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Comparison Between the Gaussian Plume Model and Solution of Advection Diffusion Equation in Linear K-Profile

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Abstract: The advection-diffusion equation was solved in three dimensions in this study using the vertical turbulent as a function of the downwind and vertical distances. contrasting the measured Iodine-131 (I^{131}) concentrations in the neutral situation with the Gaussian, projected models. The suggested model is more accurate than the Gaussian model in terms of the observed concentration. Additionally, the observed concentrations were within a factor of two of all the suggested and Gaussian concentrations.

Keywords: Advection-Diffusion Equation; Dispersion Parameter; Neutral Condition; Gaussian-Plume Model; and Iodine-131 (I^{131}).

1. Introduction

In order to determine the movement of contaminants in the atmosphere, researchers looked into the advection-diffusion equation by [1]. The efficacy of dry deposition of the analytical dispersion model on the ground surface was investigated by [2]. The impact of vertical turbulence on the advection-diffusion equation's behavior was looked by [3]. Using the Hankel transform and the vertical turbulence and wind speed as variables, was solved the diffusion equation in two dimensions by [4]. The diffusion problem was recently solved by [5] utilizing the separation technique and the Hankel Transform to compare two analytical approaches. The diffusion equation's Gradient Transport (K) and the Gaussian plume model were solved by [6]. Analytical and numerical solutions of concentration with deposition under unstable condition was studied by [7]

This work compares the observed concentrations of I^{131} under neutral conditions with the solution of the diffusion equation using a Gaussian plume model in three dimensions with varying vertical turbulent and wind speed as a function of vertical and horizontal distance.

2. Methods and Techniques

The vertical eddy diffusivity in the neutral air surface layer is observed to increase linearly with height, meaning that

$$K_z = ku_* z = c\bar{u}_r z \tag{1}$$

where, $\bar{u}_r$ is the wind speed at 10m and $c = ku_*/\bar{u}_r$ is a dimensionless coefficient associated with the surface drag coefficient $C_D = \bar{u}_*/\bar{u}_r$. The diffusion equation can be solved for various source types using the eddy diffusivity specification given above. An instantaneous planar (infinite area) source at the ground is the most straightforward scenario, and the solution is calculated as follows:

$$\bar{C}(z,t) = \frac{Q_{ia}}{ku_* t} \exp\left(\frac{-z}{ku_* t}\right) \tag{2}$$

The reflecting boundary requirement at the surface is satisfied by this.

Assuming a uniform wind profile and allowing diffusion in the x-direction to be omitted in contrast to advection by mean flow, a similar solution is achieved for an infinite continuous crosswind line source. The following is the solution [8,9]:

$$\bar{C}(x,z) = \frac{Q_{cl}}{ku_* x} \exp\left(\frac{-\bar{u}z}{ku_* x}\right) \tag{3}$$

The Gaussian plume model in three dimensions is given as follows:

$$\bar{C}(x,y,z) = \frac{1}{2\pi\sigma_y}\frac{Q_{cl}}{ku_* x} \exp\left(\frac{-y^2}{2\sigma_y{}^2}\right) \exp\left(\frac{-\bar{u}z}{ku_* x}\right)\exp\left(\frac{-\nu x}{u}\right) \tag{4}$$

where, ν=9.95*10^{-7} s^{-1} of iodine-131, $\exp(-\nu x/u)$ is the radioactive decay constant, σ_y is the lateral dispersion parameter, and Q_{cl} is the emission rate of a continuous line source.

In two dimensions, the diffusion equation is as follows:

$$u\frac{\partial C_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left[k_z \frac{\partial C_y(x,z)}{\partial z}\right] \tag{5}$$

where, u is the wind speed, K_z is the vertical turbulent parameter, and C_y (*x*, *z*) is the two-dimensional lateral concentration. The vertical turbulent parameter is calculated as a function of linear vertical height.

The following conditions are used to estimate Eq. (5):

No flux at the top of the planetary boundary layer or at its surface:

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$$\frac{\partial C_y(x,z)}{\partial z} = 0 \qquad at\ z = 0, h \qquad (a)$$

where, h is the height of mixing, and the mass that remains is:

$$uc\ (0, z) = Q\ \delta\ (z - h_s) \qquad (b)$$

where, the emission rate at stack height "h_s" is denoted by Q.

The vertical turbulent parameter is as follows:

$$k_z = \alpha\ x\ z \qquad 0 \le h_s \le h \qquad (c)$$

where, u is the wind velocity at 10m and w_* is the vertical convective velocity, with $\alpha = 0.31\left(\frac{w_*}{u}\right)^2$.

Then, three-dimensional concentration is as follows:

$$C(x,y,z) = \frac{1}{(\sqrt{2\pi}\sigma_y)} C_y(x,z) e^{\frac{-y^2}{\sigma_y^2}} \qquad (6)$$

The crosswind standard deviation, σ_y, is determined using Table (1).

Table 1: Contains the lateral dispersion parameter σ_y in all stabilities

Stability classes	Values of σ_y
A	$\sigma_y = 0.40x^{0.91}$
B	$\sigma_y = 0.40x^{0.91}$
C	$\sigma_y = 0.36x^{0.86}$
D	$\sigma_y = 0.32x^{0.78}$

Changing z to ξ using, $\xi = z^{\frac{1}{2}}$:

Eq. (5) is given as:

$$\frac{\partial^2 C_y(x,z)}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C_y(x,z)}{\partial \xi} - \frac{4u}{\alpha x}\frac{\partial C_y(x,z)}{\partial x} = 0 \qquad (6a)$$

The Hankel Transform is used to estimate Eq. (6a) as follows:

$$\mathcal{H}_m\left[f(z)\right] = \tilde{f}(s) \equiv \int_0^\infty f(z) J_m(sz) z\ dz$$

The estimation of the Bessel differential is as follows:

$$\Delta_m f(z) \equiv \frac{d^2 f(z)}{dz^2} + \frac{1}{z}\frac{df(z)}{dz} - \left(\frac{m}{z}\right)^2 f(z)$$

where, the following is the Hankel Transform:

$$\mathcal{H}_m\left[\Delta_m f(z)\right] \equiv -s^2 \tilde{f}(s)$$

when Eq. (6) is subjected to the Hankel Transform, the following is discovered:

$$\mathcal{H}_0\left[\Delta_0 C_y(x,z) = \frac{4u}{\alpha x}\frac{\partial C_y(x,z)}{\partial x}\right] \qquad (7)$$

One can get:

$$-s^2 c(x,z) = \frac{4u}{\alpha x}\frac{\partial c(x,z)}{\partial x} \qquad (8)$$

$$\tilde{C}(s,x) = \tilde{C}(0,s)\ exp\left(-\frac{\alpha}{4u}(xs)^2\right)$$

from, the "b" boundary condition.

$$uC_y\ (0, z) = Q\ \delta\ (z - h_s) \qquad (b)$$

$$\check{C}(s,x) = \frac{1}{2}\frac{Q}{u} J_m\left(s h_s^{\frac{1}{2}}\right) \exp\left(-\frac{\alpha}{8u} x^2 s^2\right)$$

The following is the inverse of the Hankel transform:

$$\mathcal{H}_m^{-1}\left[\check{C}(s,x)\right] = C(x,z) \equiv \int_0^\infty \check{C}(s,x) J_m(sz) s\ ds$$

Consequently, the crosswind integrated concentration looks like this:

$$C_y(x,z) = \frac{Q}{\alpha x^2} exp\left(-u\frac{h_s+z}{\alpha x^2}\right) I_0\left[2u\frac{(zh_s)^{\frac{1}{2}}}{\alpha x^2}\right] \qquad (9)$$

where, I_0 is a zero-order Bessel function. One can obtain the following by substituting Eq. (9) in Eq. (6):

$$C(x,y,z) = \frac{Q}{\alpha x^2 \sigma_y \sqrt{2\pi}} e^{\frac{y^2}{2\sigma_y^2}} exp\left(-u\frac{h_s+z}{\alpha x^2}\right) I_0\left[2u\frac{(zh_s)^{\frac{1}{2}}}{\alpha x^2}\right] e^{-\frac{vx}{u}} \qquad (10)$$

In order to obtain the concentration in three dimensions, the vertical turbulence parameter is assessed by [10] as a function of vertical height and downwind distance. The radioactive decay of I^{131} is represented by, $e^{-vx/u}$, where v =9.95*10^{-7} s^{-1}. I_0 is an order zero Bessel function.

3. Results

The meteorological data is provided in Table (2) based on I^{131} experimental data from [11,12]. Table (3) lists the three-dimensional Gaussian, suggested, and observed models. Figs (1 and 2) show the measured, suggested, and three-dimensional Gaussian model.

Table 2: Provides an overview of the weather during the experiments.

Exp.	u_{27} (m/s)	Atmospheric Stability	L(m)	u_*(m/s)	Mixing height (m)
1	5.80	D	∞	0.67	2680

Table 3: Predicted and observed I^{131} concentrations under neutral conditions

Distance (m)	Observed (Bq/m3)	Predicted (Bq/m3)	Gaussian (Bq/m3)
100	4.1	4.4	3.3
110	3.8	3.8	3.0
120	3.8	3.7	2.8
130	3.7	3.6	2.6
140	3.4	3.4	2.4
150	3.2	3.1	2.3
160	3.1	3.0	2.2
170	3	2.9	2.0

180	2.9	2.7	1.9
190	2.7	2.6	1.8
200	2.4	2.4	1.8
300	1.4	1.3	1.2
400	0.5	0.5	0.9

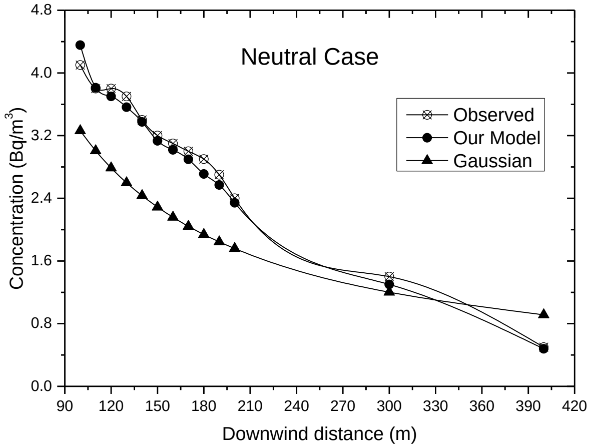

Fig. 1: Shows the differences between the measured and suggested I^{131} (Bq/m^3) concentrations over downwind distances (m).

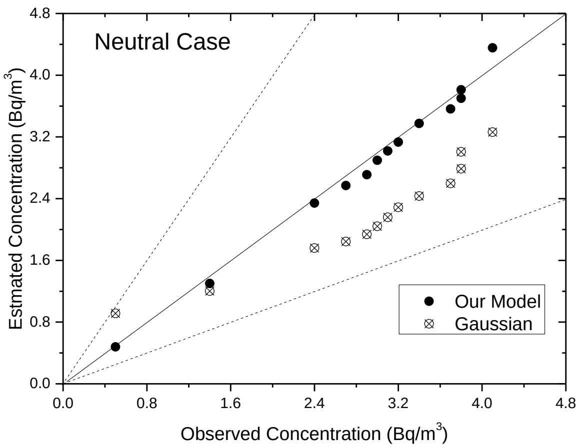

Fig. 2: Shows the correlation between the measured and suggested levels of I^{131}.

Table 4: shows the statistical differences between the concentrations measured in the neutral case and the suggested Gaussian model.

Experiments	NMSE	FB	COR	FAC2
Present model - Eq. (10)	0.002	0.02	0.99	0.98
Gaussian model - Eq. (4)	0.12	0.30	0.96	0.81

FB stands for fraction bias, COR for correction, FAC2 for factor of two, and NMSE for normalized mean square error.

4. Discussion

Compared to the Gaussian model, the suggested model is closer to the measured concentration, as seen in Fig. (1). Additionally, all of the Gaussian and suggested concentrations fell within a factor of two of the observed concentrations, as illustrated in Fig. (2).

Based on the observed data, the suggested and Gaussian models obtained 98% and 81%, respectively. As indicated in Table (4), the NMSE, FB, and COR of the suggested model are superior to those of the Gaussian plume model.

5. Conclusions

The suggested model is more accurate than the Gaussian model in terms of the observed concentration. Additionally, the observed concentrations were within a factor of two of all the suggested and Gaussian concentrations.

Based on the observed data, the suggested and Gaussian models obtained 98% and 81%, respectively. The suggested model's NMSE, FB, and COR is superior to those of the Gaussian plume model.

6. List of abbreviations

EAEA: Egyptian Atomic Energy Authority.

NMSE: the normalized mean square error.

FB: fraction bias.

COR: Correction.

FAC2: factor of two.

I_0 is a Bessel function of order zero.

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