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A New Definition of Fractal Derivative and Stability of Fractal Dynamical Systems

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Abstract: In this paper, we give a novel definition of the fractal derivative and their integral. If $r = 1$, the definition coincides with the classical definition of the first derivative. Fractal chain rule, fractal exponential functions, fractal Gronwall's inequality, fractal power series expansions, fractal Laplace transforms, and linear fractal differential systems are presented. We study the stability of dynamical systems relying on the new fractal derivative. Additionally, an example is given to compare to other definitions.

Keywords: Fractal derivative, fractal integral, fractal Gronwall's inequality, fractal Chain rule, stability, α-exponentially stability

1 Introduction

In recent years, the field of fractal and fractional calculus (FC) has experienced a remarkable surge in popularity, drawing the attention of researchers across various domains. Its versatile and robust applications span a wide spectrum, including crucial areas such as fluid dynamics, viscoelasticity, image processing, and even the modeling and prediction of complex phenomena like the dynamics of infectious diseases, exemplified by the work referenced as [1,2,3], and the intricate processes involved in cancer development and progression, as evidenced in [4].

The field of FC has seen numerous mathematicians proposing various definitions of fractional derivatives, with references such as [5], [6], [7], [8], [9], [10], and [11] serving as essential resources for understanding these diverse formulations. Foundational contributions to the field are well-documented in seminal books authored by experts like Mainardi ([12]), Podlubny [13], and Diethelm [14]. These resources provide invaluable insights into the theoretical underpinnings and practical applications of FC. The realm of fractal and fractional calculus (FC) is a vibrant and rapidly evolving field, encompassing a vast array of research endeavors that delve into its multifaceted applications and theoretical foundations. The references mentioned provide a glimpse into the diverse landscape of FC research.

In [15], the focus is on applications of fractal calculus to fractal systems, shedding light on how this mathematical framework can be harnessed to understand and model complex, self-similar phenomena. Meanwhile, [16] addresses the definitions of nabla fractional operators, offering crucial insights into the various mathematical formulations that underpin fractional calculus.

Expanding our horizons, several other notable works contribute to the rich tapestry of FC research. In [17], an exploration of the distribution of buffer over calcium within a fractional dimension provides valuable insights into

∗ Corresponding author e-mail: aliakgul00727@gmail.com real-world systems with intricate dynamics. The study in [18] ventures into comprehending the dynamics of the omicron variant, a pressing concern in the realm of infectious diseases.

In the realm of machine learning and artificial intelligence, [19] focuses on the analysis and modeling of neural networks, demonstrating the relevance of FC in cutting-edge technology. In a different direction, [20] delves into compactness results pertaining to integro-differential equations, a topic with wide-ranging applications in physics and engineering. The mathematical arsenal of FC is further enriched by contributions such as those found in [21], which introduces the Mittag-Leffler-Galerkin method for solving fractional Riccati differential equations, and [22], which presents the FBDF method for solving fractional differential matrix equations. Fractal dynamics take center stage in [23] and [24], with investigations into the stretched exponential stability of nonlinear fractal dynamical systems and the power-law stability of fractal derivative nonlinear dynamical systems, respectively. These studies deepen our understanding of how fractal behavior manifests in dynamical systems. Moreover, FC finds its way into porous media modeling in [25] and [26], with the former proposing a three-dimensional fractal derivative diffusion model for isotropic/anisotropic fractal porous media, and the latter introducing a time-space fractal derivative model for anomalous transport in porous media. These applications have significant implications in environmental science and engineering. Finally, [27] bridges the realms of fractal calculus and fractional calculus, connecting these two mathematical frameworks to predict complex systems. This interplay between fractal and fractional calculus underscores the interdisciplinary nature of FC, with far-reaching applications that span the boundaries of various scientific disciplines. In essence, the references cited in this paragraph serve as a testament to the growing relevance and impact of fractal and fractional calculus across a broad spectrum of scientific and engineering fields, exemplifying the versatility and power of these mathematical tools in understanding and solving complex problems. As FC continues to evolve, it promises to unlock new avenues of inquiry and innovation.

Furthermore, the study of control theory for fractional linear systems has flourished from multiple perspectives, encompassing research on systems utilizing conformable fractional derivatives [28], exploring the intricacies of fractal linear systems [29], and extending the analysis to fractional linear systems within infinite-dimensional spaces [30]. This diverse range of research avenues has enriched our understanding of fractional calculus and its implications in control theory. Notably, recent developments have led to the introduction of new formulations for fractal derivatives, offering alternative perspectives on the derivative of a function $x(\Theta)$. One such formulation, referenced as [8] and [31], defines the fractal derivative as:

$$T_r(x)(\Theta) = \lim_{z\to\Theta} \frac{x(z)-x(\Theta)}{z^r-\Theta^r}.$$

If the function x is differentiable, this formulation can be further expressed as:

$$T_r(x)(\Theta) = \frac{\Theta^{1-r}}{r} x'(\Theta).$$

Another approach, referred to as the conformable derivative [32], introduces an alternate expression for the derivative of a function $x(\Theta)$:

$$T_r(x)(\Theta) = \lim_{\varepsilon\to 0} \frac{x(\Theta+\varepsilon\Theta^{1-r})-x(\Theta)}{\varepsilon}.$$

When the function x is differentiable, this conformable derivative can be simplified as:

$$T_r(x)(\Theta) = \Theta^{1-r} x'(\Theta).$$

Moreover, a novel perspective on the fractal derivative of a function x has emerged, providing yet another representation:

$$T_r(x)(\Theta) = r\Theta^{\frac{r-1}{r}} x'(\Theta).$$

This innovative formulation expands the toolkit of mathematical techniques available for understanding and modeling complex phenomena through the lens of fractal and fractional calculus. As FC continues to evolve, these diverse approaches to fractional derivatives promise to enrich our ability to analyze and describe a wide array of natural and engineered systems.

We prepare our manuscript as: In Section 2 the new fractal derivatives and fractal integrals of higher orders are described, and the fractal chain rule, fractal Laplace transform, fractal power series expansions, and fractal Gronwall inequality are obtained. In Section 3, we study the stability of fractal dynamical systems. In Section 4, the example is presented to illustrate the numerical comparisons of the stability.

2 The definition of new fractal derivative and fractal integral

2.1 New fractal derivative

Definition 1. *Let a function $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. The fractal derivative T_r of function x of order $r \in (0,1]$ is described as:*

$$T_r(x)(\Theta) := \lim_{h\to\Theta} \frac{x(h)-x(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}.$$

If $T_r(x)(\Theta)$ exist for all $\Theta \in I$, then we say x a r-fractal differentiable.

Lemma 1. *We have*

1. $T_r(1) = 0$.
2. $T_r(\Theta^p) = r\Theta^{\frac{r-1}{r}} p\Theta^{p-1}$.
3. $T_r(e^{c\Theta}) = cr\Theta^{\frac{r-1}{r}} e^{c\Theta}, c \in \mathbb{R}$.
4. $T_r(\Theta^{\frac{1}{r}}) = 1$.
5. $T_r(e^{\lambda\Theta^{\frac{1}{r}}}) = \lambda e^{\lambda\Theta^{\frac{1}{r}}}$.
6. $T_r \sin(\Theta^{\frac{1}{r}}) = \cos(\Theta^{\frac{1}{r}})$.
7. $T_r \cos(\Theta^{\frac{1}{r}}) = -\sin(\Theta^{\frac{1}{r}})$.

Remark. Consider a function x to be r-fractal differentiable. In the specific case where r equals 1, it implies that function x is simply differentiable.

Theorem 1. *When a function $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is r-fractal differentiable at a point Θ_0, where r falls within the interval $(0,1]$, it implies that the function x is also continuous at that particular point Θ_0.*

Proof. We have

$$x(h)-x(\Theta) = \frac{x(h)-x(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}(h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}).$$

Then,

$$\lim_{h\to\Theta_0}[x(h)-x(\Theta)] = \lim_{h\to\Theta_0}\frac{x(h)-x(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}\lim_{h\to\Theta_0}(h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}),$$

so

$$\lim_{h\to\Theta_0}[x(h)-x(\Theta)] = T_r(x)(\Theta_0)\times 0,$$

then $\lim_{h\to\Theta_0} x(h) = x(\Theta)$, so x is continuous at Θ_0.

Theorem 2. *Let f and g are r-fractal differentiable and $a,b,c \in \mathbb{R}$. Then*

1. $T_r(af+bg)(\Theta) = aT_r(f)(\Theta)+bT_r(g)(\Theta)$.
2. $T_r(c) = 0$, *for all constant functions* $f(x) = c$.
3. $T_r(fg)(\Theta) = T_r(f)(\Theta)g(\Theta)+f(\Theta)T_r(g)(\Theta)$.
4. $T_r\left(\frac{f}{g}\right) = \frac{T_r(f)(\Theta)g(\Theta)-T_r(g)(\Theta)f(\Theta)}{g(\Theta)g(\Theta)}$.

Proof. Items (1) to (2) are straightforward consequences of the definition. As for (3): To address this point, consider a fixed value of Θ.

$$\begin{aligned}
T_r(fg)(\Theta) &= \lim_{h\to\Theta}\frac{f(h)g(h)-f(\Theta)g(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}\\
&= \lim_{h\to\Theta}\frac{f(h)g(h)-f(\Theta)g(h)+f(\Theta)g(h)-f(\Theta)g(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}\\
&= \lim_{h\to\Theta}\frac{f(h)-f(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}g(h)+\lim_{h\to\Theta}f(\Theta)\frac{g(h)-g(\Theta)}{h^{\frac{1}{r}}-\Theta^{\frac{1}{r}}}\\
&= T_r(f)(\Theta)g(h)+T_r(g)(\Theta)f(\Theta).
\end{aligned}$$

Regarding (4): Now, when fixed Θ,

$$
\begin{aligned}
T_r(\frac{f}{g})(\Theta) &= \lim_{h\to\Theta} \frac{\frac{f(h)}{g(h)} - \frac{f(\Theta)}{g(\Theta)}}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= \lim_{h\to\Theta} \frac{\frac{f(h)g(\Theta)-g(h)f(\Theta)}{g(h)g(\Theta)}}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= \lim_{h\to\Theta} \frac{f(h)g(\Theta) - f(\Theta)g(\Theta) + f(\Theta)g(\Theta) - g(h)f(\Theta)}{g(h)g(\Theta)(h^{\frac{1}{r}} - \Theta^{\frac{1}{r}})} \\
&= \lim_{h\to\Theta} \frac{f(h) - f(\Theta)}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \frac{g(\Theta)}{g(h)g(\Theta)} - \lim_{h\to\Theta} \frac{f(\Theta)}{g(h)g(\Theta)} \frac{g(h) - g(\Theta)}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}},
\end{aligned}
$$

Given that both functions f and g exhibit r-fractal differentiability,

$$
\begin{aligned}
T_r(\frac{f}{g})(\Theta) &= T_r(f)(\Theta)\frac{g(\Theta)}{g(\Theta)g(\Theta)} - \frac{f(\Theta)}{g(\Theta)g(\Theta)}T_r(g)(\Theta) \\
&= \frac{T_r(f)(\Theta)g(\Theta) - T_r(g)(\Theta)f(\Theta)}{g(\Theta)g(\Theta)}.
\end{aligned}
$$

Theorem 3. *(Fractal chain rule) Suppose function f is differentiable and function g is r-fractal differentiable. In that case, we can express the following:*

$$T_r(fog)(\Theta) = f'(g(\Theta))T_r(g)(\Theta).$$

Proof.

$$
\begin{aligned}
T_r(fog)(\Theta) &= \lim_{h\to\Theta} \frac{f(g(h)) - f(g(\Theta))}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= \lim_{h\to\Theta} \frac{f(g(h)) - f(g(\Theta))}{g(h) - g(\Theta)} \frac{g(h) - g(\Theta)}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= f'(g(\Theta))T_r(g)(\Theta).
\end{aligned}
$$

Lemma 2. *Suppose we have a function x that is differentiable, then the following statement holds:*

$$T_r(x)(\Theta) = r\Theta^{\frac{r-1}{r}}x'(\Theta).$$

Proof.

$$
\begin{aligned}
T_r(x)(\Theta) &= \lim_{h\to\Theta} \frac{x(h) - x(\Theta)}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= \lim_{h\to\Theta} \frac{x(h) - x(\Theta)}{h - \Theta} \lim_{h\to\Theta} \frac{h - \Theta}{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}} \\
&= \lim_{h\to\Theta} \frac{x(h) - x(\Theta)}{h - \Theta} \frac{1}{\lim_{h\to\Theta} \frac{h^{\frac{1}{r}} - \Theta^{\frac{1}{r}}}{h - \Theta}} \\
&= x'(\Theta)\frac{1}{\frac{1}{r}\Theta^{\frac{1}{r}-1}} \\
&= r\Theta^{\frac{r-1}{r}}x'(\Theta).
\end{aligned}
$$

2.2 Fractal integral

When dealing with integration, continuous functions form the most significant category for defining integrals.

Definition 2.

$$I_r(f)(\Theta) = I_1(\frac{1}{r}\Theta^{-\frac{r-1}{r}}f)$$
$$= \frac{1}{r}\int_0^{\Theta} \Theta^{-\frac{r-1}{r}} f(\Theta)d\Theta,$$

in this context, the integral represents the standard Riemann improper integral, and r falls within the range of $(0,1]$.

One of the nice results is the following.

Theorem 4. *Let f is any continuous function in the domain of I_r. then*

$$T_r I_r(f)(\Theta) = f(\Theta).$$

Proof. Given the continuity of f, it's evident that $I_r(f)(\Theta)$ is differentiable. Consequently,

$$T_r(I_r(f))(\Theta) = r\Theta^{\frac{r-1}{r}}\frac{d}{d\Theta}I_r(f)(\Theta)$$
$$= r\Theta^{\frac{r-1}{r}}\frac{d}{d\Theta}\int_0^{\Theta}\frac{1}{r}\frac{f(s)}{s^{\frac{r-1}{r}}}ds$$
$$= r\Theta^{\frac{r-1}{r}}\frac{f(\Theta)}{r\Theta^{\frac{r-1}{r}}}$$
$$= f(\Theta).$$

Theorem 5. *Assuming $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a differentiable function with $0 < r \leq 1$, we have the following:*

$$I_r(T_r(f)(\Theta)) = f(\Theta) - f(0).$$

Proof.

$$I_r(T_r(f)(\Theta)) = \frac{1}{r}\int_0^{\Theta}\Theta^{-\frac{r-1}{r}}T_r(f)(\Theta)d\Theta$$
$$= \frac{1}{r}\int_0^{\Theta}\Theta^{-\frac{r-1}{r}}r\Theta^{\frac{r-1}{r}}f'(\Theta)d\Theta$$
$$= \frac{1}{r}\int_0^{\Theta}f'(\Theta)d\Theta$$
$$= f(\Theta) - f(0).$$

Theorem 6. *(Fractal Gronwall's inequality) If*

$$r(\Theta) \leq \delta + \int_a^{\Theta} kr(s)(s-a)^{-\frac{r-1}{r}}ds, \quad (\Theta \in J).$$

Then for all $\Theta \in J$

$$r(\Theta) \leq \delta e^{k(\Theta-a)^{\frac{1}{r}}}.$$

Proof. Let's define $R(\Theta) = \delta + \int_a^{\Theta} kr(s)(s-a)^{-\frac{r-1}{r}}ds = \delta + I_r(kr(s))(\Theta)$. Then $R(a) = \delta$ and $R(\Theta) \geq r(\Theta)$, and

$$T_r(R)(\Theta) - kR(\Theta) = kr(\Theta) - kR(\Theta) \leq kr(\Theta) - kr(\Theta) = 0. \tag{1}$$

Multiply (1) by $e^{-k(\Theta-a)^{\frac{1}{r}}}$, we get

$$e^{-k(\Theta-a)^{\frac{1}{r}}}T_r(R)(\Theta) - e^{-k(\Theta-a)^{\frac{1}{r}}}kR(\Theta) \leq 0,$$

from 3 in Theorem 2, we get

$$T_r(e^{-k(\Theta-a)^{\frac{1}{r}}}R(\Theta)) \leq 0,$$

Since $e^{-k(\Theta-a)^{\frac{1}{r}}}R(\Theta)$ is differentiable on (a,b) then Theorem 5 implies that

$$I_rT_r(e^{-k(\Theta-a)^{\frac{1}{r}}}R(\Theta)) = e^{-k(\Theta-a)^{\frac{1}{r}}}R(\Theta) - R(a) = e^{-k(\Theta-a)^{\frac{1}{r}}}R(\Theta) - \delta \leq 0.$$

Hence

$$r(\Theta) \leq R(\Theta) \leq \frac{\delta}{e^{-k(\Theta-a)^{\frac{1}{r}}}} = \delta e^{k(\Theta-a)^{\frac{1}{r}}}.$$

2.3 Fractal Laplace transform

Definition 3. *Let $0 < r \leq 1$ and the function $f : \mathbb{R} \to \mathbb{R}$. The fractal Laplace transform of order r of function f is defined by:*

$$L_r\{f(\Theta)\}(s) = F_r(s) = \frac{1}{r}\int_0^\infty e^{-s\Theta^{\frac{1}{r}}} f(\Theta)\Theta^{-\frac{r-1}{r}} d\Theta.$$

Theorem 7. *Let $0 < r \leq 1$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable function. Thus, we have*

$$L_r\{T_r(f)(\Theta)\}(s) = sF_r(s) - f(0).$$

Proof.

$$\begin{aligned} L_r\{T_r(f)(\Theta)\}(s) &= \frac{1}{r}\int_0^\infty e^{-s\Theta^{\frac{1}{r}}} r\Theta^{\frac{r-1}{r}} f'(\Theta)\Theta^{-\frac{r-1}{r}} d\Theta \\ &= \int_0^\infty e^{-s\Theta^{\frac{1}{r}}} f'(\Theta) d\Theta \\ &= [e^{-s\Theta^{\frac{1}{r}}} f(\Theta)]_0^\infty - \int_0^\infty -\frac{s}{r}\Theta^{\frac{1-r}{r}} e^{-s\Theta^{\frac{1}{r}}} f(\Theta) d\Theta \\ &= -f(0) + \frac{s}{r}\int_0^\infty \Theta^{\frac{1-r}{r}} e^{-s\Theta^{\frac{1}{r}}} f(\Theta) d\Theta \\ &= sF_r(s) - f(0). \end{aligned}$$

Lemma 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $L_r\{f(\Theta)\}(s) = F_r(s)$ exists. So*

$$F_r(s) = \mathfrak{L}\{f(\Theta^r)\}(s),$$

with $\mathfrak{L}\{g(\Theta)\}(s) = \int_0^\infty e^{-s\Theta} g(\Theta) d\Theta$.

Proof. This result is readily proven by making the substitution $u = \Theta^{\frac{1}{r}}$.

Theorem 8. *Let $0 < r \leq 1$. So, we have*

1. $L_r\{1\}(s) = \frac{1}{s}, s > 0$.
2. $L_r\{\Theta\}(s) = \frac{\Gamma(r+1)}{s^{r+1}}, s > 0$.
3. $L_r\{\Theta^p\}(s) = \frac{\Gamma(rp+1)}{s^{rp+1}}, s > 0$.
4. $L_r\left\{e^{\Theta^{\frac{1}{r}}}\right\}(s) = \frac{1}{s-1}, s > 1$.
5. $L_r\{e^{\lambda\Theta^{\frac{1}{r}}}\} = \frac{1}{s-\lambda}$.

Proof.

1. From definition directly.
2. $L_r\{\Theta\}(s) = \frac{1}{r}\int_0^\infty e^{-s\Theta^{\frac{1}{r}}} \Theta^{\frac{1}{r}} d\Theta = \int_0^\infty e^{-s\tau}\tau^r d\tau = \Gamma(r+1)s^{-r-1}$.
3. $L_r\{\Theta^p\}(s) = \frac{\Gamma(rp+1)}{s^{rp+1}}$.
4. $L_r\left\{e^{\Theta^{\frac{1}{r}}}\right\}(s) = \frac{1}{s-1}, s > 1$.
5. $L_r\{e^{\lambda\Theta^{\frac{1}{r}}}\} = \mathfrak{L}\{e^{\lambda t}\} = \frac{1}{s-\lambda}$.

Proposition 1. *1. If the functions f and g are amenable to transformation, then the transformation of their sum equals the sum of their individual transformations, as expressed by the equation:*

$$L_r\{f + g\} = L_r\{f\} + L_r\{g\}.$$

2. If the function f can be subject to a transformation, and λ is a real number, then the transformation of the product of λ and f is equivalent to the product of λ and the transformation of f. In mathematical terms:

$$L_r\{\lambda f\} = \lambda L_r\{f\}.$$

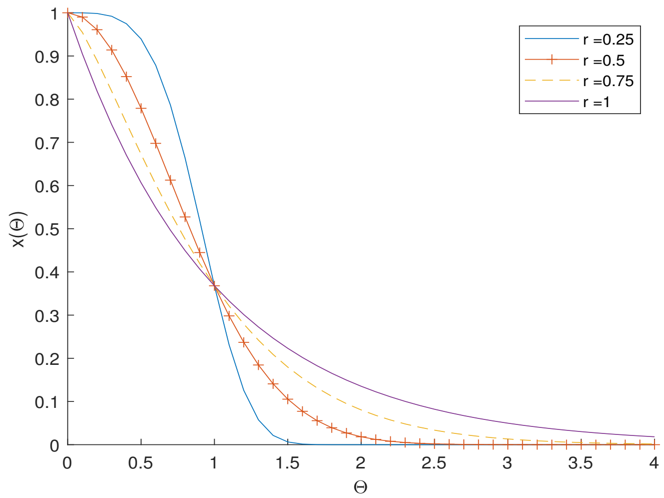

Fig. 1: The solution of (3) for different values of r where $\lambda = -1$ and $x_0 = 1$.

*Proof.*Considering the two propositions mentioned earlier, we can conclude that L_r behaves as a linear operator.

*Example 1.*Consider the fractal equation:

$$\begin{cases} T_r(x)(\Theta) = \lambda x(\Theta),\ \Theta \geq 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

the exact solution of this is $x(\Theta) = e^{\lambda \Theta^{\frac{1}{r}}} x_0$.

*Proof.*By employing the fractal Laplace Transform on both sides of the equation (2), we obtain:

$$L_r\{T_r(x)(\Theta)\}(s) = L_r\{\lambda x(\Theta)\}(s),$$

from Theorem 7 and Proposition 1, we get

$$sF_r(s) - x_0 = \lambda F_r(s).$$

Simplifying this we get

$$F_r(s) = \frac{1}{s-\lambda} x_0. \tag{3}$$

Applying the inverse fractal Laplace transform to (3), we arrive at the solution: $x(\Theta) = e^{\lambda \Theta^{\frac{1}{r}}} x_0$. The solution of (2), acquired through the fractal Laplace transformation method, is depicted in Figure 1 for various values of r.

*Example 2.*Consider the fractal system:

$$\begin{cases} T_r(x)(\Theta) = Ax(\Theta) + f(\Theta),\ \Theta \geq 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$ and $x, f : [0,b) \longrightarrow \mathbb{R}^n$. The general solution of the fractal nonhomogeneous system (4) is given as:

$$x(\Theta) = e^{A\Theta^{\frac{1}{r}}} x_0 + \frac{1}{r} \in \Theta_0^t e^{A(\Theta^{\frac{1}{r}} - s^{\frac{1}{r}})} f(s) s^{\frac{1-r}{r}} ds.$$

2.4 Fractal power series expansions

For $0 < r < 1$ and $n \in \{1,2,3,\dots\}$, we can define the nth-order sequential fractal derivative as follows:

$$(T_r x)^{(n)}(\Theta) = \underbrace{T_r T_r \dots T_r}_{n \text{ times}} x(\Theta).$$

Theorem 9. *Assume x is an infinitely r-fractal differentiable function, for $0 < r \leq 1$. Then x has the fractal power series expansion:*

$$x(\Theta) = \sum_{k=0}^{\infty} \frac{(T_r(x))^{(k)}(0)\,\Theta^{\frac{k}{r}}}{k!}, \quad 0 < t < R^r, R > 0. \tag{5}$$

Proof. Assume

$$x(\Theta) = c_0 + c_1\Theta^{\frac{1}{r}} + c_2\Theta^{\frac{2}{r}} + c_3\Theta^{\frac{3}{r}} + \cdots, 0 < t < R^r, R > 0.$$

Then, $x(0) = c_0$. Apply T_r to x and evaluating at 0 yields $(T_r x)(0) = c_1$. By proceeding inductively and repeatedly applying T_r to the function x a total of n times and then evaluating it at 0 yields $(T_r x)^{(n)}(0) = c_n n!$ and hence

$$c_n = \frac{(T_r x)^{(n)}(0)}{n!}.$$

Proposition 2. *(Fractal Taylor Inequality). Suppose we have a function x that is infinitely r-fractal differentiable, where $0 < r \leq 1$, and it has a Taylor power series representation near the point 0 as given in equation (5). In this representation, we have $|(T_r x)^{n+1}| \leq M$ for some positive M and a natural number n. Consequently, for all values in the interval $(0,R)$, the following holds:*

$$|R_n^r(\Theta)| \leq \frac{M}{(n+1)!}\Theta^{r(n+1)}, \tag{6}$$

where $R_n^r(\Theta) = \sum_{k=n+1}^{\infty} \frac{(T_r x)^{(k)}(0)\Theta^{\frac{k}{r}}}{k!} = x(\Theta) - \sum_{k=0}^{n} \frac{(T_r x)^{(k)}(0)\Theta^{\frac{k}{r}}}{k!}$.

Proof. The proof follows a similar logic to that used in traditional calculus, with the key difference being the application of the operator I_r instead of the standard integral.

Example 3. We take into consideration the fractal exponential function $x(\Theta) = e^{\Theta^{\frac{1}{r}}}$, where $0 < r < 1$. Then $(T_r x)^{(n)}(0) = 1$ for all n and hence

$$x(\Theta) = \sum_{k=0}^{\infty} \frac{\Theta^{\frac{k}{r}}}{k!}.$$

The ratio test demonstrates that this series converges to the function x over the interval $[0,\infty)$.

Example 4. Utilizing Eq. (5) and the identities $T_r \sin(\Theta^{\frac{1}{r}}) = \cos(\Theta^{\frac{1}{r}})$ and $T_r \cos(\Theta^{\frac{1}{r}}) = -\sin(\Theta^{\frac{1}{r}})$, we get

$$\sin(\Theta^{\frac{1}{r}}) = \sum_{k=0}^{\infty} (-1)^k \frac{\Theta^{\frac{2k+1}{r}}}{(2k+1)!}, \quad t \in [0,\infty), \tag{7}$$

and

$$\cos(\Theta^{\frac{1}{r}}) = \sum_{k=0}^{\infty} (-1)^k \frac{\Theta^{\frac{2k}{r}}}{(2k)!}, \quad t \in [0,\infty). \tag{8}$$

3 Stability of fractal dynamical systems

3.1 Non-autonomous linear fractal systems

Consider the non autonomous linear fractal differential system as follows:

$$\begin{cases} T_r(x)(\Theta) = Ax(\Theta), \\ x(0) = x_0, \end{cases} \tag{9}$$

Notations:

$$\sigma(A) = \{\lambda \in \mathbb{C}/\lambda \text{ eigenvalue of } A\}, \tag{10}$$

and

$$\mathbb{C}^o_- = \{\lambda \in \mathbb{C}/Rel(\lambda) < 0\}. \tag{11}$$

Definition 4.*Let x be the solution of equation (9). We say that the linear fractal differential system (9) (or A) is asymptotically stable if*

$$\lim_{\Theta \longmapsto +\infty} x(\Theta) = 0.$$

Definition 5.*We say that the linear fractal differential system (9) (or A) is r-exponentially stable if* $\exists M \geq 0, \lambda > 0$ *will*

$$\|x(\Theta)\| \leq Me^{-\lambda \Theta^{\frac{1}{r}}} \|x_0\|, \forall x_0 \in \mathbb{R}^n.$$

*Remark.*If the fractal system (9) is r-exponentially stable, then any solution of (9) is bounded.

Lemma 4.*We have*

$$\sigma(A) \subset \mathbb{C}^o_- \Leftrightarrow \text{ it exists } M > 0, \nu > 0 \text{ such as } \left\| e^{\Theta^{\frac{1}{r}} A} \right\| \leq Me^{-\nu \Theta^{\frac{1}{r}}}.$$

Proof.$\Rightarrow$)We have by Jordan's reduction theorem there exists an invertible matrix P such that $A = PJP^{-1}$ where $J = \operatorname{diag}\left(J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_r}\right)$, with $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. We have $e^{\Theta^{\frac{1}{r}} A} = Pe^{\Theta^{\frac{1}{r}} J} P^{-1}$, so

$$\begin{aligned} \left\| e^{\Theta^{\frac{1}{r}} A} \right\| &\leq \|P\| \left\| e^{\Theta^{\frac{1}{r}} J} \right\| \left\| P^{-1} \right\| \\ &\leq \|P\| \max_{1 \leq 1 \leq r} \left\| e^{\Theta^{\frac{1}{r}} J_{\lambda_i}} \right\| \left\| P^{-1} \right\|. \end{aligned}$$

Let $\nu > 0$ such that $\nu < \inf_{1 \leq i \leq r} (-Rel(\lambda_i))$, and let $\lambda \in \sigma(A)$. We have

$$\left(e^{\Theta^{\frac{1}{r}} J_\lambda} \right)_{ij} = \begin{cases} e^{\lambda \Theta^{\frac{1}{r}}} \cdot \frac{(\Theta^{\frac{1}{r}})^{j-i}}{(j-i)!} & \text{if } j \leq i, \\ 0 & \text{if } j < i, \end{cases}$$

from where

$$\left(e^{\nu \Theta^{\frac{1}{r}}} e^{\Theta^{\frac{1}{r}} J_\lambda} \right)_{ij} = \begin{cases} e^{(\lambda + \nu) \Theta^{\frac{1}{r}}} \cdot \frac{(\Theta^{\frac{1}{r}})^{j-i}}{(j-i)!} & \text{if } j \leq i, \\ 0 & \text{if } j < i. \end{cases}$$

Which give

$$\left\| e^{\nu \Theta^{\frac{1}{r}}} e^{\Theta^{\frac{1}{r}} J_\lambda} \right\| = e^{(Rel(\lambda) + \nu) \Theta^{\frac{1}{r}}} \cdot \max_{j \geq i} \frac{(\Theta^{\frac{1}{r}})^{j-1}}{(j-1)!},$$

because $(Rel(\lambda) + \nu < 0)$ so exists $M_\lambda > 0$ such as $\|e^{\Theta^{\frac{1}{r}} J_\lambda}\| \leq M_\lambda e^{\nu \Theta^{\frac{1}{r}}}$. To complete the proof we take $M = \max_{\lambda \in \sigma(A)} M_\lambda \|P\| \cdot \left\| P^{-1} \right\|$.

$\Leftarrow$)Let $\lambda \in \sigma(A)$, and x_0 be an eigenvector associated with λ. So

$$Ax_0 = \lambda x_0,$$

from where

$$e^{\Theta^{\frac{1}{r}} A} x_0 = e^{\lambda \Theta^{\frac{1}{r}}} x_0,$$

which give

$$\begin{aligned} \left\| e^{\Theta^{\frac{1}{r}} A} x_0 \right\| &= e^{Rel(\lambda)\Theta^{\frac{1}{r}}} \|x_0\| \\ &\leq M e^{-\nu \Theta^{\frac{1}{r}}} \|x_0\|, \end{aligned}$$

thus

$$e^{Rel(\lambda)\Theta^{\frac{1}{r}}} \leq M e^{-\nu \Theta^{\frac{1}{r}}} \to 0 \text{ when } \Theta \to +\infty,$$

so $Rel(\lambda) < 0$, then $\lambda \in \mathbb{C}^o_-$.

The r-exponentially stable is characterized by the following result:

Proposition 3.*The system (9) is r-exponentially stable if and only if $Rel(\lambda) < 0$ for any eigenvalue λ of A.*

*Proof.*From Lemma 4.

Corollary 1.*The system (9) is r-exponentially stable if and only if it is asymptotically stable.*

Theorem 10.*Let the Eq. (9) is asymptotically stable then for any positive definite matrix Q, there exists a unique matrix P positive definite solution of the Lyapunov equation:*

$$A^T P + PA + Q = 0. \tag{12}$$

*Proof.*Let

$$P = \frac{1}{r} \int_0^{\infty} \Theta^{\frac{1-r}{r}} e^{A^T \Theta^{\frac{1}{r}}} Q e^{A \Theta^{\frac{1}{r}}} d\Theta.$$

1)Let us show that P is well defined. We have A is asymptotically stable, then there exist two strictly positive constants M and $\nu > 0$ such that:

$$\left\| e^{A\Theta^{\frac{1}{r}}} \right\| = \left\| e^{A^T\Theta^{\frac{1}{r}}} \right\| \leq M e^{-\nu \Theta^{\frac{1}{r}}}.$$

So

$$\begin{aligned} \|P\| &\leq \|Q\| \int_0^{\infty} \frac{1}{r} M^2 e^{-2\nu \Theta^{\frac{1}{r}}} d\Theta \\ &\leq \|Q\| \frac{M^2}{2\nu r}. \end{aligned}$$

2)Let us show that the matrix P is positive definite. If $x \in \mathbb{R}^n - \{0\}$, we have:

$$\begin{aligned} x^T P x &= \frac{1}{r} \int_0^{\infty} \Theta^{\frac{1-r}{r}} x^T e^{A^T \Theta^{\frac{1}{r}}} Q e^{A \Theta^{\frac{1}{r}}} x d\Theta \\ &= \frac{1}{r} \int_0^{\infty} (\Theta^{\frac{1-r}{2r}} e^{A \Theta^{\frac{1}{r}}} x)^T Q (\Theta^{\frac{1-r}{2r}} e^{A \Theta^{\frac{1}{r}}} x) d\Theta > 0 \ \ (\text{ because } Q \text{ is positive definite}). \end{aligned}$$

3)We have

$$A^T P + PA = \frac{1}{r} \int_0^{\infty} \Theta^{\frac{1-r}{r}} \left(A^T e^{A^T \Theta^{\frac{1}{r}}} Q e^{A \Theta^{\frac{1}{r}}} + e^{A^T \Theta^{\frac{1}{r}}} Q e^{A \Theta^{\frac{1}{r}}} A \right) d\Theta,$$

but the two matrices A and $e^{A\frac{\Theta^r}{r}}$ commute, so

$$A^T P + PA = \frac{1}{r} \int_0^{\infty} \Theta^{\frac{1-r}{r}} \frac{d}{d\Theta} \left(e^{A^T \Theta^{\frac{1}{r}}} Q e^{A \Theta^{\frac{1}{r}}} \right) d\Theta.$$

Since the Eq. (9) is asymptotically stable, we have

$$A^T P + PA = -Q.$$

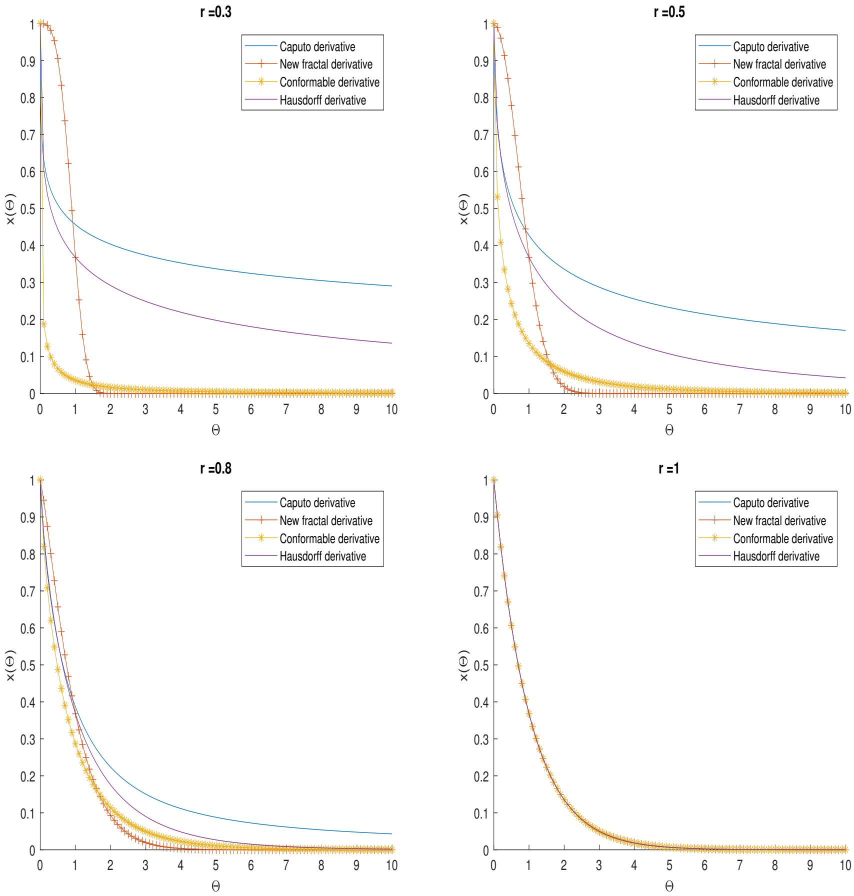

Fig. 2: Results for $x(0) = 1$ and $r = \{0.3, 0.5, 0.8, 1\}$.

4)Let us show the uniqueness of the solution. Let R be a positive definite solution matrix of the Eq. (12), so

$$\begin{aligned}\Theta^{\frac{1-r}{r}} \frac{d}{d\Theta}\left(e^{A^T\Theta^{\frac{1}{r}}} R e^{A\Theta^{\frac{1}{r}}}\right) &= \Theta^{\frac{1-r}{r}} A^T e^{A^T\Theta^{\frac{1}{r}}} R e^{A\frac{\Theta^r}{r}} + \Theta^{\frac{1-r}{r}} e^{A^T\Theta^{\frac{1}{r}}} R A e^{A\Theta^{\frac{1}{r}}} \\ &= \Theta^{\frac{1-r}{r}} e^{A^T\Theta^{\frac{1}{r}}} \left[A^T R + RA\right] e^{A\Theta^{\frac{1}{r}}} \\ &= -\Theta^{\frac{1-r}{r}} e^{A^T\Theta^{\frac{1}{r}}} Q e^{A\Theta^{\frac{1}{r}}}.\end{aligned}$$

We integrate the two terms of the equality, we get

$$-R = -\frac{1}{r}\int_0^{\infty} \Theta^{\frac{1-r}{r}} e^{A^T\Theta^{\frac{1}{r}}} Q e^{A\Theta^{\frac{1}{r}}} d\Theta,$$

that is to say $P = R$.

4 Example

In this section, we provide an example to demonstrate the numerical comparisons regarding the stability characteristics of the novel fractal derivative equation, the Hausdorff derivative equation [24], and the corresponding classical fractional-order equation (in the Caputo sense [33]). The equation under consideration is [24]:

$$T_r(x)(\Theta) = -x(\Theta), \tag{13}$$

where $r \in (0,1]$ and $x(\Theta) : [0,\infty) \to \mathbb{R}$.

A comparison of the numerical solutions of equation (13) and its corresponding equation when replacing the novel fractal derivative T_r by Caputo derivative [33], conformable derivative [32] and Chen Hausdorff derivative [8,31] is presented in Figure 2.

5 Conclusion

In the course of this research effort, we have introduced a novel and innovative definition of the fractal derivative, thereby expanding the theoretical foundation of fractional calculus. Our work has gone beyond mere introduction, as we have also presented a series of theorems that underpin this new fractal derivative, elucidating its mathematical properties and potential applications.

One of the remarkable insights emerging from our study is the concept of r-exponential stability, a concept that extends the classical notion of exponential stability. This extension broadens the applicability of stability analysis to a wider array of systems, allowing us to capture complex behaviors that may not conform to the traditional framework of exponential stability. Notably, when the stretched parameter equals 1, our r-exponential stability reduces to the classical exponential stability, demonstrating the versatility and continuity of these stability concepts.

In practical terms, the strength and significance of our proposed fractal derivative are exemplified by its successful application in real-world scenarios. Through rigorous testing and validation, we have demonstrated the applicability of this new notion in solving problems across various domains. This empirical evidence not only bolsters the theoretical framework but also underscores the practical relevance of our work. We anticipate that our work will inspire further innovation and exploration, opening new frontiers in mathematics and its practical applications.

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