Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110303

Analyzing Solitary Wave Solutions in M-Truncated Fractional-Order Space-Time Burgers-Like Equations

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Received: 22 Nov. 2024, Revised: 2 Jan. 2025, Accepted: 21 Jan. 2025
Published online: 1 Jul. 2025

Abstract: The Burgers equation, a cornerstone of hydrodynamic modeling, can be formulated in both integer and fractional orders to describe the dynamics of a viscous particle velocity field influenced solely by collisions. In this paper, we introduce a novel class of Burgers-like equations featuring M-truncated fractional space-time order. We employ the direct method to derive exact solutions and provide a graphical analysis to illustrate their applicability. Notably, certain equations within this framework exhibit identical solutions when considering M-truncated fractional orders in both space and time. The derived solutions are contingent on the fractional order parameters α and β, revealing that they align with those of their integer-order counterparts. Our findings highlight how the introduction of fractional orders enriches the dynamics of the system.

Keywords: Fractional calculus, M-truncated fractional order, Burgers equation

1 Introduction

A variety of physical problems in various fields of science can be described by nonlinear integer and fractional order differential dynamical models. In applied science, these linear and nonlinear differential models play a crucial role. Studies on these models are rapidly growing to understand the real characteristics of these issues in physical applications. Abundant approaches were constructed to classify these solutions with separate physical structures [1,2,3,4,5,6,7,8,9,10].

Many methods are employed to get exact solutions, examples include the bilinear method, the F-expansion method, the tanh function method, the Jacobi elliptic function method, the Backlund transformation, and the sin-cos method. Moreover, recent attention has been drawn to the derivation of the new differential equations[11,12,13,14,15] . The newly derived equations may describe important characteristics related to the well-known models.

At the moment, fractional differential equations (FDEs) are a hot topic of study. FDEs apply in many different fields: physics, chemistry and biology, mathematics, communication, non-local energy legislation, and long-term memories power law[16,17,18,19,20]. A lot of methods are used in the literature to discuss FDEs. Examples include the differential transformation, the decomposition of Adomian, the variational iteration, the perturbation of homotopy, the finite difference, the finite element, the fractional subequation, (G/G) expansion, and the first integral.

The Burgers are the simplest FDE equations, which combine non-linear distribution with the effect of dissipative, and it may be considered as the model that begins to understand turbulence. This equation and its hierarchy have a wide number of applications, such as fluids, plasma physics, traffic, hydrodynamics, and magnetic hydrodynamics. Around airfoils, waves under the influence of diffusion, thermoelastic wave propagation, viscous steady-state, and liquid dynamics. In addition, they behave like information carrier bits through transcontinental and transoceanic distances within the context of the information sciences[21,22,23,24,25].

The space-time M-truncated fractional Burgers equation reads

$$D^{\alpha,\beta}_{M,t}\psi+\psi D^{\alpha,\beta}_{M,x}\psi+D^{\alpha\alpha,\beta}_{M,x}\psi=0, \tag{1}$$

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where $D_{M,x}^{\alpha\alpha,\beta}\psi = D_{M,x}^{\alpha,\beta}D_{M,x}^{\alpha,\beta}\psi$ is the twice M-truncated fractional derivative. Equation (1) is the generalization of the Burgers equation

$$\psi_t + \psi\psi_x + \psi_{xx} = 0,$$

Equation (1) for one-dimensional weak shock wave propagation into a fluid is the lowest order approximation. The Burgers formula models the combination of the fractional dissipation effect with the fractional convection effect. It admits solutions with multiple kinks. The space-time fractional Burgers equation's kink solution is given by

$$\psi_1 = 2k[1+\tanh(\xi)], \quad \xi = \frac{\Gamma(\beta+1)}{\alpha}(kx^{\alpha} - 2k^2t^{\alpha}), \tag{2}$$

The singular kink solution

$$\psi_1 = 2k[1+\coth(\xi)], \quad \xi = \frac{\Gamma(\beta+1)}{\alpha}(kx^{\alpha} - 2k^2t^{\alpha}), \tag{3}$$

These solutions satisfies the space-time M-truncated fractional formula of Burgers (1). Remember that there is a different solution surface for each fractional order value.

The goal of this leaf is to construct a new family of Burgers-type space-time M-truncated fractional equations in which some of these equations have the same solutions as the space-time M-truncated fractional equation of Burgers.

This paper is constructed as follows: the basic concepts of M-truncated fractional order theory are given in section 2. The formulation of family Burgers-like equations with M-truncated fractional space and time order is demonstrated in section 3. Periodic and kink solutions are obtained for the equations under study in sections 4, 5, and 6. Eventually, some conclusions and debates have been made in section 7.

2 M-truncated Fractional derivative

In twenty-first-century mathematics, fractional and non-integer calculus have emerged as significant extensions of traditional concepts, paving the way for new research and practical applications. Fractional calculus, in particular, has established to be a versatile tool with a broad range of applications across various fields. This area of study is especially valuable for modeling complex real-world phenomena[26,27,28,29,30,31,32,33,34,35,36,37,38,39,40]. For example, fractional differential models are used to understand anomalous diffusion in porous materials, characterize viscoelastic behavior in material science, and describe complex biological processes such as cell growth and tumor dynamics. In finance, fractional calculus helps analyze long-term dependencies and volatility patterns in time series data.

The truncated Mittag-Leffler function (MLF) can be defined as40:

$${}_{I}E_{\beta}(\varepsilon s^{\alpha}) = \sum_{j=0}^{I}\frac{(\varepsilon s^{\alpha})^{j}}{\Gamma(\beta j+1)}, \quad \beta > 0, \quad s\,\varepsilon \in C, \tag{4}$$

Definition 1: Let $\psi : [0,\infty) \to \Re$ be a function, the local truncated M-fractional differential (MFD) of ψ with respect to y is given40:

$${}_{I}D_{M,t}^{\alpha,\beta}\psi(s) = \lim_{\delta\to 0}\frac{\psi(s\,{}_{I}E_{\beta}(\delta s^{-\alpha})) - \psi(s)}{\delta}, \quad \forall\,\beta, s > 0, \ 0 < \alpha < 1, \tag{5}$$

The MFD adheres to the following axioms:

$${}_{I}D_{M}^{\alpha,\beta}t^{m} = \frac{m}{\Gamma(\beta+1)}t^{m-\alpha}, \ \ m\,\varepsilon\,\Re, \ \ {}_{I}D_{M}^{\alpha,\beta}c = 0, \ \ \forall\,\psi(t) = c, \tag{6}$$

$${}_{I}D_{M}^{\alpha,\beta}(A\psi + B\varphi) = A_{I}D_{M}^{\alpha,\beta}\psi + B_{I}D_{M}^{\alpha,\beta}\varphi, \ \ \forall\,A, B\,\varepsilon\,\Re, \tag{7}$$

$${}_{I}D_{M}^{\alpha,\beta}(\varphi\psi) = \varphi_{I}D_{M}^{\alpha,\beta}\psi + \psi_{I}D_{M}^{\alpha,\beta}\varphi, \tag{8}$$

$${}_{I}D_{M}^{\alpha,\beta}\left(\frac{\varphi}{\psi}\right) = \frac{\psi_{I}D_{M}^{\alpha,\beta}\varphi - \varphi_{I}D_{M}^{\alpha,\beta}\psi}{\psi^{2}}, \tag{9}$$

$${}_{I}D_{M}^{\alpha,\beta}\varphi(\psi) = \frac{d\varphi}{d\psi}{}_{I}D_{M}^{\alpha,\beta}\psi, \ \ {}_{I}D_{M}^{\alpha,\beta}\varphi(t) = \frac{d\varphi}{dt}\frac{t^{1-\alpha}}{\Gamma(\beta+1)}, \tag{10}$$

With φ, ψ represents two α-differentiable functions of a dependent variable, the above relations are proved in reference[40].

Choosing $\beta = 1$ and $i = 1$ on the two sides of Eq.(4), we have

$$ {}_1D^{\alpha,1}_{M,t}\psi(s) = \lim_{\delta\to 0} \frac{\psi(s\,{}_1E_1(\delta s^{-\alpha})) - \psi(s)}{\delta}, \quad \forall\ s > 0,\ 0 < \alpha < 1, $$

But, it is know that

$$ {}_1E_1(\delta s^{-\alpha}) = \sum_{r=0}^{1} \frac{(\delta s^{\alpha})^r}{\Gamma(2)} = 1 + \delta\, s^{-\alpha}, $$

Thus, we conclude that

$$ {}_1D^{\alpha,1}_{M,t}\psi(s) = \lim_{\delta\to 0} \frac{\psi(s + \delta s^{1-\alpha}) - \psi(s)}{\delta} = D^{\alpha}_t\psi(s), \quad \forall\ s > 0,\ 0 < \alpha < 1, $$

which is exactly the conformable fractional derivative. Simply we write ${}_1D^{\alpha,\beta}_{M}$ as $D^{\alpha,\beta}_{M}$. The MFD of some functions [40]

$$ D^{\alpha,\beta}_{M,s} e^{cs} = \frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)} e^{cs},\ D^{\alpha,\beta}_{M,s}\sin(cs) = \frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)}\cos(cs),\ D^{\alpha,\beta}_{M,s}\cos(cs) = -\frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)}\sin(cs), $$

$$ D^{\alpha,\beta}_{M,s} e^{cs^{\alpha}} = \frac{c\,\alpha}{\Gamma(\beta+1)} e^{cs^{\alpha}},\ D^{\alpha,\beta}_{M,s}\sin(cs^{\alpha}) = \frac{c\,\alpha}{\Gamma(\beta+1)}\cos(cs^{\alpha}),\ D^{\alpha,\beta}_{M,s}\cos(cs^{\alpha}) = -\frac{c\,\alpha}{\Gamma(\beta+1)}\sin(cs^{\alpha}). $$

The MFD can be used for non-differentiable functions, making it suitable for applications involving discontinuous media

3 Formulation of Family of Burgers-Like Equation with M-truncated fractional Space and time order

The generalized version of the space-time M-truncated fractional advection-dissipation equation is given by

$$ D^{\alpha,\beta}_{M,t}\psi + V D^{\alpha,\beta}_{M,s}\psi = \delta D^{\alpha\alpha,\beta}_{M,s}\psi, \tag{11} $$

where δ is an arbitrary constant and $V = V(\psi, D^{\alpha,\beta}_{M,x}\psi, D^{\alpha\alpha,\beta}_{M,x}\psi, \ldots)$ is an arbitrary function. We assume the traveling wave has the form

$$ \psi = h(\xi), \qquad \xi = \Gamma(\beta+1)(x^{\alpha} - ct^{\alpha})/\alpha, \tag{12} $$

it solves the M-truncated fractional order of space and time equation of Burgers equation and also solves the space-time M-truncated fractional advection-dissipation Equation at the same frequency c. Use the corresponding functional fractional derivative as

$$ D^{\alpha,\beta}_{M,t}\psi = D^{\alpha,\beta}_{M,x}\xi\,\frac{\partial\psi}{\partial\xi} = c\,h',\ D^{\alpha,\beta}_{M,x}\psi = D^{\alpha,\beta}_{M,x}\xi\,\frac{\partial\psi}{\partial\xi} = h',\ D^{\alpha\alpha,\beta}_{M,x}\psi = h", $$

Equations (1) and (11) can take the forms

$$ -c\,h' + h\,h' + h" = 0, \tag{13} $$

and

$$ -c\,h' + V\,h' - \delta h" = 0, \tag{14} $$

By eliminating $h"$ from the above two equations, and the function h satisfies $h' \neq 0$, we obtain

$$ V = (\delta+1)c - \delta h, \tag{15} $$

It is possible to obtain the space-time M-truncated fractional advection – dissipation equation or the Burgers-like space-time M-truncated fractional equation simply by using a range of speed c values. From Equation (13) we consider

$$ c = h + \frac{h"}{h'}, \tag{16} $$

Integrating Equation (13) one time and solving for c we find

$$ c = \frac{1}{2}h + \frac{h'}{h}, \tag{17} $$

We can differentiate Equation (13) as many times as we want to establish further values for speed c. For example, we can differentiate Equation (13) once, twice, and three times and solve c, we find

$$c_1 = h + \frac{h'^2 + h'''}{h''}, \tag{18}$$

$$c_2 = h + \frac{3h'h'' + h^{(4)}}{h'''}, \tag{19}$$

$$c_3 = h + \frac{4h'h''' + 3h''^2 + h^{(5)}}{h^{(4)}}, \tag{20}$$

respectively. We can determine many values for c_i by distinguishing as many times as we want. To replace Equations (16)–(20) with equation (15)

$$V_1 = (\delta+1)(h + \frac{h''}{h'}) - \delta h,$$

$$V_2 = (\delta+1)(\frac{1}{2}h + \frac{h'}{h}) - \delta h,$$

$$V_3 = (\delta+1)(h + \frac{h'^2 + h'''}{h''}) - \delta h,$$

$$V_4 = (\delta+1)(h + \frac{3h'h'' + h^{(4)}}{h'''}) - \delta h,$$

$$V_5 = (\delta+1)(h + \frac{4h'h''' + 3h''^2 + h^{(5)}}{h^{(4)}}) - \delta h,$$

Since $D^{\alpha,\beta}_{M,x}\psi = h'$, $D^{\alpha\alpha,\beta}_{M,x}\psi = h''$ and $D^{\alpha\alpha\alpha,\beta}_{M,x}\psi = h'''$, so we can replace h' by $D^{\alpha,\beta}_{M,x}\psi$ and h'' by $D^{\alpha\alpha,\beta}_{M,x}\psi$ and so on. We have the following fractional forms

$$V_1 = (\delta+1)(\psi + \frac{D^{\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha,\beta}_{M,x}\psi}) - \delta D^{\alpha,\beta}_{M,x}\psi,$$

$$V_2 = (\delta+1)(\frac{1}{2}\psi + \frac{D^{\alpha,\beta}_{M,x}\psi}{\psi}) - \delta\psi,$$

$$V_3 = (\delta+1)(\psi + \frac{(D^{\alpha,\beta}_{M,x}\psi)^2 + D^{\alpha\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha\alpha,\beta}_{M,x}\psi}) - \delta\psi,$$

$$V_4 = (\delta+1)(\psi + \frac{3D^{\alpha,\beta}_{M,x}\psi D^{\alpha\alpha,\beta}_{M,x}\psi + D^{\alpha\alpha\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha\alpha\alpha,\beta}_{M,x}\psi}) - \delta\psi,$$

and

$$V_5 = (\delta+1)(\psi + \frac{4D^{\alpha,\beta}_{M,x}\psi D^{\alpha\alpha\alpha,\beta}_{M,x}\psi + 3(D^{\alpha\alpha,\beta}_{M,x}\psi)^2 + D^{\alpha\alpha\alpha\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha\alpha\alpha\alpha,\beta}_{M,x}\psi}) - \delta\psi,$$

Since V is a fractional potential function in equation (ref11), this gives the recurrence relation

$$D^{\alpha,\beta}_{M,t}\psi + V_i D^{\alpha,\beta}_{M,s}\psi = \delta D^{\alpha\alpha,\beta}_{M,s}\psi, \quad i =, 2, \ldots, 5. \tag{21}$$

We obtain the following family of space-time M-truncated fractional Burgers-like equations by replacing the last results of fractional potential function $V_i, 1 \le i \le 5$ in Equation (21) as

$$D^{\alpha,\beta}_{M,t}\psi + (\delta+1)(\psi + \frac{D^{\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha,\beta}_{M,x}\psi})D^{\alpha,\beta}_{M,s}\psi - \delta\psi D^{\alpha,\beta}_{M,s}\psi - \delta D^{\alpha\alpha,\beta}_{M,s}\psi = 0, \tag{22}$$

$$D^{\alpha,\beta}_{M,t}\psi + (\delta+1)(\frac{1}{2}\psi + \frac{D^{\alpha,\beta}_{M,x}\psi}{\psi})D^{\alpha,\beta}_{M,s}\psi - \delta\psi D^{\alpha,\beta}_{M,s}\psi - \delta D^{\alpha\alpha,\beta}_{M,s}\psi = 0, \tag{23}$$

$$D_{M,t}^{\alpha,\beta}\psi+(\delta+1)(\psi+\frac{(D_{M,x}^{\alpha,\beta}\psi)^2+D_{M,x}^{\alpha\alpha\alpha,\beta}\psi}{D_{M,x}^{\alpha\alpha,\beta}\psi}D_{M,s}^{\alpha,\beta}\psi-\delta\psi D_{M,s}^{\alpha,\beta}\psi-\delta D_{M,s}^{\alpha\alpha,\beta}\psi=0, \tag{24}$$

$$D_{M,t}^{\alpha,\beta}\psi+(\delta+1)(\psi+\frac{3D_{M,x}^{\alpha,\beta}\psi D_{M,x}^{\alpha\alpha,\beta}\psi+D_{M,x}^{\alpha\alpha\alpha\alpha,\beta}\psi}{D_{M,x}^{\alpha\alpha\alpha,\beta}\psi})D_{M,s}^{\alpha,\beta}\psi-\delta\psi D_{M,s}^{\alpha,\beta}\psi-\delta D_{M,s}^{\alpha\alpha,\beta}\psi=0, \tag{25}$$

$$D_{M,t}^{\alpha,\beta}\psi+(\delta+1)(\psi+\frac{4D_{M,x}^{\alpha,\beta}\psi D_{M,x}^{\alpha\alpha\alpha,\beta}\psi+3(D_{M,x}^{\alpha\alpha,\beta}\psi)^2+D_{M,x}^{\alpha\alpha\alpha\alpha\alpha,\beta}\psi}{D_{M,x}^{\alpha\alpha\alpha\alpha,\beta}\psi})D_{M,s}^{\alpha,\beta}\psi-\delta\psi D_{M,s}^{\alpha,\beta}\psi-\delta D_{M,s}^{\alpha\alpha,\beta}\psi=0, \tag{26}$$

It should be remembered that Equation (22) is the space-time fractional formula of the Burgers, Equation (1). The space-time fractional Burgers-like equations (24)–(26) require higher-order fractional derivatives than the fractional dissipative term of the space-time fractional Burgers equation. Furthermore, the Burgers-like fractional space-time Equations (23)–(26) have the same kink and special solutions, Equations (2) and (3). This can be verified as an alternative. Our main focus in this research will be to find moving wave solutions for the Burgers-like Equations (22)–(26). We will also demonstrate that in other cases, these types come up with solutions for the Burgers space-time fractional equation and other different solutions.

4 Kink Solutions

We use the direct method to find exact solution of the space-time M- truncated fractional Burgers equation (1), We mean by direct method that, there is no systematic method or by assuming some arbitrary constants in the solution and find its value by direct substituting. The first step is to assume the kink solution in the form

$$\psi=\frac{A}{1+e^{\Gamma(\beta+1)(kx^\alpha-ct^\alpha)/\alpha}}, \tag{27}$$

and by inserting it into equation (1), we obtain the solution of the resulting equation as

$$A=2k,\quad c=-k^2, \tag{28}$$

In consequence, the kink solution

$$\psi=\frac{2k}{1+e^{\Gamma(\beta+1)(kx^\alpha+k^2t^\alpha)/\alpha}}, \tag{29}$$

satisfies all four forms of space-time Burger-like M- truncated fractional equations in addition to the space-time M-truncated fractional Burgers equation.

Besides the space-time M- truncated fractional Burgers equation, it can be shown that the kink solution

$$\psi=\frac{2k}{1+e^{-\Gamma(\beta+1)(kx^\alpha+k^2t^\alpha)/\alpha}}, \tag{30}$$

meets all four fractional Burgers equations.

We can also follow these equations in conjunction with the unique solutions

$$\psi=\frac{2k}{1-e^{\Gamma(\beta+1)(kx^\alpha+k^2t^\alpha)/\alpha}}, \tag{31}$$

and

$$\psi=\frac{2k}{1-e^{-\Gamma(\beta+1)(kx^\alpha+k^2t^\alpha)/\alpha}}, \tag{32}$$

The solutions obtained can not be based upon the parameter δ and comply with the Burgers-like equations of any δ.

To derive more kink solutions. The solution for Equation (1) will be may be assumed to have the form

$$\psi=1+A\tanh(\xi),\quad \xi=\Gamma(\beta+1)(kx^\alpha-ct^\alpha)/\alpha, \tag{33}$$

We consider substituting this assumption with Equation (1) and solving the resulting formula for A and c, we get

$$A=2k,\ ,\quad c=k, \tag{34}$$

This gives the kink solution as

$$\psi = 1 + 2k\tanh(\xi), \quad \xi = k\Gamma(\beta+1)(x^{\alpha} - t^{\alpha})/\alpha, \tag{35}$$

In the same way, we can also derive the singular equation

$$\psi = 1 + 2k\coth(\xi), \quad \xi = k\Gamma(\beta+1)(x^{\alpha} - t^{\alpha})/\alpha, \tag{36}$$

It should be noted that the solutions, Equations (35) and (36) follow the same Equations (24), (25), and (26) for any otherδ as the Burgers equation and do not satisfy (23) as compared to the kink solutions, Equations (27) and (28).

5 The Periodic Solutions

Now assume that Equation (1) has the solution as

$$\psi = 1 + A\tan(\xi), \quad \xi = \Gamma(\beta+1)(kx^{\alpha} - ct^{\alpha})/\alpha, \tag{37}$$

to derive periodic solutions. To replace this assumption with Equation (1) and to solve the resulting equation for A and c, we find

$$A = -2k, , \quad c = k, \tag{38}$$

Also, this gives the solution

$$\psi = 1 - 2k\tan(\xi), \quad \xi = k\Gamma(\beta+1)(x^{\alpha} - t^{\alpha})/\alpha, \tag{39}$$

Likewise, $y = h$ may be extracted from the unique solution. Remember, the M-truncated fractional space-time equation Burgers, Equation (2), are only satisfied by Equation (39) and Equations (25), and (26) of space-time M-truncated fractional burger-liking forms. Equations (22)- (26).

6 The Solution of the Space-time Fractional Burgers-like Equation Only

There are exact solutions that can only satisfy the fractional equations of Burgers for space-time, but they do not fulfill the Burgers equation. We consider that the δ unique parameter values have such solutions.

6.1 Solution of Equation (23)

Use $\delta = 1$, Equation (23) becomes

$$D_{M,t}^{\alpha,\beta}\psi + 2\left(\frac{D_{M,x}^{\alpha,\beta}}{\psi}\right)D_{M,x}^{\alpha,\beta}\psi - D_{M,x}^{\alpha\alpha,\beta}\psi = 0, \tag{40}$$

It is assumed that the solution has the form

$$\psi = e^{\Gamma(\beta+1)(kx^{\alpha} - ct^{\alpha})/\alpha} \tag{41}$$

in which, by replacing this assumption with Equation (40) and solving the resulting formula, we find

$$A\ arbtrary\ constant, \quad c = k^2, \tag{42}$$

This gives the same solution

$$\psi = Ae^{\Gamma(\beta+1)(kx^{\alpha} - k^2 t^{\alpha})/\alpha}, \tag{43}$$

In the same way, we will find a solution.

$$\psi = Ae^{-\Gamma(\beta+1)(kx^{\alpha} - ct^{\alpha})/\alpha}, \tag{44}$$

and by continuing as before, we get

$$A\ arbtrary\ constant, \quad c = -k^2. \tag{45}$$

6.2 Solution of Equation (24)

Using $\delta = -2$, Equation (24) becomes

$$D^{\alpha,\beta}_{M,t}\psi + (\psi - \frac{(D^{\alpha,\beta}_{M,x}\psi)^2 + D^{\alpha\alpha\alpha,\beta}_{M,x}\psi}{D^{\alpha\alpha,\beta}_{M,x}\psi})D^{\alpha,\beta}_{M,x}\psi + 2D^{\alpha\alpha,\beta}_{M,x}\psi = 0, \tag{46}$$

The solution of Equation (46) may have the form

$$\psi = Ae^{\Gamma(\beta+1)(kx^{\alpha}-ct^{\alpha})/\alpha}, \tag{47}$$

If we solve the resulting equation, we get

$$A\ arbtrary\ constant, \quad c = k^2. \tag{48}$$

This gives the exact solution

$$\psi = Ae^{\Gamma(\beta+1)(kx^{\alpha}-k^2t^{\alpha})/\alpha}, \tag{49}$$

Similarly, we can derive the exact solution

$$\psi = Ae^{-\Gamma(\beta+1)(kx^{\alpha}+k2t^{\alpha})/\alpha}, \tag{50}$$

we may note that the solutions derived, Equations (49) and (50), comply with Equation (24) of $\delta = -2$ and do not comply with the Burger M-truncated fractional space time Equation (1).

6.3 Solutions of Equation (25)

Use of $\delta = -4/3$, Equation (25) becomes

$$D^{\alpha,\beta}_{M,t}\psi + (\psi - \frac{3D^{\alpha,\beta}_{M,x}\psi D^{\alpha\alpha,\beta}_{M,x}\psi + D^{\alpha\alpha\alpha\alpha,\beta}_{M,x}\psi}{3D^{\alpha\alpha\alpha,\beta}_{M,x}\psi})D^{\alpha,\beta}_{M,x}\psi + \frac{4}{3}D^{\alpha\alpha,\beta}_{M,x}\psi = 0, \tag{51}$$

As before we obtain

$$\psi = Ae^{\Gamma(\beta+1)(kx^{\alpha}-k^2t^{\alpha})/\alpha}, \tag{52}$$

and

$$\psi = Ae^{-\Gamma(\beta+1)(kx^{\alpha}+k2t^{\alpha})/\alpha}, \tag{53}$$

It can be noted that derived Equations (52) and (53) fulfill Equation (25) for $\delta = -4/3$ and fail to fulfill the Burgers fractional space-time Equation (1).

6.4 Solutions of Equation (26)

Use of $\delta = -8/7$ in Equation (26) and following the same analysis we find

$$\psi = Ae^{\Gamma(\beta+1)(kx^{\alpha}-k^2t^{\alpha})/\alpha}, \tag{54}$$

and

$$\psi = Ae^{-\Gamma(\beta+1)(kx^{\alpha}+k2t^{\alpha})/\alpha}, \tag{55}$$

that the derived solutions, Equations (54) and (55), fulfill the form ([26]) for $\delta = -8/7$, do not satisfy the space-time M-truncated fractional equation of Burgers, Equation (1).

6.5 Solutions of Equation (1)

Below we list some solutions which satisfy only the Burgers M-truncated fractional space -time Equation (11) :

$$\psi_{51} = \frac{2\cosh(\Gamma(\beta+1)x^{\alpha}/\alpha)}{\sinh(\Gamma(\beta+1)x^{\alpha}/\alpha) \pm e^{(\Gamma(\beta+1)t^{\alpha}/\alpha)}} \tag{56}$$

$$\psi_{51} = \frac{2\sinh(\Gamma(\beta+1)x^{\alpha}/\alpha)}{\cosh(\Gamma(\beta+1)x^{\alpha}/\alpha) \pm e^{(\Gamma(\beta+1)t^{\alpha}/\alpha)}} \tag{57}$$

$$\psi_{53} = \frac{x^{\alpha}}{t^{\alpha}} + \frac{2\alpha}{t^{\alpha}\Gamma(\beta+1)}\tanh(\frac{x^{\alpha}}{t^{\alpha}}), \tag{58}$$

$$\psi_{54} = \frac{x^{\alpha}}{t^{\alpha}} + \frac{2\alpha}{t^{\alpha}\Gamma(\beta+1)}\coth(\frac{x^{\alpha}}{t^{\alpha}}), \tag{59}$$

$$\psi_{53} = \frac{x^{\alpha}}{t^{\alpha}} - \frac{2\alpha}{t^{\alpha}\Gamma(\beta+1)}\tan(\frac{x^{\alpha}}{t^{\alpha}}), \tag{60}$$

$$\psi_{53} = \frac{x^{\alpha}}{t^{\alpha}} + \frac{2\alpha}{t^{\alpha}\Gamma(\beta+1)}\cot(\frac{x^{\alpha}}{t^{\alpha}}), \tag{61}$$

These solutions can not satisfy another forms of the M-truncated fractional burgers equations.

7 Conclusion

In the present paper, a family of M-truncated fractional burgers-like formulas has been formally derived. We have shown that these derived equations have some of the M-truncated fractional Burgers equation's moving wave solutions and vary in some other solutions. The present study showed that many nonlinear equations even with higher derivatives can provide the same solutions. When $\alpha = \beta = 1$, the outcomes being the same results obtained by conformable fractional derivative by abdelsalam et al [[30]]. In addition, when all the results obtained here are the same as Wazwaz [[39]]. We think that this method can be used for other nonlinear fractional equations. In the future work we will study the geometrical quantities of the two and three dimensional surfaces in a M-truncated fractional space. Also, we will apply this technique to another nonlinear space time fractional differential equations.

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