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Biconjugate Gradient Stabilized and V-cycle Multigrid Methods for Discretized Helmholtz Equation: A Comparative Study

N. H. Sweilam[1,], Ibrahim M. Hanafy[2], Moutaz Ramadan[2,3], and Azza Eseily[2]*

[1]Mathematics Department, Faculty of Science, Cairo University, Cairo, Egypt
[2]Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt
[3]Department of Mathematics, College of Sciences, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia

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Abstract: In this work, a comparison between two numerical techniques for solving the discretized scheme of Helmholtz equation is conducted: the first technique is the V-cycle multigrid method combined with the Generalized Minimal Residual Method as a smoother and the second technique is the Biconjugate Gradient Stabilized Method. On the other side, Helmholtz equation will be discretized using the finite difference method, converting the continuous problem to a linear system of equations. The multigrid method leverages a hierarchical, multi-level framework to accelerate convergence by addressing errors across various spatial scales, while Biconjugate Gradient Stabilized Method is an independent iterative technique known for its stability and convergence efficiency. Through this comparative study, we evaluate the performance of both approaches based on the convergence rate, computational cost, and iteration count to achieve a specified accuracy. Numerical experiments are conducted across multiple grid sizes to assess effectiveness.

Keywords: Helmholtz equation; Finite difference method; V-Cycle multigrid method; Generalized minimal residual method; Biconjugate gradient stabilized method.

1 Introduction

One of the important equations in the applied mathematics is Helmholtz equation, it represents a significant type of partial differential equations (PDEs) widely used in modeling time-harmonic phenomena such as wave propagation, acoustics, electromagnetics, and elasticity. Applications include seismic wave modeling, acoustic scattering, electromagnetic wave transmission, and noise suppression. Numerical solutions often involve discretization, leading to large, sparse linear systems, particularly for high-frequency problems with fine grid resolutions and large wave numbers. While traditional methods struggle with slow convergence, modern techniques, though more effective, can be computationally expensive. These challenges are especially significant in fields like exploration seismology and acoustic scattering as in [1].

The three-dimensional Helmholtz equation is defined as follows:

$$\begin{cases} \nabla^2 u(x,y,z) + \lambda^2 u(x,y,z) = f(x,y,z), & \text{in } R, \\ u(x,y,z) = g(x,y,z), & \text{on } \partial R, \end{cases} \tag{1}$$

where R is a square domain in 2D or a cubic one in 3D, ∂R represents its boundary,λ is the wave number, $f(x,y,z)$ is the forcing function and $g(x,y,z)$ is the boundary function,The Laplacian operator in three dimensions is given by:

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The exact solution $u(x,y,z)$, along with the functions $f(x,y,z)$ and $g(x,y,z)$, is assumed to be continuously differentiable. In this study, a finite-difference approximation on uniform grids is employed to investigate the numerical behavior of equation (1), where the step sizes are taken to be equal in all directions, i.e.,

* Corresponding author e-mail: nsweilam@sci.cu.edu.eg

$\Delta x = \Delta y = \Delta z$. For further details on such numerical schemes, see [2] and the references therein.

Multigrid methods have been widely studied for solving a linear system derived from an elliptic PDE, see for example [3,4,5,6,7]. Many traditional iterative methods significantly reduce the variability of the approximation error. Although the error itself might not be small, it becomes smoother. This is referred to as smoothing, and it represents the fundamental concept of MG methods. The second principle is coarse grid correction, which involves representing smooth features from a fine grid on a coarser grid with high accuracy using suitable techniques. When applying MG techniques, especially in the context of finite difference problems, is determining the minimum number of grid points per wavelength on the coarsest grid. This parameter is critical for ensuring solution accuracy and stability. Classical MG theory has been extensively discussed in references such as [8,9]. Early studies [10, 11,12,13] demonstrated that standard MG techniques can effectively solve Helmholtz equation when the wave number l is relatively small compared to the mesh size. However, it requires the coarsest grids in the hierarchy to resolve waves associated with l.

The V-cycle is strategy used in MG methods to accelerate the solution process by smoothing the error, particularly for large-scale problems like Helmholtz equations. This iterative method is refined through a series of smoothing and correction steps applied across different grid levels. In the V-cycle, the process begins with smoothing on the finest grid to reduce high-frequency errors. Residuals, which represent the difference between the left- and right-hand sides of the discretized equations at a given iteration, are subsequently transferred to successively coarser grids. On the coarser grids, the problem is approximated, and the solution is corrected. After the correction step, the solution is interpolated back to finer grids, where further smoothing is performed. This iterative process is carried out in a "V" shape, with steps moving from fine grids down to coarse grids and back up to the fine grids. Each traversal of the V-cycle helps reduce errors at different scales, accelerating convergence discussed in [14,15].

The Biconjugate Gradient Stabilized Method (Bi-CGSTAB) method is a highly significant and widely used algorithm for solving linear systems of equations. Numerous efforts have been made to develop more efficient methods by re-structuring the original Bi-CG algorithm as in [16,17]. Among these, one of the most notable advancements was introduced by van der Vorst in [18], resulting in the Bi-CGSTAB method, which has become one of the most successful improvements to the Bi-CG approach.

The most important step in some of iterative methods like Generalized minimal residual method (GMRES) focus on the residual $r_k = b - Ax_k$, where x_k represents the k^{th} approximation of the solution to equation $Ax = b$. These methods aim to reduce the sequence of residual norms. GMRES, in particular, is a widely used for solving linear systems of equations [19]. Various implementations of GMRES have been developed, each designed to achieve specific goals, with their own strengths and limitations. In this context, different GMRES variants are applied to ill-posed linear problems to evaluate their effectiveness in solving nearly singular systems and to identify those that are less suitable.

2 Finite Difference Method

PDEs play a crucial role in addressing various scientific challenges, including Boundary Value Problems (BVPs). This paper focuses on the numerical solutions for Helmholtz equations (1). We begin by considering a square unit domain. Let the grid spacing be $h = \frac{b-a}{n}$, and define $x_i = ih$, $y_j = jh$, and $z_k = kh$ where $(i, j, k = 1, 2, \ldots, n)$. Equation (1) can be approximated at the grid point (x_i, y_j, z_k) using the widely adopted full-sweep finite difference (FD) approximation method, yielding the corresponding approximation equation [20]:

$$\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{h^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{h^2} + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{h^2} + l^2 u_{i,j,k} = f_{i,j,k}. \tag{2}$$

$$u_{i,j,k} = \frac{1}{-6 + h^2 l^2}\Big(h^2 f_{i,j,k} - u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - u_{i,j,k-1}\Big). \tag{3}$$

For $1 \le i \le m-1$, $1 \le j \le n-1$, and $1 \le k \le l-1$, the boundary conditions are introduced by:

$$\begin{aligned} u_{0,j,k} &= g(a, y_j, z_k), & u_{m,j,k} &= g(b, y_j, z_k), \\ u_{i,0,k} &= g(x_i, a, z_k), & u_{i,n,k} &= g(x_i, b, z_k), \\ u_{i,j,0} &= g(x_i, y_j, a), & u_{i,j,l} &= g(x_i, y_j, b). \end{aligned} \tag{4}$$

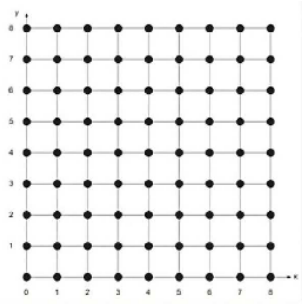


Fig. 1: Uniform mesh grid for a 2D finite grid network with $N = 8$.

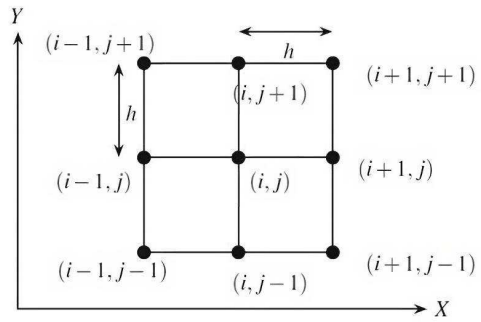
The algebraic equations for Problem (3) are derived using a second-order central finite difference scheme. The solution domain in two dimensions is uniformly divided in all directions with a constant mesh size h, defined as:

$$\Delta h = \frac{b-a}{N}.$$

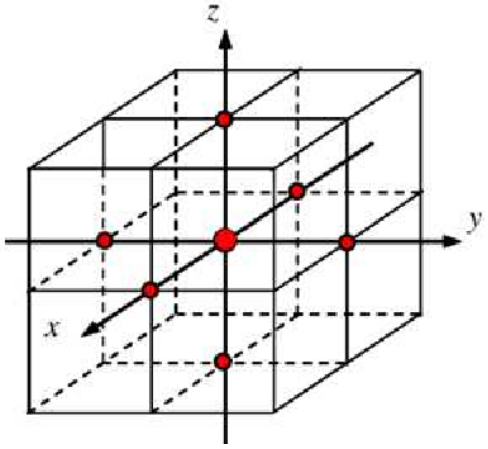
When discretizing the Helmholtz equation, we obtain a linear system of equations of the form:

$$Ax = b, \tag{5}$$

where the matrix A is typically sparse and indefinite. The vector x is the unknown vector whose values are to be determined, and b is the right-hand side vector, representing the outcome or constants of the system.



(a)



(b)

Fig. 2: (a) 2D centered finite difference approximation with 9-point stencil. (b) 3D uniform grid with 7-point stencil.

The linear system (5) will be solved using two iterative methods: the V-cycle multigrid method, which employs the MG with GMRES as smoother (MG-GMRES), and the BiCGSTAB method. The MG method is a multi-level strategy designed to accelerate convergence by addressing errors at different spatial scales. In contrast, BiCGSTAB is an independent iterative solver that works directly on the linear system and is widely used due to its stability and excellent convergence properties.

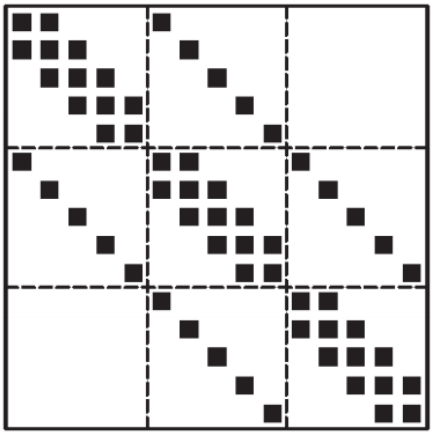


Fig. 3: Sparsity pattern of the matrix A.

3 The Multigrid Method (V-cycle)

The results from the second-order finite-difference scheme can be efficiently solved using MG methods, particularly in sparse linear systems. To eliminate high-frequency errors, MG methods employ relaxation techniques. These methods also utilize coarse grid corrections to smooth out the errors. A successful implementation of the MG method for solving the 2D and 3D Helmholtz equation discretized by standard second-order finite-difference schemes has been demonstrated in previous studies [21].

In the MG-GMRES is used as the smoother instead of Gauss-Seidel relaxation. GMRES is an iterative method that minimizes the residual over the Krylov subspace, which makes it an effective smoother, especially for problems with poor conditioning. Bilinear interpolation is used to transfer corrections from the coarse grid to the fine grid, and a full-weighting scheme is applied to update the residuals on the coarse grid.

The MG method is an iterative technique for solving systems of equations derived from the discretization of elliptic PDEs. It is founded on two fundamental principles [22,23,24].The first principle highlights the strong error-smoothing capability of many traditional iterative methods, such as Jacobi and Gauss-Seidel. These methods efficiently eliminate the high-frequency of the error within small number of iterations, while leaving the smoother, low-frequency components largely unaffected.

These classical methods are slow to converge as a result of their inability to eliminate these smooth error components quickly. The coarse-grid principle is the second principle asserts that smooth error components on a fine grid can be accurately represented on a coarser grid without significant loss of information. These two strategies where a coarse grid is employed to address the smooth error effectively. This process begins by relaxing the system on the fine grid with the help of a smoother. After a certain number of iterations, relaxation is applied, and it slows down, indicating that smooth error components become more dominant. At this point, the process transitions to a coarser grid, where the smoother can more efficiently address these components due to their increased oscillatory nature. This approach forms the basis of the coarse-grid correction scheme, which is applied recursively in the MG method.

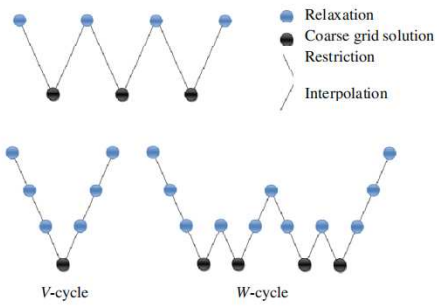


Fig. 4: Comparison of the V-cycle and W-cycle in MG methods, illustrating the transition between fine and coarse grids. The blue nodes represent relaxation steps, downward arrows indicate restriction, and upward arrows denote interpolation.

4 The GMRES Method.

In 1986, the GMRES method was designed for solving linear systems of equations and was introduced in [25] by Saad and Schultz. They presented an efficient formulation known as the "Standard GMRES," which is based on the Arnoldi process [26]. Since then, various GMRES variants have been developed, focusing on simplicity, stability, or speed, and some of these implementations are briefly discussed in this section. The GMRES algorithm typically consists of two primary steps: First, the Arnoldi process is a numerical method used to generate an orthogonal basis, and second, a least squares problem is solved to refine the approximation using the orthogonal vectors produced. To solve a system like (5), GMRES starts with initial approximation $x_0 \in \mathbb{R}^n$ and the k^{th} iteration in GMRES is represented as $x_k = x_0 + z_k$, where z_k minimizes the residual norm $\|r_k\|$. This can be expressed as:

$$\|r_k\| = \|r_0 - Az_k\| = \min_{z \in x_0 + \mathscr{K}_k(r_0)} \|r_0 - Az\|. \tag{6}$$

Where $r_0 = b - Ax_0$ and $\mathscr{K}_k(v) = \text{span}\{v, Av, \ldots, A^{k-1}v\}$

In the first step, GMRES generally uses the Arnoldi process to construct a set of basis vectors for the Krylov subspace as $K_{k+1}(r_0)$:

Arnoldi Process – Modified Gram–Schmidt (Algorithm 1)

Start with a vector v_1 such that $\|v_1\| = 1$

For $j = 1, \ldots, k$:

Compute $v_{j+1} = Av_j$ and for $i = 1, \ldots, j$,

$h_{i,j} = \langle v_{j+1}, v_j \rangle$, $v_{j+1} = v_{j+1} - h_{i,j}v_j$. End

Set $h_{j+1,j} = \|v_{j+1}\|$; then normalize v_{j+1} as $v_{j+1} = v_{j+1}/h_{j+1,j}$. End

In short, steps 2a and 2b are represented as

$$v_{j+1} = \Pi_j^{\perp} Av_j / \|Av_j\|.$$

From this process, the following significant relationship is derived:

$$AV_k = V_{k+1}\bar{H}_k. \tag{7}$$

The GMRES method relies on the fact that the columns of V_{k+1}, namely $v_1, v_2, \ldots, v_k$ form an orthogonal and normalized vectors for the subspace $K_k(r_0)$. The matrix of Hessenberg $\bar{H}_k = h_{i,j} \in \mathbb{R}^{(k+1)\times k}$ represents the matrix form of A in the subspace $K_k(v_1)$. With respect to V_k. From equations (6) and (7), the fundamental formula for GMRES is derived as:

$$\min_{z \in x_0 + K_k(r_0)} \|r_0 - Az\| = \min_{y \in \mathbb{R}^k} \|r_0 - A\tilde{V}_k y\| = \min_{y \in \mathbb{R}^k} \|\beta e_1 - \bar{H}_k y\|. \tag{8}$$

Let $\beta = \|r_0\|$. It is important to note that the Arnoldi process fails at step k if and only if $h_{k+1,k} = 0$, indicating that A is the singular matrix. If y_k minimizes the right side of the least squares problem in equation (8), then $z_k = V_k\tilde{y}$ is the optimal solution on the left side of (8) within the Krylov subspace $K_k(r_0)$.

The GMRES (Algorithm 2):

Start with an initial guess x_0, compute $r_0 = b - Ax_0$, and set $v_1 = r_0/\|r_0\|$.

Construct $(k+1)$ orthogonal vectors $v_1, v_2, \ldots, v_{k+1}$ to serve as a basis $K_{k+1}(r_0)$.

Solve the least squares and find $\tilde{y} \in \mathbb{R}^k$.

Update $x_k = x_0 + V_k\tilde{y}$.

If it does not meet the desired criteria, set $x_0 = x_k$ and return to step 1.

The GMRES algorithm starts with an initial guess x_0. By orthogonalizing the vector v_{k+1} with respect to $v_1, v_2, \ldots, v_k$ and by solving a least squares problem, the subsequent GMRES approximation is determined, recursively reducing the residual norms.

5 Multigrid V-Cycle with GMERS Smoother

The MG V-cycle algorithm is an efficient solver for discretized PDEs. In this variant, the GMERS smoother is employed for pre- and post-smoothing to effectively damp high-frequency error components. The algorithm operates recursively on a hierarchy of grids.

A_h: Matrix on the fine grid Ω_h.
b_h: Right-hand side vector on the fine grid Ω_h.
α_1: Pre-smoothing iterations.
α_2: Post-smoothing iterations.
μ: Number of recursive V-cycles on coarser grids.
v_0: Initial guess for the solution v_h.
v_h: Approximate solution on the fine grid Ω_h.

Algorithm 3: Multigrid V-Cycle with GMERS Smoother

Step 1: Pre-smoothing

Relax α_1 times on the equation $A_h u_h = b_h$, on Ω_h using the GMERS smoother with an arbitrary initial guess v_0, and compute the updated approximation v_h.
Relax: $v_h \leftarrow \text{GMERS}(A_h, b_h, v_0, \alpha_1)$.

Step 2: Coarse-Grid Correction

2.1. Compute the residual:

Calculate the residual r_h as:

$$r_h = b_h - A_h v_h \tag{9}$$

2.2. Restrict the residual to the coarser grid:

Restrict r_h to the coarser grid using the restriction operator R_h^{2h}:

$$r_{2h} = R_h^{2h} r_h \tag{10}$$

2.3. Solve the coarse grid residual equation:

Solve the equation $A_{2h} e_{2h} = r_{2h}$:
If Ω_h is the coarsest grid, solve directly: $e_{2h} = (A_{2h})^{-1} r_{2h}$.
Otherwise, initialize $e_{2h} = 0$ and perform recursive calls to the MG method for μ times:
$e_{2h} = \text{MGM}(A_{2h}, r_{2h}, \alpha_1, \alpha_2, \mu, e_{2h})$.

2.4. Interpolate the error to the fine grid:

Use the interpolation operator P_h^{2h} to transfer the error e_{2h} back to the fine grid:

$$e_h = P_h^{2h} e_{2h} \tag{11}$$

2.5. Correct the fine-grid solution:

Update the fine grid solution v_h by adding the interpolated error e_h:
$v_h \leftarrow v_h + e_h$.

Step 3: Post-Smoothing

Relax α_2 times on the equation $A_h u_h = b_h$, on Ω_h using the GMER smoother with the updated initial guess v_h, and compute the final solution:
Relax: $v_h \leftarrow \text{GMERS}(A_h, b_h, v_h, \alpha_2)$.

6 Biconjugate Gradient Stabilized

The BiCGStab method is an iterative solver designed for large, sparse, and non-symmetric systems of linear equations. Such systems often arise in numerical simulations, particularly when employing discretization techniques like the FDM. FDM transforms partial differential equations into algebraic equations, producing matrices that are typically sparse and large. These characteristics make direct solvers computationally prohibitive, prompting the need for efficient iterative methods like BiCGStab. BiCGStab improves upon the classical Bi-Conjugate Gradient (BiCG) method by addressing its erratic convergence behavior and enhancing numerical stability. BiCGStab stabilizes the iteration process by minimizing the residual in a least-squares sense. The algorithm begins with an initial approximate and iteratively refines the solution by computing residuals, performing sparse matrix-vector operations, and applying a stabilization mechanism. Its effectiveness is further enhanced when coupled with suitable preconditioning techniques, which improve convergence by reducing the condition number of the system. The Bi-CGSTAB method is a highly important and effective algorithm for solving non-Hermitian linear systems of equations [27].

Algorithm 4: Biconjugate Gradient Stabilized

–Initialization:
–Choose an initial guess x_0 for the solution.
–Compute the initial residual: $r_0 = b - Ax_0$.
–Pick an arbitrary vector $\tilde{r}_0$ (for example, $\tilde{r}_0 = r_0$).
–Initialize: $v_0 = p_0 = 0$; $\rho_0 = \alpha_1 = \omega_0 = 1$.
–Iteration: For $n = 1, 2, \ldots$, until convergence:
–Compute the scalar $\rho_n = \langle r_{n-1}, \tilde{r}_0 \rangle$.
–Compute the scalar $\beta_n = \left(\frac{\rho_n}{\rho_{n-1}}\right)\left(\frac{\alpha_n}{\omega_{n-1}}\right)$.
–Update the search direction: $p_n = r_{n-1} + \beta_n(p_{n-1} - \omega_{n-1} v_{n-1})$.
–Compute $v_n = A\rho_n$.
–Compute the scalar $\sigma_n = \langle v_n, \tilde{r}_0 \rangle$.
–Compute $\alpha_n = \frac{\rho_n}{\sigma_n}$.
–Update $s_n = r_{n-1} - \alpha_n v_n$.
–Compute $t_n = As_n$.
–Compute $\omega_n = \frac{\langle s_n, t_n \rangle}{\langle t_n, t_n \rangle}$.
–Update the residual $r_n = s_n + \omega_n t_n$.
–Update the solution $x_n = x_{n-1} + \alpha_n p_n + \omega_n s_n$.
–Convergence Check:
–Check the norm of the residual $\|r_0\|$. If it is below a tolerance threshold, stop the iteration.

7 Numerical Tests

To evaluate the accuracy of the current approach, we consider three model problems governed by the Helmholtz equation. The systems of equations derived from the difference scheme are solved using both the Bi-CGSTAB and MG-GMRES methods. Furthermore, we provide plots of the numerical errors and the solutions for these model problems. There are parameters such as the number of iterations (Iter.No), computational time in

seconds (CPU) are taken into account for numerical comparison. Throughout the simulations, the convergence test was conducted with a tolerance error of $\varepsilon = 10^{-10}$ applied to various values of the number of grid points N. We employ the L_2 and L_∞ error norms to methodically measure the difference between the numerical and analytical solutions. This method enables a thorough evaluation of the numerical solution's accuracy in representing the system's behavior. The definitions of the L_2 and L_∞ norms of the solution are as follows:

$$L_2 = \|U^{\text{exact}} - U^n\|_2 = \left[\sum_{i=0}^{N} \left|U_i^{\text{exact}} - U_i^n\right|^2\right]^{1/2}. \quad (5)$$

$$L_\infty = \|U^{\text{exact}} - U^n\|_\infty = \max_i \left|U_i^{\text{exact}} - U_i^n\right|. \quad (6)$$

Example 1

Consider the following two-dimensional Helmholtz-type equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - U = 0, \quad 0 < x < 1, \quad 0 < y < 1. \quad (7)$$

The boundary conditions are given by:

$$\begin{aligned} &U(x,0) = x, \quad U(x,1) = e^x + x\cosh(1), \\ &U(0,y) = y, \quad U(1,y) = ye^1 + \cosh(y). \end{aligned} \quad (8)$$

The exact solution for this problem is:

$$U(x,y) = ye^x + x\cosh(y). \quad (9)$$

Table 1: Computational results for Example 1.

N	Methods	Iter. No	CPU (s)	L_∞	L_2
32	BiCGSTAB	72	0.084	9.13×10^{-6}	1.52×10^{-4}
	MG-GMRES	2	0.011	9.13×10^{-6}	1.53×10^{-4}
64	BiCGSTAB	140	0.126	1.96×10^{-6}	6.22×10^{-5}
	MG-GMRES	3	0.068	1.69×10^{-6}	5.46×10^{-5}
128	BiCGSTAB	270	0.386	2.20×10^{-6}	9.79×10^{-5}
	MG-GMRES	4	0.278	3.76×10^{-7}	2.38×10^{-5}
256	BiCGSTAB	594	2.807	4.22×10^{-6}	3.95×10^{-4}
	MG-GMRES	4	1.921	8.28×10^{-8}	1.07×10^{-5}
512	BiCGSTAB	930	12.80	2.87×10^{-5}	5.48×10^{-4}
	MG-GMRES	6	8.470	3.36×10^{-8}	9.18×10^{-6}

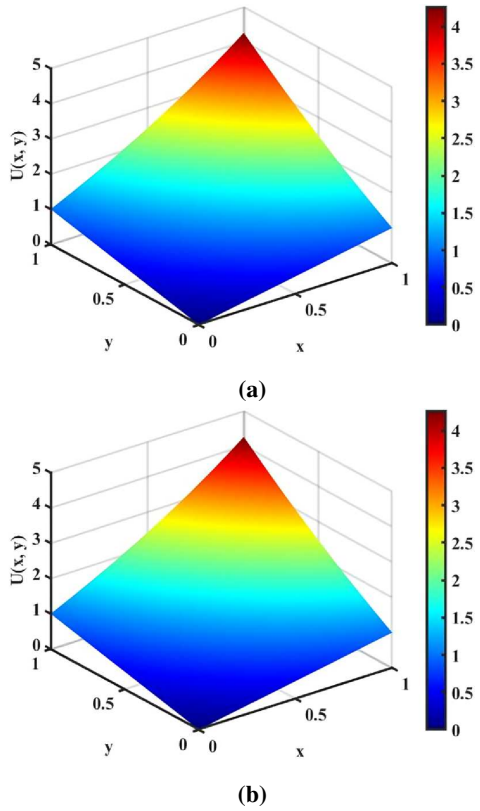


Fig. 5: (a) Numerical solution using MG-GMRES for $N = 512$. (b) Corresponding exact solution.

Example 2

Consider the following two-dimensional Helmholtz-type equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + U = x, \quad 0 < x < 1, \quad 0 < y < 1. \quad (10)$$

The boundary conditions are given by:

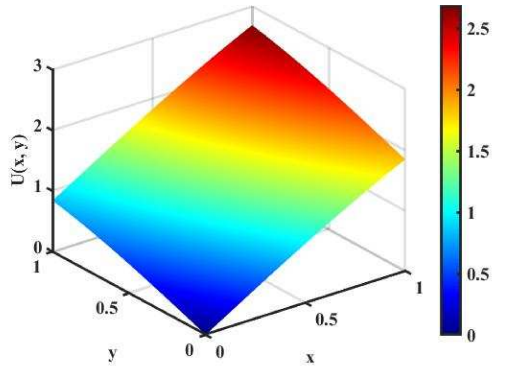
$$\begin{aligned} &U(x,0) = \sin(x) + x, \quad U(x,1) = \sin(x) + \sin(1) + x, \\ &U(0,y) = \sin(y), \quad U(1,y) = \sin(1) + \sin(y) + 1. \end{aligned} \quad (11)$$

The exact solution for this problem is:

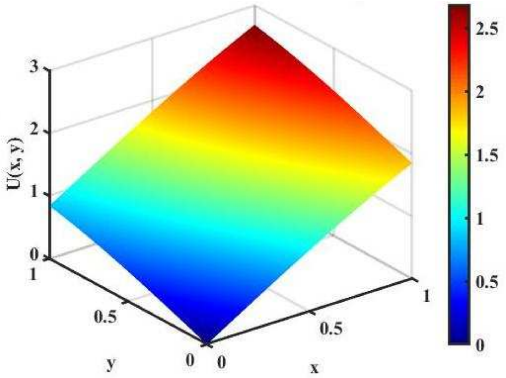
$$U(x,y) = \sin(x)\sin(y) + x. \quad (12)$$

Table 2: Computational results for Example 2.

N	Methods	Iter. No	CPU (s)	L_∞	L_2
32	BiCGSTAB	65	0.082	5.722×10^{-6}	1.017×10^{-4}
	MG-GMRES	4	0.011	5.193×10^{-6}	9.097×10^{-5}
64	BiCGSTAB	134	0.106	1.329×10^{-7}	3.839×10^{-5}
	MG-GMRES	3	0.034	8.592×10^{-7}	2.626×10^{-5}
128	BiCGSTAB	245	0.303	2.379×10^{-6}	1.152×10^{-4}
	MG-GMRES	4	0.289	3.402×10^{-7}	2.316×10^{-5}
256	BiCGSTAB	470	1.041	8.910×10^{-6}	9.111×10^{-4}
	MG-GMRES	4	0.871	6.942×10^{-8}	9.141×10^{-6}
512	BiCGSTAB	918	9.328	1.148×10^{-5}	4.230×10^{-4}
	MG-GMRES	5	8.644	2.080×10^{-8}	5.614×10^{-6}



(a)



(b)

Fig. 6: (a) Numerical solution using MG-GMRES for $N = 512$. (b) Corresponding exact solution.

Example 3

Consider the 3D partial differential equation:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+5U=0,\quad 0<x<1,\ 0<y<1,\ 0<z<1. \tag{13}$$

Subject to the boundary conditions:

$$U(x,0,z)=z\cos(\sqrt{5}x), \tag{14}$$
$$U(x,1,z)=(1+z)\cos(\sqrt{5}x), \tag{15}$$
$$U(0,y,z)=y+z, \tag{16}$$
$$U(1,y,z)=(y+z)\cos(\sqrt{5}x), \tag{17}$$
$$U(x,y,0)=y\cos(\sqrt{5}x), \tag{18}$$
$$U(x,y,1)=(1+y)\cos(\sqrt{5}x). \tag{19}$$

The exact solution is:

$$U(x,y,z)=y+z+\cos(\sqrt{5}x). \tag{20}$$

Table 3: Computational results for Example 3.

N	Methods	Iter. No	CPU (s)	L_∞	L_2
8	BiCGSTAB	25	0.020	9.440×10^{-4}	2.901×10^{-4}
	MG-GMRES	4	0.011	7.225×10^{-4}	2.103×10^{-4}
16	BiCGSTAB	49	0.031	2.719×10^{-4}	8.870×10^{-5}
	MG-GMRES	2	0.029	1.951×10^{-4}	5.359×10^{-5}
32	BiCGSTAB	96	0.290	7.256×10^{-5}	2.462×10^{-5}
	MG-GMRES	5	0.248	5.108×10^{-5}	1.597×10^{-5}
64	BiCGSTAB	182	3.420	1.872×10^{-5}	6.486×10^{-5}
	MG-GMRES	15	2.920	1.519×10^{-5}	4.990×10^{-6}

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+k^2U=0,\quad 0<x<1,\ 0<y<1,\ 0<z<1. \tag{21}$$

The boundary conditions are given by:

$$U(x,0,z)=\cos(2x)\cdot\frac{\sinh(\sqrt{5}z)}{\sinh(\sqrt{5}\pi)}, \tag{22}$$
$$U(x,1,z)=\cos(2x)\cos\left(\sqrt{1+k^2}\right)\cdot\frac{\sinh(\sqrt{5}z)}{\sinh(\sqrt{5}\pi)}, \tag{23}$$
$$U(0,y,z)=\cos\left(\sqrt{1+k^2}\cdot y\right)\cdot\frac{\sinh(\sqrt{5}z)}{\sinh(\sqrt{5}\pi)}, \tag{24}$$
$$U(1,y,z)=\cos(2)\cos\left(\sqrt{1+k^2}\cdot y\right)\cdot\frac{\sinh(\sqrt{5}z)}{\sinh(\sqrt{5}\pi)}, \tag{25}$$
$$U(x,y,0)=0, \tag{26}$$
$$U(x,y,1)=\cos(2x)\cos\left(\sqrt{1+k^2}\cdot y\right)\cdot\frac{\sinh(\sqrt{5})}{\sinh(\sqrt{5}\pi)}. \tag{27}$$

The exact solution for this problem is:

$$U(x,y,z)=\cos(2x)\cos\left(\sqrt{1+k^2}\cdot y\right)\times\frac{\sinh(\sqrt{5}z)}{\sinh(\sqrt{5}\pi)}. \tag{28}$$

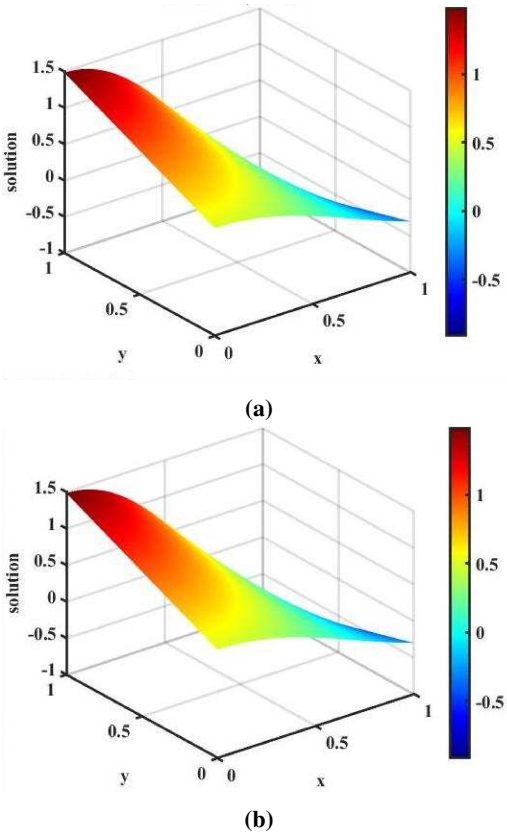


Fig. 7: (a) Numerical solution using MG-GMRES for $N = 64$ and $h = 0.5$. (b) Corresponding exact solution.

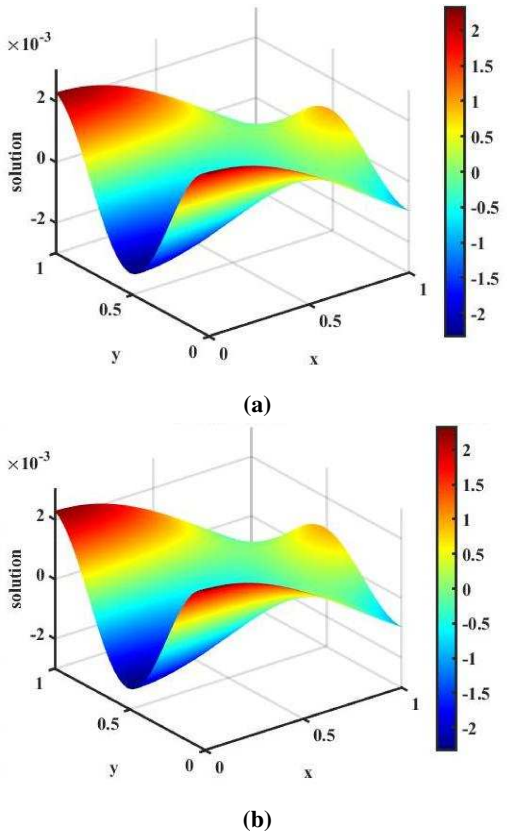


Fig. 8: (a) Numerical solution using BiCGSTAB, $N = 32$, $k^2 = 36$, and $z = 0.5$. (b) Corresponding exact solution.

Table 4: Computational results for Example 4. Here, "div." denotes divergence and "Cond. No." stands for condition number.

N	Methods	k^2	Iter. No	CPU	L_∞	L_2	Cond. No.
32	BiCGSTAB	4	90	0.7	3.657×10^{-7}	9.612×10^{-8}	865.60
	MG-GMRES	4	10	0.4	3.293×10^{-7}	7.668×10^{-8}	–
32	BiCGSTAB	9	100	0.168	6.036×10^{-7}	1.711×10^{-7}	1108.83
	MG-GMRES	9	7	0.2	5.953×10^{-7}	1.639×10^{-7}	–
32	BiCGSTAB	16	104	0.172	2.722×10^{-6}	7.313×10^{-7}	1761.31
	MG-GMRES	16	18	0.61	2.621×10^{-6}	6.860×10^{-7}	–
32	BiCGSTAB	25	110	1.35	1.487×10^{-5}	4.830×10^{-6}	5599.49
	MG-GMRES	25	30	1.5	1.052×10^{-5}	3.229×10^{-6}	–
32	BiCGSTAB	36	132	1.9	1.795×10^{-5}	6.024×10^{-6}	4465.36
	MG-GMRES	36	div.	div.	div.	div.	–

Conclusion

In this paper FDM is applied to derive the discrete scheme of Helmholtz equation, then MG-GMRES and BiCGSTAB methods are used to solve the resultant system of equations. In general, we can conclude that MG-GMRES uses a multilevel approach, where coarse grids are used to handle low-frequency errors and fine grids focus on smoothing high-frequency errors. This combination helps MG-GMRES solve the problem faster and with fewer iterations. On the other hand, BiCGSTAB directly solves the system without using a multilevel strategy. While it is a reliable method, it often requires more iterations and takes longer to achieve the same degree of precision accuracy as MG-GMRES. It is noted that when the wave number is a big number, it is preferable to use BiCGSTAB method because the MG-GMRES method will be diverge. All results, including plots and figures, were generated using MATLAB.

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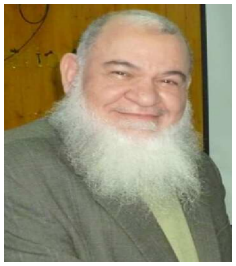
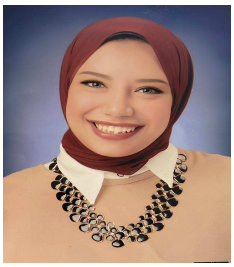
Nasser Sweilam is a Professor of Numerical Analysis at the Department of Mathematics, Faculty of Science, Cairo University, Egypt. He obtained his Ph.D. in 1994 through a channel program between Cairo University and TU-Munich, Germany, focusing on optimal control of variational inequalities. Prof. Sweilam has served as Head of the Mathematics Department at Cairo University (2012–2018), and is Secretary-General of the National Committee of Mathematics (IMU–Egypt). He has supervised over 40 M.Sc. and Ph.D. theses, delivered more than 80 invited lectures, and published over 180 research articles in areas including optimal control, variational inequalities, fractional calculus, and high-performance computing. Among his recognitions are the Cairo University Excellence Award in Mathematics (2012), the Obada Prize (2021), and the 325 Years of Fractional Calculus Award (2020). He is an active member of AMS, EMS, and editorial boards of several international journals.



Ibrahim Hanafy is a Professor of Pure Mathematics (Topology) at the Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Egypt. He is involved in both teaching and research in the fields of general and algebraic topology, and he contributes actively to academic and institutional development within the department.

Moutaz Ramadan is an Assistant Professor of Applied Mathematics at Port Said University, Egypt, and Prince Sattam bin Abdulaziz University, Saudi Arabia. He earned his Ph.D. in 2017 from Port Said University, specializing in optimal control applied to cosmology and computational fluid mechanics. He has broad teaching experience in mathematics and physics-related courses and has supervised several postgraduate theses. Dr. Ramadan has held administrative roles in quality assurance and planning and serves on various academic committees. He has published over 18 research papers in areas such as numerical PDEs, fractional calculus, optimal control, and cosmology. He received a DAAD-funded research grant with Brandenburg University, Germany, and is a member of the Egyptian Mathematical Society and the Syndicate of Scientific Professions.



Azza Eseily is a Demonstrator at the Faculty of Science at Port Said University, Egypt. She specializes in Numerical Linear Algebra and is actively engaged in research related to numerical methods for solving linear systems, particularly in the context of partial differential equations. Her academic interests include iterative solvers, multigrid methods, and their applications in scientific computing.