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Kolmogorov Numbers for Relatively Bounded Operators

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Abstract: In this manuscript, we extend the notion of Kolmogorov numbers of bounded linear operators to a class of unbounded operators, namely; relatively bounded operators with respect to a densely defined closed linear operator T. We get many interesting results about T-Kolmogorov numbers, for example; we show that a T-bounded operator is relatively T-compact if and only if its sequence of T-Kolmogorov numbers converges to zero. Moreover we prove that a T-bounded operator is of finite rank at most n if and only if its nth T-Kolmogorov number vanishes.

Keywords: Relatively bounded operators; relative boundedness; relatively compact operators; Kolmogorov diameters of bounded subsets; Kolmogorov numbers of bounded operators.

1 Introduction

In 1964, the concept of Kolmogorov numbers for bounded linear operators between Banach spaces was introduced by I. A. Novosel'skij based on A. N. Kolmogorov's notion of diameters (see [1], p. 193). These numbers are one of the important examples of s-numbers, which was invented by E. Schmidt in 1907. The sequence of Kolmogorov numbers d_n has many interesting properties. For example, a bounded linear operator is compact if and only if its sequence of Kolmogorov numbers tends to zero. For a brief discussion of the algebraic and analytic properties of d_n, and other characteristics of bounded linear operators; please, see [1]. In this work, we extend the notion of Kolmogorov numbers to a class of unbounded operators, namely; the class of relatively bounded operators with respect to a densely defined closed linear operator. Parallel to the above mentioned fact, we prove that a relatively bounded operator is relatively compact if and only if its sequence of T-Kolmogorov numbers converges to zero. In [2], the compactness of operators was classified according to rate of convergence to zero of its Kolmogorov numbers. We apply these results to relative compactness.

2 Notations, basic definitions and propositions

Let us agree henceforth that X and Y are two Banach spaces, where $\mathscr{B}(X,Y)$ denote the space of all bounded linear operators from X into Y, and let $\dim(F)$ denote the dimension of a given subspace F of Y.

Definition 1[3] *An unbounded operator $T : X \to Y$ with domain $D(T) \subset X$ is a pair $(D(T), T)$, where $D(T)$ is a linear subspace of X, and T is a linear map from $D(T)$ to Y.*

Definition 2[3] *Let T be an operator from X to Y. A sequence $\{x_n\} \subset D(T)$ is called T-convergent to $x \in X$ (and write $x_n \xrightarrow{T} x$) if both $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences (and $x_n \to x$).*

Definition 3[3,4,5] *A linear operator T from X to Y is said to be closed if $x_n \xrightarrow{T} x$ implies $x \in D(T)$, and $Tx = \lim Tx_n$.*

We denote the class of all closed densely defined linear operators from X into Y by $\mathscr{C}(X,Y)$.

Definition 4[3,6] *Let X and Y be Banach spaces and $T \in \mathscr{C}(X,Y)$. Set*

$$\|x\|_T := \|x\| + \|Tx\|, \qquad x \in D(T). \tag{1}$$

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Then, $\|\cdot\|_T$ defines a norm on $D(T)$ which is called the graph norm (or simply T-norm).

Further, it is easy to show that $(D(T), \|\cdot\|_T)$ is a Banach space, the completeness of $D(T)$ is a direct consequence of the closedness of T.

Definition 5[7] *A sequence $\{x_n\} \subset D(T)$ is said to be T-bounded if there exists a constant $c > 0$ such that*

$$\|x_n\|_T \le c \text{ for every } n \in \mathbb{N}.$$

Definition 6[8] *Let T be an operator from X into Y. A T-Cauchy sequence in $D(T)$ is a Cauchy sequence with respect to the T-norm.*

Definition 7[3, 6, 9] *Let T be densely defined closed linear operator with domain $D(T) \subset X$. A linear operator A is called relatively bounded with respect to T or simply T-bounded, if $D(T) \subset D(A)$, and*

$$\|Ax\| \le a\|x\| + b\|Tx\|, \qquad x \in D(T), \tag{2}$$

where a, b are non-negative constants.

The greatest lower bound b_0 of all possible constants b in (2) is called the relative bound of A with respect to T or simply the T-bound of A.

Remark.[7, 10]

1. An unbounded operator A is said to be T-bounded, or relatively bounded if and only if it is bounded with respect to the graph norm.
2. The set of all T-bounded operators forms a vector space.

A notion analogous to relative boundedness is that of relative compactness.

Definition 8[3, 6, 9] *Let T be a closed operator from X to Y, the linear operator A where $D(T) \subset D(A) \subset X$ is said to be relativity compact with respect to T, or simply; T-compact, if for any sequence $\{x_n\}$ in $D(T)$ with both $\{x_n\}$ and $\{Tx_n\}$ are bounded, the sequence $\{Ax_n\}$ contains a convergent sub-sequence.*

Remark.[11] A T-bounded operator A is T-compact if and only if it translates any T-bounded set into a relatively compact set.

Remark.[10] The set $L_c^T(X,Y)$ of all T-compact operators from X to Y forms a vector space.

Remark.[3] If an operator A is T-compact, then it is T-bounded.

For further details about the relatively bounded and relatively compact operators, we refer the reader to [3, 10, 12, 13].

Definition 9[4] *Let K be a subset of a normed space X, for a given $\varepsilon > 0$, a set $M \subset X$ is said to be an ε- net for K if for every point $x \in K$ there is a point $x_\varepsilon \in M$ such that $\|x - x_\varepsilon\| < \varepsilon$.*

Lemma 10[4] *A subset K of a Banach space X is relatively compact (its closure is compact) if and only if for every $\varepsilon > 0$, K has a finite ε-net.*

2.1 The n-th Kolmogrov diameter of a bounded subset of a normed space

Definition 11[2] *Let K be a bounded subset of a Banach space X with the closed unit ball U_X. For $n \in \mathbb{N}$, the n-th diameter $\delta_n(K)$, is defined as the infimum of all positive numbers c such that there is a linear subspace F with dimension at most n such that $K \subseteq cU_X + F$; that is,*

$$\delta_n(K) := \inf\{c > 0 : K \subseteq cU_X + F, \dim F \le n\}.$$

These diameters were first introduced by A. N. Kolmogorov.

Remark.[14, 15]

1. $\delta_0(K) \ge \delta_1(K) \ge \delta_2(K) \ge \cdots$.
2. A bounded subset K of a normed space X is precompact (has a finite ε-net for every $\varepsilon > 0$) if and only if
$$\lim_{n\to\infty} \delta_n(K) = 0.$$
3. A bounded subset K of a normed space X lies in a linear subspace of dimension at most n if and only if
$$\delta_n(K) = 0.$$

2.2 The s-numbers of bounded linear operators

Definition 12[1] *A map s, which assigns to every operator $T \in \mathscr{B}(X,Y)$ a unique sequence $(s_n(T))_{n=0}^{\infty}$ of real numbers, is called s-function, if the following conditions are satisfied:*

1. *$\|T\| = s_0(T) \ge s_1(T) \ge \cdots \ge 0$ for all $T \in \mathscr{B}(X,Y)$ (monotonicity);*
2. *$s_{m+n}(T_1 + T_2) \le s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in \mathscr{B}(X,Y)$ and $m, n \in \mathbb{N}$ (additivity);*
3. *$s_n(RST) \le \|R\| \cdot s_n(S) \cdot \|T\|$ for all $T \in \mathscr{B}(X_0, X), S \in \mathscr{B}(X,Y)$, and $R \in \mathscr{B}(Y, Y_0)$ (multiplicativity);*
4. *$s_n(\lambda T) = |\lambda| \cdot s_n(T)$ for all $T \in \mathscr{B}(X,Y), \lambda \in \mathbb{R}$;*
5. *if $rank(T) < n$, then $s_n(T) = 0$ for all $T \in \mathscr{B}(X,Y)$ ($rank(T)$ is the dimension of $range(T)$) (rank property);*
6. $s_r(I_n) = \begin{cases} 1, & \text{for } r < n, \\ 0, & \text{for } r \ge n, \end{cases}$
where I_n is the identity operator of the Euclidean space $\ell_n^2 =: \{x \in \ell^2 : x_i = 0 \quad \text{if } i > n\}$ to itself (property of norming);
7. *if $\dim X \ge n$, then $s_n(I_X) = 1$ (norm-determining property).*
We call $s_n(T)$, the n-th s-number of the operator T.

There are many examples of s-numbers of operators acting between Banach spaces, namely, the approximation numbers, the Kolmogorov numbers, the Gelfand numbers, the Tichomirov numbers, the Weyl numbers, the Chang numbers, and the Hilbert numbers, they are defined as follows:

1.The n-th approximation number, denoted by $\alpha_n(T)$, is defined as

$$\alpha_n(T) = \inf\{\|T - A\| : A \in \mathscr{B}(X,Y), \text{rank}(A) \leq n\}.$$

These numbers measure the closeness by which a bounded linear map may be approximated by similar maps but with finite-dimensional range.

2.The n-th Kolmogorov number, denoted by $d_n(T)$, is defined as

$$d_n(T) = \delta_n(T(U_X)).$$

Roughly speaking, the Kolmogorov numbers $d_n(T)$ deal with that part of the set $T(U_X)$ which lies outside a certain finite dimensional subspace.

3.The n-th Gelfand number, denoted by $c_n(T)$, is defined as

$$c_n(T) = a_n(J_Y T),$$

4.The n-th Tichomirov number, denoted by $d_n^{\star}(T)$, is defined as

$$d_n^{\star}(T) = d_n(J_Y T),$$

where J_Y is a metric injection from the space Y into a higher space $\ell^{\infty}(\Omega)$ for adequate index set Ω (a metric injection is a one to one operator with closed range and with norm equal one).

5.The n-th Weyl number, denoted by $x_n(T)$, is defined as

$$x_n(T) = \inf\{\alpha_n(TA) : \|A : \ell^2 \to X\| \leq 1\}.$$

6.The n-th Chang number, denoted by $y_n(T)$, is defined as

$$y_n(T) = \inf\{\alpha_n(ST) : \|S : Y \to \ell^2\| \leq 1\}.$$

7.The n-th Hilbert number, denoted by $h_n(T)$, is defined by

$$h_n(T) = \sup\{\alpha_n(STA) : \|S : Y \to \ell^2\| \leq 1 \text{and } \|A : \ell^2 \to X\| \leq 1\}.$$

For more informations about those numbers, we refer the reader to [1,2,16].

In the following, we list some basic proprieties of Kolmogorov numbers.

Proposition 1.[15] *For two mappings S, $T \in \mathscr{B}(X,Y)$, we always have*

1.$d_{m+n}(S+T) \leq d_m(S) + d_n(T)$.
2.$|d_m(S) - d_m(T)| \leq \|S - T\|$.
3.$d_n(T) = 0$ if and only if $T \in \mathscr{A}_n(X.Y)$, where $\mathscr{A}_n$ is a finite dimensional space of dimension at most n.
4.$d_{m+n}(ST) \leq d_m(S) \cdot d_n(T)$.

*Remark.*Proposition (1) is also valid for the approximation numbers.

3 Main Results

Let us introduce the closed relative unit ball of a Banach space X, by

$$U_X^T = \{x \in D(T) : \|x\|_T \leq 1\}.$$

Definition 13*Let X and Y be two Banach spaces, and let $T \in \mathscr{C}(X,Y)$. Set*

$$|||x|||_T := \max_{x \in D(T)} \{\|x\|, \|Tx\|\}.$$

$|||x|||_T$ defines a norm on $D(T)$ which is equivalent to the norm $\|x\|_T$, and so $(D(T), \|.\|_T)$ becomes a Banach space.

By $\mathfrak{D}_X^T$, we denote the relative T-unit ball which is the unit ball related to the norm $|||x|||_T$, i.e.,

$$\mathfrak{D}_X^T := \{x : |||x|||_T \leq 1 \ , \ x \in D(T)\}.$$

Lemma 14*Let X and Y be two Banach spaces, and let $T \in \mathscr{C}(X,Y)$. The unit balls U_X, U_Y, U_X^T and $\mathfrak{D}_X^T$, are related as follows:*

1.$\mathfrak{D}_X^T = U_X \cap T^{-1}U_Y$.
2.$U_X^T \subseteq \mathfrak{D}_X^T$.
3.$\bigcup_{0 \leq \lambda \leq 1} \left(\lambda U_x \cap (1-\lambda) T^{-1} U_Y\right) \subseteq U_X^T$.

It is well known that a bounded linear operator A is compact if and only if it translates the unit ball U_X of its domain X into a relatively compact subset $A(U_X)$ in the co-domain Y. In the following proposition, by reformulating definition (8), we show that a relatively bounded operator A with respect to T is relatively compact if and only if it translates the relative T-unit ball $\mathfrak{D}_X^T$ of its domain X into a relatively compact subset $A\left(\mathfrak{D}_X^T\right)$ in the co-domain Y.

Proposition 2.*A T-bounded operator A is a T-compact if and only if $A\left(\mathfrak{D}_X^T\right)$ is relatively compact.*

*Proof.*Let A be a T-bounded operator which translates the relative T-unit ball $\mathfrak{D}_X^T$ of its domain X into a relatively compact subset $A\left(\mathfrak{D}_X^T\right)$, and suppose that $\{x_n\}$ is a T-bounded sequence in X, if there exist two positive numbers c_1 and c_2 such that $\|x_n\| \leq c_1$ and $\|Tx_n\| \leq c_2$. In this case,

$$\{y_n\} = \left\{\frac{x_n}{\max\{c_1, c_2\}}\right\}$$

is a relatively T-bounded in $\mathfrak{D}_X^T$. From the compactness of $A\left(\mathfrak{D}_X^T\right)$, then there exists a sequence $\{n_k\}$ of integers such that $\{A(y_{n_k})\}$ is a convergent sub-sequence in $A\left(\mathfrak{D}_X^T\right)$, and so is $\{A(x_{n_k})\}$. This is the relatively compactness in the sense of definition (8). On the other hand, by taking $\{x_n\}$ in $\mathfrak{D}_X^T$, hence the relative compactness of the operator A in sense of definition (8) implies that $A\left(\mathfrak{D}_X^T\right)$ is relatively compact.

Definition 15 *To each T-bounded operator A, we can assign a non-negative number $|||A|||_T$ defined by*

$$|||A|||_T = \sup_{\substack{x\in D(T)\\ x\neq 0}} \frac{\|Ax\|}{|||x|||_T} = \sup_{\substack{x\in D(T)\\ x\neq 0}} \frac{\|Ax\|}{\max\{\|x\|,\|Tx\|\}} = \inf_{x\in D(T)} \{c>0 : \|Ax\| \le c|||x|||_T\}.$$

Proposition 3. *The set of all relatively compact operators is closed in the set of the relatively bounded operators.*

Proof. Suppose that $\{A_n\}$ is a sequence of T- compact operators from X into Y with $A_n \to A$, where A is a T-bounded operator. Then we want to show that the limit operator $A \in L_c^T(X,Y)$. It is sufficient from proposition (2) to show that $A\left(\mathfrak{D}_X^T\right)$ is a relatively compact set. To do this, fix $\varepsilon > 0$. Since $A_n \to A$, so there exists n such that

$$|||A_n - A|||_T < \frac{\varepsilon}{4}. \tag{3}$$

By assumption A_n is T-compact, and so $\overline{A_n\left(\mathfrak{D}_X^T\right)}$ is compact. Therefore, there are a finite number of vectors $y_k \in \overline{A_n\left(\mathfrak{D}_X^T\right)} (k = 1,\cdots,m)$ such that

$$A_n\left(\mathfrak{D}_X^T\right) \subseteq \bigcup_{k=1}^{m} \left(\frac{\varepsilon}{4}A_n\left(\mathfrak{D}_X^T\right) + y_k\right). \tag{4}$$

If we let $y \in A\left(\mathfrak{D}_X^T\right)$, then $y = Ax$ for some $x \in \mathfrak{D}_X^T$. By (4)

$$A_n x \in \bigcup_{k=1}^{m} \left(\frac{\varepsilon}{4}A_n\left(\mathfrak{D}_X^T\right) + y_k\right).$$

From (3) and since $|||x|||_T \le 1$, we get

$$|||y_k - y|||_T \le |||y_k - A_n x|||_T + |||A_n x - Ax|||_T \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4}|||x|||_T < \frac{\varepsilon}{2}.$$

So that $y \in \left(\frac{\varepsilon}{2}A\left(\mathfrak{D}_X^T\right) + y_k\right)$. Since $y \in A\left(\mathfrak{D}_X^T\right)$ was arbitrary, then this shows that

$$A\left(\mathfrak{D}_X^T\right) \subseteq \bigcup_{k=1}^{m} \left(\frac{\varepsilon}{2}A_n\left(\mathfrak{D}_X^T\right) + y_k\right).$$

Therefore,

$$\overline{A\left(\mathfrak{D}_X^T\right)} \subseteq \overline{\bigcup_{k=1}^{m} \left(\frac{\varepsilon}{2}A\left(\mathfrak{D}_X^T\right) + y_k\right)} \subseteq \bigcup_{k=1}^{m} \left(\varepsilon A_n\left(\mathfrak{D}_X^T\right) + y_k\right).$$

Since ε was arbitrary, then $A\left(\mathfrak{D}_X^T\right)$ has an ε-net. Hence, from Lemma (10), the proof is done.

Theorem 16. *Let $A_1, A_2, \ldots, A_n$ be a sequence of T-compact operators, then the summation $\sum_{i=1}^{n} A_i$ is T-compact.*

Proof. Let $\{x_n\}$ be a sequence in $\mathfrak{D}_X^T$. Since $A_i\left(\mathfrak{D}_X^T\right)$ is relatively compact, for all $i = 1,\ldots,n$, there exists a sub-sequence of indices n_k such that $A_i x_{n_k}$ is convergent for $i = 1,\ldots,n$ and so, $\sum_{i=1}^{m}\{A_i x_n\}$ has a convergent sub-sequence.

Theorem 17. *The product of a relatively compact operator with a relatively bounded operator is relatively compact.*

Proof. More precisely, let S be a T-bounded operator from X_0 to X, and let $A \in L_c^T(X,Y)$, where X_0, X, and Y are Banach spaces, then we want to prove that the composition AS is relatively compact with respect to T. Given $\{z_n\} \subset U_{X_0}, \|z_n\|_T \le 1$, Since S is a T-bounded operator, from Remark (2), we have $\|Sz_n\|_T \le \|z_n\|_T$, then the sequence $\{Sz_n\}$ is T- bounded, Therefore, it follows from Remark (2) that $\{ASz_n\}$ is a relatively compact set. Hence, AS is T- compact.

Lemma 18 *Let S be a T-bounded operator from X_0 to X, and $A \in L_c^T(X,Y)$. If $R \in \mathscr{B}(Y,Y_0)$, where X_0, X, Y, and Y_0 are Banach spaces; that is,*

$$X_0 \xrightarrow{S} X \xrightarrow[T]{A} Y \xrightarrow{R} Y_0,$$

then $RAS \in L_c^T(X_0,Y_0)$.

Proof. Given $\{x_n\} \subset U_X$. Since A is a T-compact operator, then from Remark (2), there is a sub-sequence $\{Ax_{n_k}\}$ of $\{Ax_n\}$ which converges. Hence, from Remark (2), we have

$$\|RAx_{n_k}\|_T \le c\|Ax_{n_k}\|_T.$$

Therefore, from the continuity of R, the sequence $\{RAx_{n_k}\}$ converges. Thus, $RA \in L_c^T(X,Y_0)$, and from Theorem (17), we finish the proof.

Now, we introduce a definition for the n-th T-Kolmogorov number for the T-bounded operator A.

Definition 19 *Let A be an arbitrary T-bounded operator. The infimum of all positive numbers c such that there is a linear subspace F of Y with dimension at most n for which*

$$A\left(\mathfrak{D}_X^T\right) \subseteq cU_Y + F$$

is called the n-th T-Kolmogorov number for T-bounded operator A, and we will denote it by $d_n^T(A)$. That is;

$$d_n^T(A) = \delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) = \inf_c\left\{c>0 : A\left(\mathfrak{D}_X^T\right) \subseteq cU_Y + F,\ \dim F \le n\right\}.$$

Accordingly, we can give a definition for the T-Tichomirov numbers for T-bounded operators.

Definition 20 *Let A be an arbitrary T-bounded operator from a normed space X into a normed space Y. Let J be a canonical embedding of the Banach space Y into a super universal space $\ell^\infty(I)$ for a suitable subset I. Then,*

$$d_n^{JT}(JA) = \delta_n\left(JA(\mathfrak{D}_X^{JT})\right).$$

*Remark.*One can prove that the number $d_n^{JT}(JA)$ is independent of the choice of the embedding J.

In the following proposition, we give an equivelant definition for the n -th T-Kolmogorov numbers of a relatively bounded operator A with respect to a densely closed operator T.

Proposition 4.*Let X and Y be two normed spaces, with unit balls U_X and U_Y respectively, and let A be a linear operator relatively bounded with respect to a densely defined operator T from X to Y such that $D(T) \subset D(A) \subset X$, the n-th T-Kolmogorov number for A could be written as*

$$d_n^T(A) = \inf_{\substack{F \subset Y \\ \dim F \le n}} \sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F} \|A(x) - f\|_T.$$

*Proof.*From definition 19 then, for any positive ε and for any finite dimensional subspace $F \subset Y$ with $\dim F \le n$, we have

$$A\left(\mathfrak{D}_X^T\right) \nsubseteq \left(\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) - \varepsilon\right) U_Y + F.$$

Otherwise; if

$$A\left(\mathfrak{D}_X^T\right) \subseteq \left(\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) - \varepsilon\right) U_Y + F_0$$

for some F_0, then $\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) \le \delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) - \varepsilon$, and this gives a contradiction. Thus, for any finite dimensional subspace $F \subset Y$ with $\dim F \le n$, there exists $x_0 \in \mathfrak{D}_X^T$ such that

$$\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) - \varepsilon < \inf_{f \in F} \|A(x_0) - f\|_T.$$

Consequently, for any finite dimensional subspace $F \subset Y$ with $\dim F \le n$, we get

$$\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) - \varepsilon \le \sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F} \|Ax - f\|_T.$$

Since ε is arbitrary, then

$$\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) \le \inf_{\dim F \le n} \sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F} \|Ax - f\|_T. \quad (5)$$

Now, for every $\eta > 0$, there exists $F_0 \subset Y$ such that $\dim F_0 \le n$, and

$$A\left(\mathfrak{D}_X^T\right) \subseteq \left(\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) + \eta\right) U_Y + F_0.$$

Hence, for every $x \in \mathfrak{D}_X^T$,

$$Ax = \left(\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) + \eta\right) u + f, f \in F_0, u \in U_Y.$$

Therefore,

$$Ax - f = \left(\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) + \eta\right) u.$$

Henceforth,

$$\sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F_0} \|Ax - f\|_T \le \delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) + \eta. \quad (6)$$

From (5) and (6), we get for every η positive, there exists F_0 such that $\dim F_0 \le n$, and

$$\delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) \le \sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F_0} \|Ax - f\|_T \le \delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) + \eta.$$

*Remark.*Let A be an arbitrary T-bounded operator, from a normed space X into a normed space Y. As an easy consequence of the definition of the n-th T Kolmogorov number, we obtain that

1.$d_0^T(A) = \sup_{x \in \mathfrak{D}_X^T} \|Ax\| = \|A\|_T$.
2.$\|A\|_T = d_0^T(A) \ge d_1^T(A) \ge d_2^T(A) \ge \cdots \ge 0$.
3.$d_n^T(\lambda A) = |\lambda| d_n^T(A)$ for all $\lambda \in \mathbb{R}$.

Proposition 5.*Let A_1 and A_2 be two T-bounded operators. Then, for every $n_1, n_2 \in \mathbb{N}$, we have*

$$d_{n_1+n_2}^T(A_1 + A_2) \le d_{n_1}^T(A_1) + d_{n_2}^T(A_2).$$

*Proof.*For an arbitrary positive number ε, there exist two finite dimensional sub-spaces $F_1, F_2 \subseteq Y$ such that $\dim F_1 \le n_1$, and $\dim F_2 \le n_2$ with

$$d_{n_i}^T(A_i) = \inf_{\dim F_i \le n_i} \sup_{x \in \mathfrak{D}_X^T} \inf_{f \in F_i} \|A(x) - f_i\|_T, i = 1, 2. \quad (7)$$

Therefore,

$$d_{n_1+n_2}^T(A_1 + A_2) = \inf_{\dim F \le n_1+n_2} \sup_{x \in \mathfrak{D}_X^T} \inf_{\in F} \|(A_1 + A_2)(x) - f\|_T$$

$$\le \inf_{\substack{\dim F_1 \le n_1 \\ \dim F_2 \le n_2}} \sup_{x \in \mathfrak{D}_X^T} \inf_{\substack{f_1 \in F_1 \\ f_2 \in F_2}} \left(\|A_1(x) - f_1\|_T + \|A_2(x) - f_2\|_T\right)$$

$$\le \inf_{\dim F_1 \le n_1} \sup_{x \in \mathfrak{D}_X^T} \inf_{f_1 \in F_1} \|A_1(x) - f_1\|_T$$

$$+ \inf_{\dim F_2 \le n_2} \sup_{x \in \mathfrak{D}_X^T} \inf_{f_2 \in F_2} \|A_2(x) - f_2\|_T$$

$$= d_{n_1}^T(A_1) + d_{n_2}^T(A_2) + \varepsilon \qquad (\text{ from } (7)).$$

The arbitrariness of $\varepsilon > 0$ completes the proof.

Theorem 21.*A T-bounded operator A is T-compact if and only if $d_n^T(A)$ tends to zero as n goes to infinity.*

*Proof.*Let $A \in L_c^T(X,Y)$. Then, from proposition (2), $A\left(\mathfrak{D}_X^T\right)$ is a relatively compact set in Y. Therefore, for every $\varepsilon > 0$, there exists a finite number of elements $y_1, y_2, \cdots, y_m \in Y$ such that

$$A\left(\mathfrak{D}_X^T\right) \subseteq \bigcup_{i=1}^{m} \{y_i + \varepsilon U_Y\}.$$

Hence,

$$A\left(\mathfrak{D}_X^T\right) \subseteq G + \varepsilon U_Y,$$

where G is the linear sub-space of Y with $\dim G \le m$ spanned by the elements y_i $(i = 1, \cdots, m)$. On the other hand,

$$d_n^T(A) = \delta_n\left(A\left(\mathfrak{D}_X^T\right)\right) < \varepsilon \quad \text{for every } n \ge m.$$

Therefore,

$$\lim_{n \to \infty} d_n^T(A) = 0.$$

Conversely, if

$$\lim_{n\to\infty} d_n^T(A) = 0$$

for the T-bounded operator A, then for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$d_m^T(A) < \varepsilon \text{ for every } m \geq k.$$

By the definition of the n-th T-Kolmogorov number, there exists a subspace F of Y with $\dim F \leq k$ such that

$$A\left(\mathfrak{D}_X^T\right) \subseteq \varepsilon U_Y + F.$$

Since $U_Y \cap F = \{y \in F : \|y\| \leq 1\}$ is a bounded subset in a finite dimensional subspace, then it is relatively compact. So, it has a finite ε-net (say) $\{z_1, z_2, \cdots, z_m\}$ such that

$$\left(d_0^T(A) + \varepsilon\right)(U_Y \cap F) \subseteq \bigcup_{i=1}^{m} \{z_i + \varepsilon U_Y\}.$$

Hence, for every $x \in \mathfrak{D}_X^T$, we can represent Ax in the form

$$Ax = \varepsilon y + z, \quad \text{where} \quad y \in U_Y \text{ and } \quad z \in F.$$

Thus,

$$\|z\| \leq \|Ax\| + \varepsilon\|y\| \leq |||Ax|||_T + \varepsilon = d_0^T(A) + \varepsilon.$$

Consequently,
$Ax = \varepsilon y + z \in \varepsilon U_Y + \left(d_0^T(A) + \varepsilon\right)(U_Y \cap F) \subseteq \bigcup_{i=1}^{m} \{z_i + 2\varepsilon U_Y\}.$

That is;

$$A\left(\mathfrak{D}_X^T\right) \subseteq \bigcup_{i=1}^{m} \{z_i + 2\varepsilon U_Y\}.$$

Thus, $A\left(\mathfrak{D}_X^T\right)$ has a finite 2ε-net for every $\varepsilon > 0$. Therefore, it is relatively compact.

Proposition 6. *The n-th T-Kolmogorov number for T-bounded operator A vanishes if and only if A is a finite rank operator with rank$(A) \leq n$. That is;*

$$d_n^T(A) = 0 \text{ if and only if } rank(A) = \dim(range(A)) \leq n. \quad (8)$$

Proof. We consider a T-bounded operator A acting between two normed spaces X and Y. Let $d_n^T(A) = 0$. Suppose contrarily that the dimension of the range of the A is larger than n. Then, there exist x_i $(i = 1, \cdots, n+1)$ elements in $D(T)$ where Ax_i's are linearly independent elements in range(A). By Hahan Banach theorem, we can determine $n+1$ linear functionals g_k with $\langle Ax_i, g_k \rangle = \delta_{ik}$. Since any determinant is a continuous function for all its arguments, and since the determinant

$$\det(\delta_{ik}) = \det(g_k(Ax_i)) = 1,$$

then there exists a positive number η such that

$$\det\{\alpha_{ik}\} \neq 0 \quad \text{for } |\delta_{ik} - \alpha_{ik}| < \eta, (i,k = 1, \cdots, n+1).$$

Set

$$\gamma = \frac{\eta}{\max_k \|g_k\|}, \quad k = 1, \cdots, n+1.$$

Since Ax_i $(i = 1, \cdots, n+1)$ are linearly independent, then x_i's are linearly independent. Without loss of generality, we can take $\|x_i\| \leq 1$ and $\|Tx_i\| \leq 1$ (by dividing each x_i by $|||x_i|||_T = \max(\|x_i\|, \|Tx_i\|)$, if necessary). Therefore, $x_i \in \mathfrak{D}_X^T$ for every $i = 1 \cdots, n+1$. Since by hypothesis

$$\inf\left\{c > 0 : A\left(\mathfrak{D}_X^T\right) \subseteq cU_Y + F, \dim F \leq n\right\} = 0,$$

then there exists a finite dimensional subspace F of Y with $\dim F \leq n$ such that $Ax_i \in \gamma U_Y + F$. Hence, for every $i = 1, \cdots, n+1$, we get

$$Ax_i = \gamma y_i + z_i, \quad x_i \in \mathfrak{D}_X^T, \quad y_i \in U_Y, \quad \text{and } z_i \in F.$$

Consequently,

$$\|Ax_i - z_i\| \leq \gamma.$$

Since the elements $z_1, z_2, \cdots, z_{n+1}$ are linearly dependent in F, we have $\det\{\langle z_i, g_k \rangle\} = 0$. On the other hand,

$$|g_k(Ax_i) - g_k(z_i)| \leq \|g_k\| \|Ax_i - z_i\| = \|g_k\| \frac{\eta}{max\|g_k\|} \leq \eta.$$

We have the assertion

$$\det\{\langle z_i, g_k \rangle\} = \det\{\langle A(x_i), g_k \rangle - \eta \langle y_i, g_k \rangle\} \neq 0.$$

This contradiction shows that the assumption $\dim(\text{range}(A)) > n$ is false and this completes the proof.

Conclusion In this work we give an extension of the concept of Kolmogorov numbers to a class of unbounded operators, namely relatively bounded operators with respect to a densely defined closed linear operator. We prove that a relatively bounded operator is relatively compact if and only if its sequence of relative Kolmogorov numbers converges to zero. Moreover, the n-th relative Kolmogorov number of an operator vanishes if and only if it is of finite rank (with rank at most n). These results about relative Kolmogorov numbers for relatively bounded operators are similar to the known results concerning Kolmogorov numbers for bounded linear operators.

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