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A New Fractal Derivative and its Properties

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Abstract: This study introduces a novel fractal derivative alongside its corresponding integral, delving into essential properties such as the fractal Laplace transform, the fractal chain rule, and derivative operations. We also explore the solution of a linear fractal differential system. Furthermore, we provide two illustrative examples that allow us to compare the proposed fractal differential equation to existing definitions, including the Hausdorff derivative, Caputo derivative, and Yang derivative. This comparative analysis underscores the efficacy of the extended definition for addressing non-integer order differential equations.

Keywords: Fractal derivative, fractal chain rule, fractal integral, fractal Laplace transform.

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1 Introduction

Fractional and fractal calculus, including the Caputo fractional calculus [1], the nabla fractional calculus [2], cotangent fractional calculus [3], Hilfer cotangent fractional calculus [4], and fractal-fractional calculus [5], play a pivotal role in both applied and theoretical fields across science and engineering. Notably, Wen Chen introduced the innovative Hausdorff derivative, which has found practical application in mathematical models addressing real-world challenges [6] [7] and anomalous transport phenomena [9]. Akgul also contributed to this realm by presenting a novel fractional derivative with distinct kernels [10].

The literature boasts an array of research endeavors related to fractional and fractal calculus. For instance, Pandey mathematically modeled the COVID-19 pandemic using the Caputo-Fabrizio fractional derivative [11] and the Maize Streak Virus Epidemic Model Using Caputo-Fabrizio Fractional Derivative in [12]. In another context, Evirgen examined the dynamics of the Nipah virus using the Caputo fractional derivative [13], while Sadekotman delved into the fractional modeling of TiO_2 nanopowder synthesis employing the Caputo fractional derivative [14]. Furthermore, Kumar explored alkali-silica chemical reactions using the Caputo fractional derivative [15], and Sadek addressed the observability and controllability of fractal linear dynamical systems [16]. Kolebaje ventured into mathematical modeling of COVID-19 with the Atangana-Baleanu fractional derivative [17], and Sadek89 examined the observability and controllability of fractional linear systems using conformable derivatives [18]. The realm of research also includes topics like the conformable finite element method for solving conformable fractional partial differential equations [19], stability analysis of conformable linear infinite-dimensional systems [20], and the generalization of the BDF methods for solving

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matrix fractional differential equations [21]. These are just a few examples among the multitude of research works in this field. The definition of the Hausdorff derivative for a function $x(\ell)$ with a fractional order $\alpha \in]0,1]$ (as introduced in [6, 7]) can be expressed as follows:

$$C^{\alpha}(x)(\ell) = \lim_{z \to t} \frac{x(z) - x(\ell)}{z^{\alpha} - \ell^{\alpha}}, \tag{1}$$

if x is differentiable, then

$$C^{\alpha}(x)(\ell) = \frac{1}{\ell^{\alpha-1}\alpha} x'(\ell).$$

The Yang derivative, proposed in [22], for a function $x(\ell)$ with a fractional order $\alpha \in]0,1]$, is given by:

$$\mathbb{F}^{\alpha}(x)(\ell) = \frac{1}{\alpha e^{\alpha \ell}} x'(\ell).$$

Now, we introduce a novel fractal derivative for the function x, defined as:

$$F^{\alpha}(x)(\ell) = \lim_{h \to t} \frac{x(h) - x(\ell)}{e^{h^{\alpha}} - e^{\ell^{\alpha}}},$$

If x is differentiable, then this fractal derivative can be expressed as:

$$F^{\alpha}(x)(\ell) = \frac{\ell^{1-\alpha}}{\alpha e^{\ell^{\alpha}}} x'(\ell).$$

The structure of this paper is as follows: In Section 2, we introduce the novel fractal derivative and explore its some properties, such as the fractal chain rule. Section 3 delves into fractal integrals, and in Section 4, we examine the fractal Laplace transform, discussing how fractal derivatives and integrals interact with each other. To provide a numerical basis for comparing the stability characteristics of the new fractal derivative equation, the Hausdorff derivative equation, and their corresponding fractional-order equations (in the Caputo sense), we present examples in example 3 and example 4.

2 New fractal derivative

First, we introduce a truncated exponential function, denoted as:

$$e_k^{\ell} = \sum_{i=1}^{k} \frac{\ell^i}{i!},$$

With this concept in place, we can proceed to define another fractional derivative, which is presented below.

Definition 1. *Consider a function x defined on a subset $I \subset \mathbb{R}$, where $x : I \longrightarrow \mathbb{R}$. We introduce the fractal derivative F_k^{α} of x with respect to order $\alpha \in (0,1]$ as follows:*

$$F_k^{\alpha}(x)(\ell) = \lim_{h \to t} \frac{x(h) - x(\ell)}{e_k^{h^{\alpha}} - e_k^{\ell^{\alpha}}},$$

We characterize x as α-differentiable when $F_k^{\alpha}(x)(\ell)$ exists for all $\ell \in I$.

So,

$$F_1^{\alpha}(x)(\ell) = \lim_{h \to t} \frac{x(h) - x(\ell)}{h^{\alpha} - \ell^{\alpha}}, \tag{2}$$

and

$$F_{\infty}^{\alpha}(x)(\ell) = \lim_{h \to t} \frac{x(h) - x(\ell)}{e^{h^{\alpha}} - e^{\ell^{\alpha}}}, \tag{3}$$

The expression in Eq. (2) corresponds to the fractal derivative as defined in Eq. (1) in the works of Chen [6,7]. In contrast, the expression in Eq. (3) represents the fractal derivative introduced in the current paper. It's worth noting that when $\alpha = 1$, Eq. (2) reduces to the conventional definition of the first derivative of a function x at a specific point ℓ.

Definition 2. *Consider a function x defined on the interval $I \subset \mathbb{R}$ and taking real values. The fractal derivative, denoted as F^{α}, of x with respect to an order $\alpha \in (0,1]$ is given by the following expression:*

$$F^{\alpha}(x)(\ell) = \lim_{h\to t} \frac{x(h)-x(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}.$$

We refer to x as α-differentiable if the value of $F^{\alpha}(x)(\ell)$ exists for all $\ell \in I$.

Lemma 1. *Here are the fractal derivatives of a few functions:*

1. $F^{\alpha}(1) = 0.$
2. $F^{\alpha}(\ell^{p}) = \frac{p}{\alpha e^{\ell^{\alpha}}}\ell^{p-\alpha}.$
3. $F^{\alpha}(e^{e^{\ell^{\alpha}}}) = 1.$
4. $F^{\alpha}(e^{\lambda e^{\ell^{\alpha}}}) = \lambda e^{\lambda e^{\ell^{\alpha}}}.$

Remark. A function x is considered α-differentiable, and if $\alpha = 1$, it implies that x is differentiable.

Theorem 1. *Suppose we have an interval $I \subset \mathbb{R}$, and within this interval, there is a specific point $a \in I$. If a function $x : I \longrightarrow \mathbb{R}$ is α-differentiable at the point a for some $\alpha \in (0,1]$, it implies that the function x exhibits continuity at this particular point a.*

Proof. We have

$$x(h)-x(a) = \frac{x(h)-x(a)}{e^{h^{\alpha}}-e^{a^{\alpha}}}(e^{h^{\alpha}}-e^{a^{\alpha}}).$$

Then,

$$\lim_{h\to a}[x(h)-x(a)] = \lim_{h\to a}\frac{x(h)-x(a)}{e^{h^{\alpha}}-e^{a^{\alpha}}}\lim_{h\to a}(e^{h^{\alpha}}-e^{a^{\alpha}}),$$

so

$$\lim_{h\to a}[x(h)-x(a)] = F^{\alpha}(x)(a).0,$$

this suggests that as the variable h approaches the point a, the function $x(h)$ converges to $x(a)$, thus establishing that function x maintains continuity at point a.

Theorem 2. *Consider the interval I contained within the set of real numbers. Let $a,b,c \in I$ and suppose $x : I \to \mathbb{R}$ and $y : I \to \mathbb{R}$ are both α-differentiable functions. In this context, we can express the following relationships:*

1. $F^{\alpha}(ax+by) = aF^{\alpha}(x)+bF^{\alpha}(y).$
2. $F^{\alpha}(c) = 0$, *for all* $x(\ell) = c.$
3. $F^{\alpha}(xy)(\ell) = F^{\alpha}(x)(\ell)y(\ell)+x(\ell)F^{\alpha}(y)(\ell).$
4. $F^{\alpha}\left(\frac{x}{y}\right)(\ell) = \frac{F^{\alpha}(x)(\ell)y(\ell)-F^{\alpha}(y)(\ell)x(\ell)}{y(\ell)^{2}}.$

Proof. Relationships (1) to (2) are derived based on the principles outlined in Definition 2. As for (3): Let's proceed by considering a fixed value of t.

$$\begin{aligned}
F^{\alpha}(xy)(\ell) &= \lim_{h\to\ell}\frac{x(h)y(h)-x(\ell)y(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\\
&= \lim_{h\to\ell}\frac{x(h)y(h)-x(\ell)y(h)+x(\ell)y(h)-x(\ell)y(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\\
&= \lim_{h\to\ell}\frac{x(h)-x(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}y(h)+\lim_{h\to t}x(\ell)\frac{y(h)-y(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\\
&= F^{\alpha}(x)(\ell)y(h)+F^{\alpha}(y)(\ell)x(\ell).
\end{aligned}$$

As for (4): Next, with a constant value of t,

$$\begin{aligned}
F^{\alpha}(\frac{x}{y})(\ell) &= \lim_{h\to\ell}\frac{\frac{x(h)}{y(h)}-\frac{x(\ell)}{y(\ell)}}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\\
&= \lim_{h\to\ell}\frac{\frac{x(h)y(\ell)-y(h)x(\ell)}{y(h)y(\ell)}}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\\
&= \lim_{h\to\ell}\frac{x(h)y(\ell)-x(\ell)y(\ell)+x(\ell)y(\ell)-y(h)x(\ell)}{y(h)y(\ell)(e^{h^{\alpha}}-e^{\ell^{\alpha}})}\\
&= \lim_{h\to\ell}\frac{x(h)-x(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}}\frac{y(\ell)}{y(h)y(\ell)}-\lim_{h\to t}\frac{x(\ell)}{y(h)y(\ell)}\frac{y(h)-y(\ell)}{e^{h^{\alpha}}-e^{\ell^{\alpha}}},
\end{aligned}$$

since x and y are α-differentiable so

$$\begin{aligned} F^{\alpha}(\frac{x}{y})(\ell) &= F^{\alpha}(x)(\ell)\frac{y(\ell)}{y(\ell)y(\ell)} - \frac{x(\ell)}{y(\ell)y(\ell)}F^{\alpha}(y)(\ell) \\ &= \frac{F^{\alpha}(x)(\ell)y(\ell) - F^{\alpha}(y)(\ell)x(\ell)}{y(\ell)^2}. \end{aligned}$$

Theorem 3.*(Fractal chain rule) Let $I \subset \mathbb{R}$ and $x : I \longrightarrow \mathbb{R}$ differentiable and y $x : I \longrightarrow \mathbb{R}$ α-differentiable. Then the function xoy is α-differentiable,*

$$F^{\alpha}(xoy)(\ell) = x'(y(\ell))F^{\alpha}(y)(\ell).$$

*Proof.*We have,

$$\begin{aligned} F^{\alpha}(xoy)(\ell) &= \lim_{h\to\ell}\frac{x(y(h)) - x(y(\ell))}{e^{h^{\alpha}} - e^{\ell^{\alpha}}} \\ &= \lim_{h\to\ell}\frac{x(y(h)) - x(y(\ell))}{y(h) - y(\ell)}\frac{y(h) - y(\ell)}{e^{h^{\alpha}} - e^{\ell^{\alpha}}} \\ &= x'(y(\ell))F^{\alpha}(y)(\ell). \end{aligned}$$

Lemma 2.*Let $I \subset \mathbb{R}$ and the function $x : I \longrightarrow \mathbb{R}$ differentiable at ℓ, then*

$$F^{\alpha}(x)(\ell) = \frac{t^{1-\alpha}}{\alpha e^{t^{\alpha}}}x'(\ell).$$

*Proof.*We get

$$\begin{aligned} F^{\alpha}(x)(\ell) &= \lim_{h\to\ell}\frac{x(h) - x(\ell)}{e^{h^{\alpha}} - e^{\ell^{\alpha}}} \\ &= \lim_{h\to\ell}\frac{x(h) - x(\ell)}{h - \ell}\lim_{h\to\ell}\frac{h - \ell}{e^{h^{\alpha}} - e^{\ell^{\alpha}}} \\ &= \lim_{h\to\ell}\frac{x(h) - x(\ell)}{h - \ell}\frac{1}{\lim_{h\to\ell}\frac{e^{h^{\alpha}} - e^{\ell^{\alpha}}}{h - \ell}} \\ &= x'(\ell)\frac{1}{\alpha\ell^{\alpha-1}e^{\ell^{\alpha}}}. \end{aligned}$$

Yang in [22] generalized derivative is defined:

$$\mathbb{F}^{\alpha}(x)(\ell) = \frac{1}{h'(\ell)}x'(\ell), \;\; h'(\ell) > 0.$$

–The derivative for the power-law function, represented in [22] as $h(\ell) = \ell^{\alpha}$, is defined as follows:

$$\mathbb{F}^{\alpha}(x)(\ell) = \frac{1}{\alpha\ell^{\alpha-1}}x'(\ell).$$

–The derivative concerning the exponential function (not the Yang derivative), denoted as in [22] by $h(\ell) = e^{\alpha\ell}$, is defined as follows:

$$\mathbb{F}^{\alpha}(x)(\ell) = \frac{1}{\alpha e^{\alpha\ell}}x'(\ell).$$

Remark.

1.If $\alpha = 1$, we have $\mathbb{F}^{\alpha} = F^{\alpha}$.
2.Definition 1 establishes the equivalence between the general derivative in [22] ($h(\ell) = e_k^{\ell^{\alpha}}$).

3 Fractal integral

In this section, we present the fractal integral and two theorems' important role.

Definition 3. *Consider a continuous function $x : I \longrightarrow \mathbb{R}$, where $I \subset \mathbb{R}$. Then, the fractal integral I_α of x for $\alpha \in (0,1]$ is defined as follows:*

$$I_\alpha(x)(\ell) = \int_0^\ell \alpha s^{\alpha-1} e^{s^\alpha} x(s)ds, \ \ \ell \in I.$$

Theorem 4. *Let $I \subset \mathbb{R}$ and the continuous function $x : I \longrightarrow \mathbb{R}$. Then*

$$F^\alpha I_\alpha(x) = x.$$

Proof. Let $t\imath I$. $I_\alpha(x)(\ell)$ is clearly differentiable. Hence,

$$\begin{aligned} F^\alpha \left(I_\alpha(x)\right)(\ell) &= \frac{1}{\alpha \ell^{\alpha-1} e^{\ell^\alpha}} \frac{d}{d\ell} I_\alpha(x)(\ell) \\ &= \frac{1}{\alpha \ell^{\alpha-1} e^{\ell^\alpha}} \frac{d}{dt} \int_0^\ell \alpha s^{\alpha-1} e^{s^\alpha} x(s)ds \\ &= \frac{1}{\alpha \ell^{\alpha-1} e^{\ell^\alpha}} \alpha \ell^{\alpha-1} e^{\ell^\alpha} x(\ell) \\ &= x(\ell). \end{aligned}$$

Theorem 5. *Let $0 < \alpha \leq 1$ and $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. We have*

$$I_\alpha(F^\alpha(x)(\ell)) = x(\ell) - x(0).$$

Proof.

$$\begin{aligned} I_\alpha(F^\alpha(x)(\ell)) &= \int_0^\ell \alpha s^{\alpha-1} e^{s^\alpha} F^\alpha(x)(s)ds. \\ &= \int_0^\ell \alpha s^{\alpha-1} e^{s^\alpha} \frac{1}{\alpha s^{\alpha-1} e^{s^\alpha}} x'(s)ds \\ &= \int_0^\ell x'(s)ds \\ &= x(\ell) - x(0). \end{aligned}$$

4 The fractal Laplace transform

Definition 4. *Let $I \subset \mathbb{R}$, $0 < \alpha \leq 1$ and the function $x : I \longrightarrow \mathbb{R}$. We define the fractal Laplace transform of order α for the function x as follows:*

$$L_\alpha\{x(\ell)\}(s) = \alpha \int_0^\infty e^{-se^{\ell^\alpha}} x(\ell)\ell^{\alpha-1} e^{\ell^\alpha} d\ell.$$

We pose $X_\alpha(s) := L_\alpha\{x(\ell)\}(s)$.

Theorem 6. *Let $0 < \alpha \leq 1$ the function $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. We have*

$$L_\alpha \left\{F^\alpha(x)(\ell)\right\}(s) = sX_\alpha(s) - e^{-s}x(0).$$

Proof.

$$\begin{aligned} L_\alpha \left\{F^\alpha(x)(\ell)\right\}(s) &= \alpha \int_0^\infty e^{-se^{\ell^\alpha}} F^\alpha(x)(\ell)\ell^{\alpha-1} e^{\ell^\alpha} d\ell \\ &= \alpha \int_0^\infty e^{-se^{\ell^\alpha}} \frac{1}{\alpha \ell^{\alpha-1} e^{\ell^\alpha}} x'(\ell)\ell^{\alpha-1} e^{\ell^\alpha} d\ell \\ &= \int_0^\infty e^{-se^{\ell^\alpha}} x'(\ell)d\ell \\ &= [e^{-se^{\ell^\alpha}} x(\ell)]_0^\infty - \alpha s \int_0^\infty -e^{-se^{\ell^\alpha}} x(\ell)\ell^{\alpha-1} e^{\ell^\alpha} d\ell \\ &= -e^{-s}x(0) + sX_\alpha(s). \end{aligned}$$

Lemma 3. *Let the function $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ such that $L_\alpha\{x(\ell)\}(s) = X_\alpha(s)$ exists. We have*

$$X_\alpha(s) = \mathfrak{L}\{x((\log(\ell))^{\frac{1}{\alpha}})\}(s),$$

with $\mathfrak{L}\{y(\ell)\}(s) = \int_1^\infty e^{-s\ell} y(\ell) d\ell$.

Proof. The proof can be readily established by introducing the variable h as $h = e^{\ell^\alpha}$.

Theorem 7. *Let $0 < \alpha \leq 1$. Then*

1. $L_\alpha\{1\}(s) = \frac{e^{-s}}{s}, s > 0.$
2. $L_\alpha\left\{e^{e^{\ell^\alpha}}\right\}(s) = \frac{e^{-s+1}}{s-1}, s > 1.$
3. $L_\alpha\{e^{\lambda e^{\ell^\alpha}}\} = \frac{e^{\lambda-s}}{s-\lambda}, s > \lambda.$

Proof. The proof follows directly from the definition.

Proposition 1.

1. *Let the functions x and y are fractal transformable, then*

$$L_\alpha\{x+y\} = L_\alpha\{x\} + L_\alpha\{y\}.$$

2. *Let the function x is farctal transformable and $\lambda \in \mathbb{R}$, then*

$$L_\alpha\{\lambda x\} = \lambda L_\alpha\{x\}.$$

Proof. Considering the two preceding propositions, it can be affirmed that L_α qualifies as a linear operator.

Example 1. Consider the fractal problem:

$$\begin{cases} F^\alpha(x)(\ell) = \lambda x(\ell), \; \ell > 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

the exact solution is $x(\ell) = e^{\lambda e^{\ell^\alpha}} e^{-\lambda} x_0$.

Proof. Applying the fractal Laplace Transform to Eq. (4), we have

$$L_\alpha\{F^\alpha(x)(\ell)\}(s) = L_\alpha\{\lambda x(\ell)\}(s),$$

Based on Theorem 6 and Proposition 1, we obtain:

$$sX_\alpha(s) - e^{-s} x_0 = \lambda X_\alpha(s).$$

Simplifying this we get

$$X_\alpha(s) = \frac{e^{-s}}{s-\lambda} x_0. \tag{5}$$

Applying the inverse fractal Laplace transform to Eq. (5), we obtain the solution $x(\ell) = e^{\lambda e^{\ell^\alpha}} e^{-\lambda} x_0$. The solution of Eq. (4), which is derived using the fractal Laplace transformation method, is illustrated in Figure 1 for various values of α.

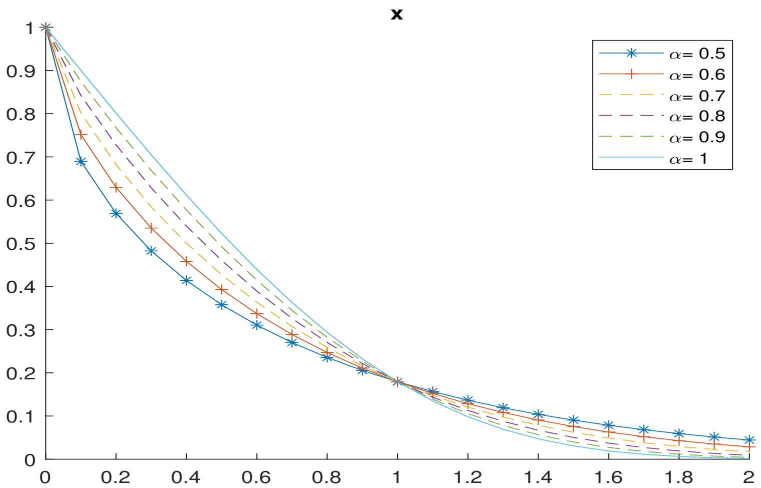

Fig. 1: The solution of Eq. (4) for different values of α where $\lambda = -1$ and $x_0 = 1$.

Example 2. Let the linear fractal differential system

$$\begin{cases} F^{\alpha}(x)(\ell) = Ax(\ell) + y(\ell),\ \ell \geq 0, \\ x(0) = x_0. \end{cases} \tag{6}$$

with $x, y : [0, a) \longrightarrow \mathbb{R}^n$ are vector functions and A is an $n \times n$ matrix. We utilize the fractal Laplace transform to represent the solution of a linear fractal differential system. The precise solution of the linear fractal equation (6) is given by:

$$x(\ell) = e^{Ae^{\ell^{\alpha}}} e^{-A} x_0 + \alpha \int_0^{\ell} e^{A(e^{\ell^{\alpha}} - e^{s^{\alpha}})} y(s) s^{\alpha-1} e^{s^{\alpha}} ds.$$

Example 3. In this example, we provide essential numerical comparisons to assess the stability of the fractal differential equation using the new fractal derivative, the Hausdorff derivative [23], the Yang derivative [22], and the Caputo derivative [24]. These comparisons are made for the following dynamical system [23]:

$$\begin{cases} F^{\alpha}(x)(\ell) = -x(\ell),\ \ell \geq 0, \\ x(0) = x_0, \end{cases} \tag{7}$$

where $x(\ell) : [0, \infty) \to \mathbb{R}$. A comparison of the solutions of Eq. (7) and its corresponding system when replacing the novel fractal derivative by Caputo derivative [24], Hausdorff derivative [6,7] and Yang derivative [22] is presented in Figure 2. From Figure 2, it is clear that the new fractal derivative system converges to zero much faster than its corresponding fractional-order system.

Example 4. In this example, we present necessary numerical comparisons of the fractal differential equation using the new fractal derivative, Hausdorff derivative [23], Yang derivative [22] and the Caputo derivative [24]). For the following dynamical system [14]:

$$\begin{cases} F^{\alpha}(x_1)(\ell) = -k_1 x_1(\ell) x_2(\ell) - k_2 x_1(\ell) x_3(\ell), \\ F^{\alpha}(x_2)(\ell) = -k_1 x_1(\ell) x_2(\ell) + k_3 x_3(\ell)^2, \\ F^{\alpha}(x_3)(\ell) = k_1 x_1(\ell) x_2(\ell) - k_2 x_1(\ell) x_3(\ell) - k_3 x_3(\ell)^2, \\ F^{\alpha}(x_4)(\ell) = k_1 x_1(\ell) x_2(\ell) + k_2 x_1(\ell) x_3(\ell), \\ F^{\alpha}(x_5)(\ell) = k_2 x_1(\ell) x_3(\ell) + k_3 x_3(\ell)^2, \end{cases} \tag{8}$$

with initial conditions:

$$x_1(1) = 12.35, x_2(1) = 24.7, x_3(1) = 0, x_4(1) = 0, x_5(1) = 0.$$

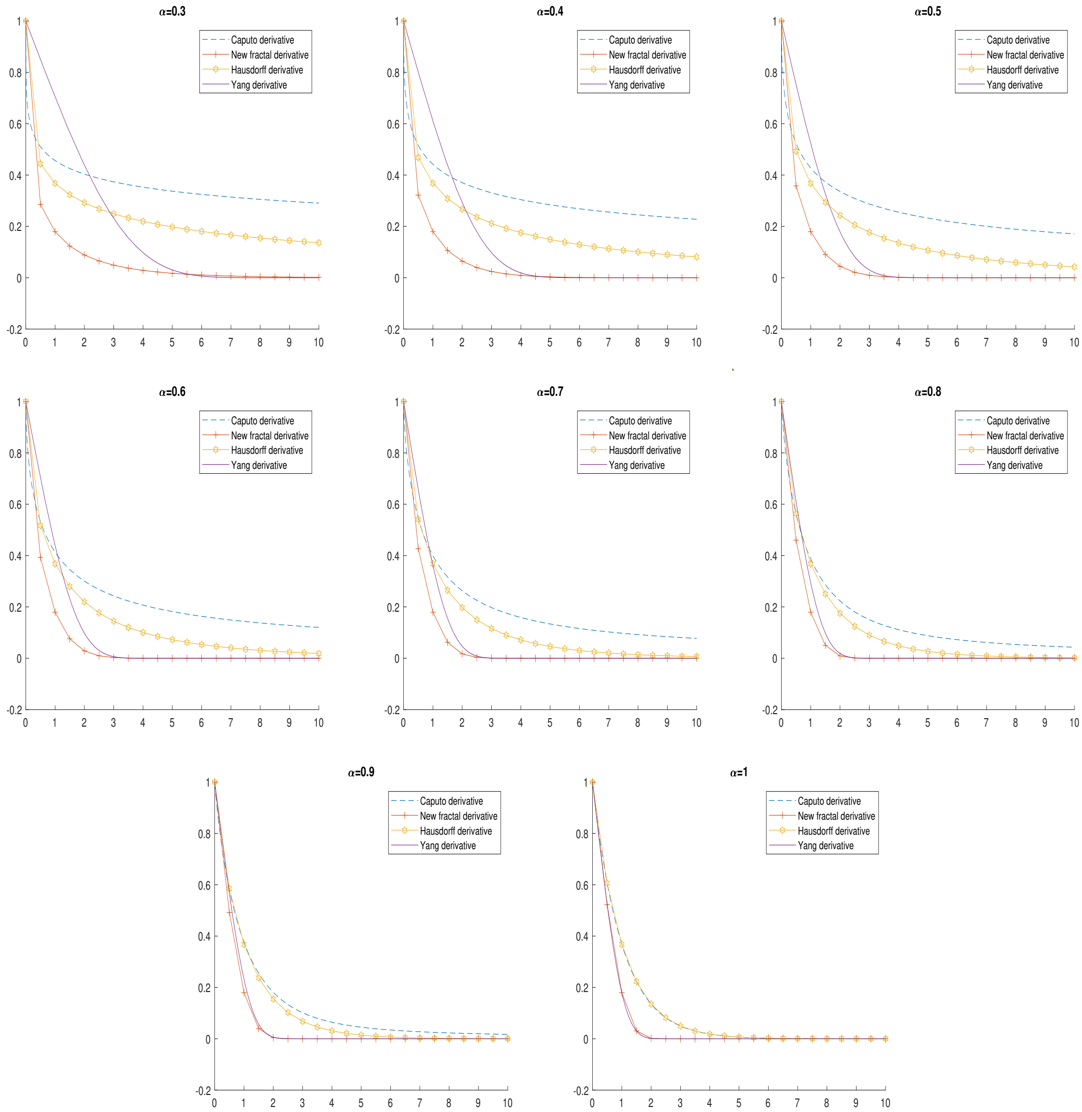

Fig. 2: Comparisons of numerical solutions of the dynamical system using the four definitions when they have the same initial condition $x(0) = 1$ and $\lambda = -1$ for $\alpha = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

Figure 3 illustrates a comparison between the solutions of a nonlinear system (8) and its corresponding system. This comparison is made by substituting the new fractal derivative F^{α} with the Yang derivative, Caputo derivative, and Hausdorff derivative.

5 Conclusion

In this study, we introduced a novel concept of the fractal derivative along with several associated theorems and the integral of this new fractal derivative. These developments have yielded significant outcomes, especially in enhancing the stability of non-integer order differential equations. As we look ahead, our forthcoming research will delve into the practical applications of this innovative definition.

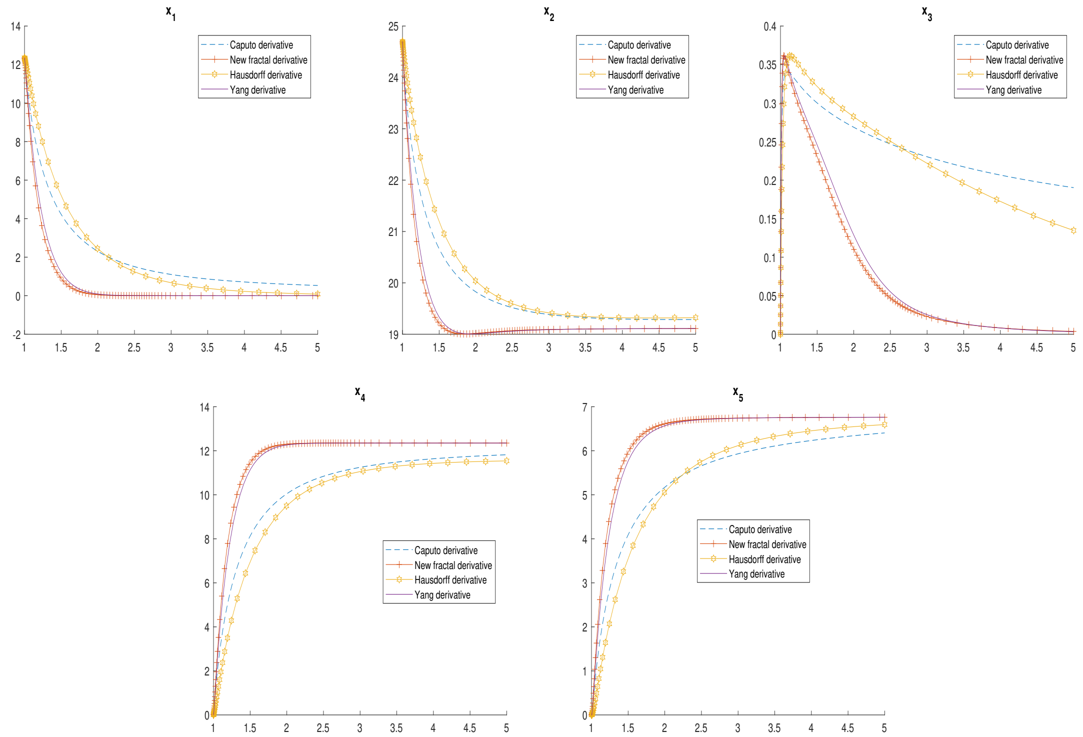

Fig. 3: Comparisons of numerical solutions of the nonlinear dynamical system using the four definitions for $\alpha = 0.85$.

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