma(1+2\beta)\Gamma(1+5\beta)}{\Gamma(1+\beta)\Gamma(1+4\beta)\Gamma(1+7\beta)}\tau^{7\beta}+\dots\right). \quad (50)$$

Following the pattern in Equation (50), the solution to Problem (42)-(43) can be expressed as follows:

$$\omega(\tau)=a\left(1+\sum_{n=1}^{\infty}k^{2n}\prod_{i=1}^{n}\frac{\Gamma(1+(3i-2)\beta)}{\Gamma(1+3i\beta)}\tau^{3n\beta}\right)+\frac{b}{\Gamma(1+\beta)}\left(\tau^{\beta}+\sum_{n=1}^{\infty}k^{2n}\prod_{i=1}^{n}\frac{\Gamma(1+(3i-1)\beta)}{\Gamma(1+(3i+1)\beta)}\tau^{(3n+1)\beta}\right), \quad (51)$$

For $\beta=1$, the expansion (51) becomes as follows:

$$\omega(\tau)=a\left(1+\sum_{n=1}^{\infty}k^{2n}\frac{(1)!(4)!\dots(3n-2)!}{(3)!(6)!\dots(3n)!}\tau^{3n}\right)+b\left(\tau^{\beta}+\sum_{n=1}^{\infty}k^{2n}\frac{(2)!(5)!\dots(3n-1)!}{(4)!(7)!\dots(3n+1)!}\tau^{(3n+1)\beta}\right). \quad (52)$$

By using Mathematica Software and choosing $a=\frac{\sqrt{3}+1}{3^{2/3}\Gamma(2/3)}$ and $b=\frac{(\sqrt{3}-1)\mathrm{k}^{2/3}}{3^{1/3}\Gamma(1/3)}$ makes Equation (52) in the following closed form:

$$\omega(\tau)=\mathrm{Ai}(k^{2/3}\tau)+\mathrm{Bi}(k^{2/3}\tau), \quad (53)$$

where Ai(•) and Bi(•) are Airy functions.

To analyze the 10th approximate solution, we discuss two types of errors, the actual and relative errors, which are defined as follows:

$$Act.\,err.(\tau)=\left|\omega(\tau)-\omega_{j}(\tau)\right|,$$

$$Rel.\,err.(\tau)=\left|\frac{Act.\,err.(\tau)}{\omega(\tau)}\right|.$$

In Table 4, comparisons are shown between the 10th approximate and the exact solutions of Problem (42)-(43) at $\beta=1$, $a=\frac{(\sqrt{3}+1)\mathrm{k}^{2/3}}{3^{2/3}\Gamma(2/3)}$, $b=\frac{(\sqrt{3}-1)\mathrm{k}^{2/3}}{3^{1/3}\Gamma(1/3)}$ and $k=1$. The table also presents the actual and relative errors. According to the results, the approximate solution is mathematically acceptable.

Table 4. The actual and relative errors of the 10th approximate solution of Problem (42) and (43) when $\beta=1$.

τ	$\omega(\tau)$	$\omega_{10}(\tau)$	$Act.\,err.(\tau)$	$Rel.\,err.(\tau)$
0.0	0.969955	0.969955	1.11022×10^{-16}	1.14461×10^{-16}
0.2	1.009167	1.009167	2.44249×10^{-15}	2.42030×10^{-15}
0.4	1.056515	1.056515	9.69447×10^{-12}	9.17589×10^{-12}
0.6	1.120863	1.120863	1.27048×10^{-9}	1.13348×10^{-9}
0.8	1.212268	1.212268	4.05333×10^{-8}	3.34359×10^{-8}
1.0	1.342716	1.342715	5.96583×10^{-7}	4.44310×10^{-7}

Notably, the closed form of the fraction PS (63) has not been found. Consequently, the residual error is computed in place of the actual and relative errors of the approximate solution when $\beta\neq1$. The residual error of the jth approximation is the absolute value of the jth residual function, which is denoted as:

$$Res.\,err.(\tau)=\left|\mathscr{R}f_{j}(\tau)\right|.$$

Table 5 displays the residual error at $\beta=0.7,\ 0.8,\ 0.9,$ and 1. The findings reveal that the error escalates with decreased values of β and increased values of τ.

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Table 5. The residual error of the 10th approximate solution of Problem (42) and (43) when $\beta = 0.7,\ 0.8,\ 0.9,\ 1$.

τ	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
0.0	0	0	0	0
0.2	6.07132×10^{-8}	3.27389×10^{-9}	1.64435×10^{-10}	7.74925×10^{-12}
0.4	7.91505×10^{-6}	8.51491×10^{-7}	8.53204×10^{-8}	8.02297×10^{-9}
0.6	1.37333×10^{-4}	2.21314×10^{-5}	3.3212×10^{-6}	4.67696×10^{-7}
0.8	1.04302×10^{-3}	2.23943×10^{-4}	4.47631×10^{-5}	8.39492×10^{-6}
1.0	5.03618×10^{-3}	1.35112×10^{-3}	3.37364×10^{-4}	7.90192×10^{-5}

Example 4. Given the following fractional Lane-Emden differential equation [20]:

$$\mathscr{D}^{2\beta}\omega(\tau) + \frac{2}{\tau^{\beta}}\mathscr{D}^{\beta}\omega(\tau) + \omega^{3}(\tau) = \tau^{9\beta} + 3\tau^{8\beta} + 3\tau^{7\beta} + \tau^{6\beta} + 12\tau^{\beta} + 6, \quad 0 < \beta \leq 1, \quad \tau \geq 0, \tag{54}$$

subject to the initial condition

$$\omega(0) = 0, \quad \mathscr{D}^{\beta}\omega(0) = 0. \tag{55}$$

As the previous examples and using the initial conditions (55), the jth approximate solution of the Problem (54)-(55) is:

$$\omega_j(\tau) = \sum_{n=2}^{j} a_n\, \tau^{n\beta}, \tag{56}$$

and the jth residual function is

$$\mathscr{R}f_j(\tau) = \mathscr{D}^{2\beta}\omega_j(\tau) + \frac{2}{\tau^{\beta}}\mathscr{D}^{\beta}\omega_j(\tau) + \omega_j^{3}(\tau) - \left(\tau^{9\beta} + 3\tau^{8\beta} + 3\tau^{7\beta} + \tau^{6\beta} + 12\tau^{\beta} + 6\right), \quad \tau \geq 0. \tag{57}$$

Solving the following algebraic equations iteratively for a_j supports the value of the coefficients of the approximate solution (56):

$$\lim_{\tau \to 0} \frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}} = 0 \ , \quad j = 2, 3, \ldots. \tag{58}$$

The coefficients of the 10th approximate solution of the given Lane-Emden equation are summarized in Table 6.

Table 6. The coefficients of the 10th approximate solution of $\omega(\tau)$ for the Problem (54)-(55).

j	a_j
2	$\frac{6\,\Gamma(\beta+1)}{\Gamma(2\beta+1)(\Gamma(\beta+1)+2)}$
3	$\frac{12\Gamma(2\beta+1)\Gamma(\beta+1)}{(2\Gamma(\beta+1)+\Gamma(2\beta+1))\Gamma(3\beta+1)}$
4,5,6,7	0
8	$\frac{\Gamma(6\beta+1)\Gamma(7\beta+1)}{(2\Gamma(6\beta+1)+\Gamma(7\beta+1))\Gamma(8\beta+1)}\left(1-\left(\frac{6\Gamma(\beta+1)}{(2+\Gamma(\beta+1))\Gamma(2\beta+1)}\right)^{3}\right)$
9	$\frac{3\Gamma(7\beta+1)\Gamma(8\beta+1)}{(2\Gamma(7\beta+1)+\Gamma(8\beta+1))\Gamma(9\beta+1)}\left(1-\frac{432\Gamma(\beta+1)^{3}}{(2+\Gamma(\beta+1))^{2}\left(2\Gamma(\beta+1)\Gamma(2\beta+1)+\Gamma(2\beta+1)^{2}\right)\Gamma(3\beta+1)}\right)$
10	$\frac{3\Gamma(8\beta+1)\Gamma(9\beta+1)}{(2\Gamma(8\beta+1)+\Gamma(9\beta+1))\Gamma(10\beta+1)}\left(1-\frac{864\Gamma(\beta+1)^{3}\Gamma(2\beta+1)}{(2+\Gamma(\beta+1))(2\Gamma(\beta+1)+\Gamma(2\beta+1))^{2}\Gamma(3\beta+1)^{2}}\right)$

Therefore, the 10th approximate solution of the Lane-Emden Problem (54)-(55) is

$$\omega(\tau) = \frac{6\,\Gamma(\beta+1)}{(\Gamma(\beta+1)+2)}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \frac{12\Gamma(2\beta+1)\Gamma(\beta+1)}{(2\Gamma(\beta+1)+\Gamma(2\beta+1))\Gamma(3\beta+1)}\frac{\tau^{3\beta}}{\Gamma(3\beta+1)}$$
$$+\frac{\Gamma(6\beta+1)\Gamma(7\beta+1)}{(2\Gamma(6\beta+1)+\Gamma(7\beta+1))}\left(1-\left(\frac{6\Gamma(\beta+1)}{(2+\Gamma(\beta+1))\Gamma(2\beta+1)}\right)^{3}\right)\frac{\tau^{8\beta}}{\Gamma(8\beta+1)}$$

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$$+\frac{3\Gamma(7\beta+1)\Gamma(8\beta+1)}{(2\Gamma(7\beta+1)+\Gamma(8\beta+1))}\left(1-\frac{432\Gamma(\beta+1)^3}{(2+\Gamma(\beta+1))^2\left(2\Gamma(\beta+1)\Gamma(2\beta+1)+\Gamma(2\beta+1)^2\right)\Gamma(3\beta+1)}\right)\frac{\tau^{9\beta}}{\Gamma(9\beta+1)}$$

$$+\frac{3\Gamma(8\beta+1)\Gamma(9\beta+1)}{(2\Gamma(8\beta+1)+\Gamma(9\beta+1))}\left(1-\frac{864\Gamma(\beta+1)^3\Gamma(2\beta+1)}{(2+\Gamma(\beta+1))(2\Gamma(\beta+1)+\Gamma(2\beta+1))^2\Gamma(3\beta+1)^2}\right)\frac{\tau^{10\beta}}{\Gamma(10\beta+1)}. \quad (59)$$

For $\beta=1$, the approximate solution (59) becomes as follows:

$$\omega(\tau)=\tau^2+\tau^3, \quad (60)$$

which is the exact solution of the Problem (54)-(55) at $\beta=1$.

Example 5. Given the following fractional Bessel's equation [29].

$$\left(1-\tau^{2\beta}\right)\mathscr{D}^{2\beta}\omega(\tau)-2\tau^{\beta}\,\mathscr{D}^{\beta}\omega(\tau)+2\omega(\tau)=0,\ \ 0<\beta\leq 1,\ \ \tau\geq 0, \quad (61)$$

subject to the initial conditions

$$\omega(0)=1,\ \ \mathscr{D}^{\beta}\omega(0)=-1. \quad (62)$$

Applying the LRM approach to Problem (61)-(62), we find that the jth approximate solution of $\omega(\tau)$ is

$$\omega_j(\tau)=1-\frac{\tau^{\beta}}{\Gamma(\beta+1)}+\sum_{n=2}^{j}a_n\,\tau^{n\beta}, \quad (63)$$

the jth residual function is

$$\mathscr{R}f_j(\tau)=\left(1-\tau^{2\beta}\right)\mathscr{D}^{2\beta}\omega_j(\tau)-2\tau^{\beta}\,\mathscr{D}^{\beta}\omega_j(\tau)+2\omega_j(\tau),\quad \tau\geq 0, \quad (64)$$

and the recurrence relation that generates the coefficients of the series (63) is

$$\lim_{\tau\to 0}\frac{\mathscr{R}f_j(\tau)}{\tau^{(j-2)\beta}}=0\ ,\ j=2,3,\ldots. \quad (65)$$

Table 7 displays the first few coefficients of the series (63).

Table 7. The first few coefficients of the series solution of $\omega(\tau)$ for the Problem (61)-(62).

j	a_j
2	$\frac{-2}{\Gamma(2\beta+1)}$
3	$\frac{2(1-\Gamma(\beta+1))}{\Gamma(3\beta+1)}$
4	$\frac{4\Gamma(\beta+1)-2(2+\Gamma(\beta+1))\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(4\beta+1)}$
5	$\frac{2(-1+\Gamma(\beta+1))(2\Gamma(\beta+1)\Gamma(2\beta+1)-(2\Gamma(\beta+1)+\Gamma(2\beta+1))\Gamma(3\beta+1))}{\Gamma(\beta+1)\Gamma(2\beta+1)\Gamma(5\beta+1)}$
6	$\frac{2(-2\Gamma(\beta+1)+(2+\Gamma(\beta+1))\Gamma(2\beta+1))(2\Gamma(2\beta+1)\Gamma(3\beta+1)-(2\Gamma(2\beta+1)+\Gamma(3\beta+1))\Gamma(4\beta+1))}{\Gamma(\beta+1)\Gamma(2\beta+1)\Gamma(3\beta+1)\Gamma(6\beta+1)}$

So, the first few terms of the series solution of the fractional Bessel's Equation (61) subject to the conditions (62) is

$$\omega(\tau)=1-\frac{\tau^{\beta}}{\Gamma(\beta+1)}-2\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}-2(\Gamma(\beta+1)-1)\frac{\tau^{3\beta}}{\Gamma(3\beta+1)}-\frac{2(2+\Gamma(\beta+1))\Gamma(2\beta+1)-4\Gamma(\beta+1)}{\Gamma(\beta+1)\Gamma(4\beta+1)}\tau^{4\beta}$$

$$+\frac{2(-1+\Gamma(\beta+1))(2\Gamma(\beta+1)\Gamma(2\beta+1)-(2\Gamma(\beta+1)+\Gamma(2\beta+1))\Gamma(3\beta+1))}{\Gamma(\beta+1)\Gamma(2\beta+1)}\frac{\tau^{5\beta}}{\Gamma(5\beta+1)}$$

$$+\left(-8+4\Gamma(2\beta+1)+\frac{8\Gamma(2\beta+1)}{\Gamma(\beta+1)}-2\Gamma(4\beta+1)-\frac{4\Gamma(4\beta+1)}{\Gamma(\beta+1)}+\frac{4\Gamma(4\beta+1)}{\Gamma(2\beta+1)}+\frac{8\Gamma(4\beta+1)}{\Gamma(3\beta+1)}\right.$$

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$$-\frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{\Gamma(3\beta+1)}-\frac{8\Gamma(2\beta+1)\Gamma(4\beta+1)}{\Gamma(\beta+1)\Gamma(3\beta+1)}\Bigg)\frac{\tau^{6\beta}}{\Gamma(6\beta+1)}+\ldots. \quad (66)$$

At $\beta=1$, the series solution (66) has the following form:

$$\omega(\tau)=1-\tau-\tau\left(\tau+\frac{\tau^3}{3}+\frac{\tau^5}{5}+\ldots\right), \quad (67)$$

which is the expansion of the exact solution $\omega(\tau)=1-\tau-\tau\arctan(\tau)$.

5 Conclusion

In this article, we introduced a novel analytical iterative technique, the LRF method, for dealing with linear and nonlinear differential equations of fractional order in the Caputo sense. The current approach makes it easy to discover the exact solution pattern with less complexity and computational costs. The main advantage of this method is the simplicity in computing the coefficients of the series solution by evaluating the limit for a form that includes the residual function for the given equation and not as the other well-known analytic techniques that need differential and integral operators which is difficult in the fractional case. In future work, we aim to adapt our new method to deal with linear and nonlinear integral equations.

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