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Comparative Analysis of Three Distributions under Studying the Regression Competing Risks Model with HIV to AIDS Infection Application

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Abstract: This paper discusses the statistical analysis of unknown parameters in competing risk data when covariates are present. The Cox regression model examines how covariates impact time-to-event data, specifically when lifetimes follow the Akshaya, exponential, and Rayleigh sub-distributions. The Bayesian method estimates and compares these unknown parameters with estimates obtained through the maximum likelihood method. Additionally, the reliability measures of the three models and relative risks are calculated. The applicability of the model is demonstrated through a comprehensive analysis of a real data set involving 329 patients transitioning from HIV infection to AIDS.

Keywords: Akshaya distribution; Exponential distribution; Rayleigh distribution; Cox's regression; Maximum likelihood estimator; Bayesian method; Markov Chain Monte Carlo method; and survival data.

1 Introduction

In numerous situations in reliability or survival analysis, the object (a system, a human being, an animal, etc.) may die/fail due to more than one cause of death/failure. In the works of literature, this problem is called a competing risks model. One of the main objectives of the competing risk models is to evaluate the probability of one cause of death in the presence of all other risks. Also, it is interesting to evaluate the reliability measures of the individual risks as well as those for the object under study. In competing risks models, we observe the time to event (failure/censored), an indicator for failure/censored, and the cause of failure in the failure event case. Different studies have used competing risk models to analyze data using parametric and non-parametric setups.

Nonparametric methods do not require distributions when risks from different causes might occur. Several authors applied the non-parametric techniques for competing risk models, such as [1], [2], [3],[4], [5],[6], [7], and [8].

Parametric setup requires distributions when risks from different causes might occur. Other studies used different lifetime distributions such as exponential, Weibull with known shape parameters, Weibull with equal but unknown shape parameters, and two-parameter bathtub distributions; see, for example, [9], [10], [11] and [12].

Many examples of the competing risk models can be found in [13], [14], and [15]. Competing risk models when the risk times follow gamma, exponential, and Weibull with either equal or known shape parameters and generalized exponential distribution are discussed in [16], [17], [18], [19], and [20].

Referring to the competing risk studies, the researchers may be interested in different effects according to the study field; for example, in medical science, the researchers may be focused on the following effects: infection time, age of infection, social state, sex type, ..., etc. The studies of

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all the previous effects in the competing risks are called regression competing risks.

In a survival analysis with competing risks, two different regression modeling strategies are possible: regression models for time-to-event data and regression approaches for the competing risks setting (such as cause-specific hazards regression and sub-distribution hazards regression); see [21]-[22]. The regression models that can be applied in the presence of censored observations were developed to investigate the influence of covariates on the event times. The Cox's and parametric regression models for time-to-event data with one possible endpoint are the most used. Also according to Cox's regression model, the assumption shows that the hazard ratios are constant over time and that each of the number of covariates under consideration has a linear effect on the logarithm of the hazard rate, given the other covariates; see [23] and [24]. The Weibull distribution is one of the most important distributions in survival analysis. Because it has a constant, increasing, or decreasing hazard rate function, see [25] and [12]. Also, the Weibull distribution properties are studied by [26] and [18]. The competing risks model when the cause of failure follows the Weibull distribution with an unknown shape and scale parameter is discussed by [20].

This paper aims to introduce a comparison between three-lifetime distributions with one parameter in survival analysis. The Cox regression model analyzes the effects of covariates on time-to-event data, particularly in cases when lifetimes exhibit the Rayleigh, exponential, and Akshaya sub-distributions. To estimate these unknown parameters and compare them with estimates derived using the maximum likelihood method, the Bayesian approach is employed.

The outline of the paper is as follows:.
The model assumptions will be described in Section 2. In Section 3, we will study the likelihood function and get the maximum likelihood estimates and the two-sided confidence intervals of the unknown parameters included in the model. In Section 4, we calculate the risks in the presence of the covariates using maximum likelihood and Bayesian methods in the study model. Bayesian analysis and Markov Chain Monte Carlo (MCMC) techniques have been considered in Section 5. An analysis of a simulation example is provided in Section 6 to illustrate the use of the technique. In Section 7, we conclude with the reviewed methodologies illustrated with data from HIV infection to AIDS, SI switch, and death in 329 men who have sex with men (MSM) in this paper. Section 8 concludes this paper.

2 Model Assumptions

The statistical analysis of the competing risks model deals only with time to events without covariates discussed by Sarhan et al. [27] and [24]. The covariates may describe the infection time, age of infection, social state, sex type, or other effects. We assume that there are n independent and identical items put on the life test. Every item is referred to as one of $k; (k \geq 2)$, independent causes of failure. We assume the following notations:

T_{ji} is the failure time of the i-th item destroyed by j-th cause j $(j = 1,2,...,k)$ of failure.

$F_j(.)$ is the cumulative distribution function of the failure times T_{ji}.

$\bar{F}_j(.)$ is the survival function of the random variables T_{ji}.

$f_j(.)$ is the probability density function of the failure time T_{ji}.

$h_j(.)$ is the hazard rate function.

m is the number of covariates.

X is a $m \times 1$ vector of covariates.

λ_j is a $m \times 1$ vector of regression coeffcients for each cause of failure.

$\delta_i = j$: the indicator variable denoting the cause of failure of the i-th individual.

T_i is $\min_{1 \leq j \leq k} \{T_{ji}\}$; $j = 1,2,...,k$ and $i = 1,2,...,n$.

h_{0j} is the baseline hazard rate function.

S_{0j} is the baseline survival function $j = 1,2,...,k$.

Following Cox's regression model, the hazard rate function for the risk j at time t for an individual with covariates X (not including a constant) and covariates coefficients λ_j is

$$h_j(t;\phi_j|X) = h_{0j}(t;\vartheta_j)e^{\lambda_j' X}, \tag{1}$$

and the survival function is given by

$$S_j(t;\phi_j|X) = [S_{0j}(t;\vartheta_j)]^{e^{\lambda_j' X}}. \tag{2}$$

Each of the above functions depends on a vector of unknown parameters, ϕ_j, which itself contains two vectors of parameters: the first is ϑ_j which includes the parameters of the baseline distribution, and the other is λ_j which contains the regression coefficients and λ_j' is a transpose vector of regression coefficients λ_j for each cause of failure. That is, ϕ_j has $p+m$ parameters, where p is the number of baseline distribution parameters and m is the number of covariates coefficients.

The survival function for the system [individual or item] at time t, given covariates X, is

$$S(t;\theta|X) = \prod_{j=1}^{k} [S_{0j}(t;\vartheta_j)]^{e^{\lambda_j' X}}. \tag{3}$$

where θ is the vector of $(p+m)k$ parameters. One of the main goals of this paper is to estimate such parameters.

3 Maximum-likelihood Estimation

In this section, the likelihood function is derived in a general setup. In competing risks with covariates, the data

consists of three parts. The individuate time to event T (censored or failure), an indicator variable δ that takes either 0 for censoring or a value in $j \in \{1,2,\cdots,k\}$ for failure due to cause j, and the values of the covariates' vector X. Since there are N independent and identical individuals on the life test, the data can be expressed as data $= (T_1,\delta_1,X_1),(T_2,\delta_2,X_2),\cdots,(T_N,\delta_N,X_N)$.

3.1 The likelihood function

Based on the data mentioned earlier, the likelihood function is

$$L(\text{data}|\theta) = \prod_{i=1}^{N}[f(t_i;\theta|X_i)]^{I(\delta_i\neq 0)}[S(t_i;\theta|X_i)]^{I(\delta_i=0)}. \quad (4)$$

From the relations between the survival, hazard rate, and probability density functions, the likelihood function can be expressed as

$$\begin{aligned} L(\text{data}|\theta) &= \prod_{i=1}^{N}\left\{[h(t_i;\theta|X_i)]^{I(\delta_i\neq 0)}\,S(t_i;\theta|X_i)\right\} \\ &= \prod_{i=1}^{N}\left\{\left[\prod_{j=1}^{k}[h_j(t_i;\phi_j|X_i)]^{I(\delta_i=j)}\right]S(t_i;\theta|X_i)\right\} \quad (5) \end{aligned}$$

whereabouts $I(B)$ is an identifying indicator function defined as

$$I(B) = \begin{cases} 1 & \text{if B is true,} \\ 0 & \text{if otherwise.} \end{cases} \quad (6)$$

Substituting from (1) and (2) into (5), the likelihood function becomes

$$\begin{aligned} L(\text{data}|\theta) &= \prod_{i=1}^{N}\left\{\prod_{j=1}^{k}[h_{0j}(t_i;\vartheta_j)e^{\lambda_j' X_i}]^{I(\delta_i=j)}\prod_{j=1}^{k}[S_{0j}(t_i;\vartheta_j)]^{e^{\lambda_j' X_i}}\right\} \\ &= \prod_{i=1}^{N}\prod_{j=1}^{k}\left\{\left[h_{0j}(t_i;\vartheta_j)e^{\lambda_j' X_i}\right]^{I(\delta_i=j)}\left[S_{0j}(t_i;\vartheta_j)\right]^{e^{\lambda_j' X_i}}\right\} \quad (7) \end{aligned}$$

Taking the natural logarithm for (7), the log-likelihood function is

$$\mathscr{L}(\text{data}|\theta) = \sum_{i=1}^{N}\sum_{j=1}^{k}\left\{I(\delta_i=j)\left[\ln h_{0j}(t_i;\vartheta_j)+\lambda_j' X_i\right] + e^{\lambda_j' X_i}\ln S_{0j}(t_i\vartheta_j)\right\}. \quad (8)$$

3.2 The Akshaya distribution case

We assume that T_{ji} follows an Akshaya distribution with an unknown parameter θ_j, represented as Akshaya(θ_j), for $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,k$. Essentially, this means that the survival function T_{ji} is given by:

$$S_{0j}(t) = \left[1+\frac{\theta_j^3 t^3+3\theta_j^2(\theta_j+1)t^2+3\theta_j(\theta_j^2+2\theta_j+2)t}{\theta_j^3+3\theta_j^2+6\theta_j+6}\right]e^{-\theta_j t}, \quad (9)$$

and the hazard rate function is expressed as

$$h_{0j}(t) = \frac{\theta_j^4(1+t)^3}{\theta_j^3 t^3+3\theta_j^2(\theta_j+1)t^2+3\theta_j(\theta_j^2+2\theta_j+2)t+(\theta_j^3+3\theta_j^2+6\theta_j+6)}, \quad (10)$$

where θ_j represents the shape parameter. The probability density function (PDF) and hazard rate function plots for the Akshaya model have been included in Figure 1 to provide a clearer visual representation of their behavior and facilitate model comparison. Substituting from Eqs. (9) and (10) into Eq. (8), then the log likelihood function is

$$\ell = \sum_{i=1}^{n}\sum_{j=1}^{k}\left[\begin{array}{c} I(\delta_i=j)\left(\begin{array}{c} 4\,n\theta_j+3\ln(1+t_i)+\lambda_j X_i \\ -\,n(A1)\end{array}\right) \\ +e^{\lambda_j X_i}\left(\begin{array}{c} -\theta_j t_i-\ln\left(\theta_j^3+3\theta_j^2+6\theta_j+6\right) \\ +\,n(A1)\end{array}\right)\end{array}\right]. \quad (11)$$

where $A1 = \theta_j^3 t_i^3+3\theta_j^2(\theta_j+1)t_i^2+3\theta_j(\theta_j^2+2\theta_j+2)t_i+(\theta_j^3+3\theta_j^2+6\theta_j+6)$

$$\frac{\partial\ell}{\partial\theta_l} = \sum_{i=1}^{n}\sum_{j=1}^{k}\delta_{lj}\left[\begin{array}{c} I(\delta_i=j)\left(\frac{4}{\theta_j}-\frac{B1}{A1}\right) \\ +e^{\lambda_j X_i} \\ \left(-t_i+\frac{B1}{A1}-\frac{3\theta_j^2+6\theta_j+6}{\theta_j^3+3\theta_j^2+6\theta_j+6}\right)\end{array}\right], \quad (12)$$

where $B1 = 3\theta_j^2 t_i^3+6\theta_j(\theta_j+1)t_i^2+3\theta_j^2 t_i^2+3(\theta_j^2+2\theta_j+2)t_i+6\theta_j(\theta_j+1)t_{i/}+(3\theta_j^2+6\theta_j+6)$

and

$$\frac{\partial\ell}{\partial\lambda_{jl}} = \sum_{i=1}^{n}\sum_{j=1}^{k}\delta_{lj}\,X_i\left[I(\delta_i=j)+e^{\lambda_j X_i}\left(-\theta_j t_i+\ln\left(1+\frac{\theta_j^3 t_i^3+3\theta_j^2(\theta_j+1)t_i^2+3\theta_j(\theta_j^2+2\theta_j+2)t_i}{\theta_j^3+3\theta_j^2+6\theta_j+6}\right)\right)\right], \quad (13)$$

The maximum likelihood point estimate of *theta* is obtained by maximizing the log-likelihood function. Then, by differentiating the log-likelihood function ℓ with respect to θ_l and λ_l, $l = 1,2,...,k$. Thus we have the likelihood equation for θ_l and λ_l, $l = 1,2,...,k$ respectively, as

where $\delta_{lj}, l, j = 1,2,...,k$ is the Kronecker delta function.

To obtain the maximum likelihood estimates for the parameters, we set Eqs. (12) and (13) equal to zero and solve it with respect to θ_j and λ_{jl}, $j = 1,2,...,k$.

An exact solution is not easy to obtain. So, we should use a numerical technique to find an approximate solution. The obtained solutions act as the maximum likelihood point estimators of the unknown parameters.

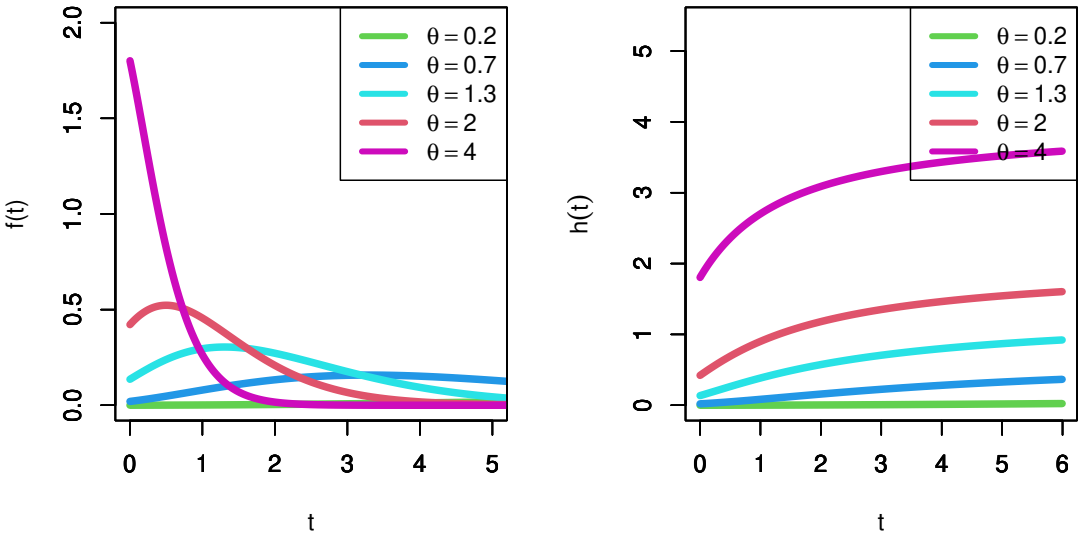


Fig. 1: Density, hazard curves for Akshaya distribution

Also, we propose to use the asymptotic normality of $\Phi = (\theta_1, \theta_2, ..., \theta_k, \lambda_{j1}, ..., \lambda_{jm}), j = 1, 2, ..., k$.

To obtain the information matrix, we need the second partial derivatives of ℓ with respect to θ_l and $\lambda_{jl}, l = 1, 2, ..., k$, which are $\frac{\partial^2 \ell}{\partial \Phi_l \partial \Phi_m}$, where $l, r = 1, 2, ..., k$.

3.3 The Rayleigh distribution case

We assume that it T_{ji} follows the Rayleigh distribution with unknown parameters, σ_j, say $Rayleigh(\sigma_j)$, for $i = 1, 2, ..., N$ and $j = 1, 2, ..., k$. That is, T_{ji} has the survival rate function is given by

$$S_{0j}(t) = e^{-\frac{t^2}{2\sigma_j}}, \tag{14}$$

and the hazard rate function is

$$h_{0j}(t) = \frac{t}{\sigma_j}, \tag{15}$$

where σ_j is the shape parameter. The PDF and hazard rate function plots for the Rayleigh model have been included in Figure 2 to provide a clearer visual representation of their behavior and facilitate model comparison.

Substituting from Eqs. (14) and (15) into Eq. (8), then the log-likelihood function is

$$\ell = \sum_{i=1}^{n} \sum_{j=1}^{k} \left[I(\delta_i = j) \left(\ln t_i - \ln \sigma_j + \lambda_j X_i \right) - \frac{t_i^2}{2\sigma_j} e^{\lambda_j X_i} \right]. \tag{16}$$

The maximum likelihood point estimate of σ is obtained by maximizing the log-likelihood function. Then, by differentiating the log-likelihood function ℓ with respect to σ_l and λ_l, $l = 1, 2, ..., k$. Thus we have the likelihood equation for σ_l and λ_l, $l = 1, 2, ..., k$ respectively, as

$$\frac{\partial \ell}{\partial \sigma_l} = \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{lj} \left[I(\delta_i = j) \left(\frac{-1}{\sigma_j} \right) + \frac{t_i^2}{2\sigma_j^2} e^{\lambda_j X_i} \right], \tag{17}$$

and

$$\frac{\partial \ell}{\partial \lambda_{jl}} = \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{lj} X_i \left[I(\delta_i = j) - \frac{t_i^2}{2\sigma_j} e^{\lambda_j X_i} \right], \tag{18}$$

whereabouts $\delta_{lj}, l, j = 1, 2, ..., k$ is the Kronecker delta function.

To get the maximum likelihood point estimates for the parameters, we set Eqs. (17) and (18) equal to zero and solve it with respect to σ_j and λ_{jl}, $j = 1, 2, ..., k$.

An exact solution is not easy to obtain. So, we should use a numerical technique to find an approximate solution. The obtained solutions act as the maximum likelihood point estimators of the unknown parameters.

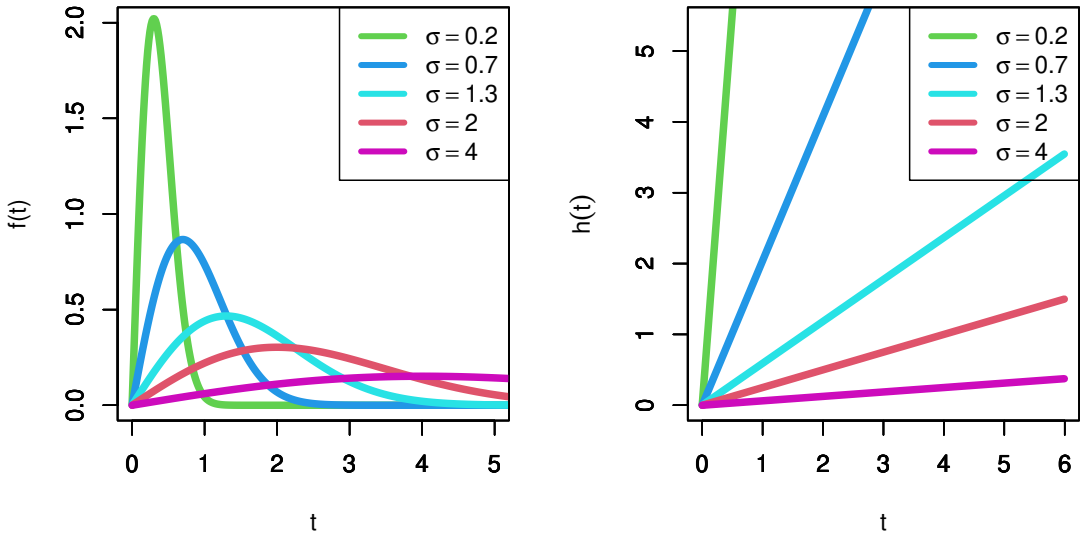


Fig. 2: Density, hazard curves for Rayleigh distribution

Also, we propose to use the asymptotic normality of $\Phi = (\sigma_1, \sigma_2, ..., \sigma_k, \lambda_{j1}, ..., \lambda_{jm}), j = 1, 2, ..., k$.

To calculate the information matrix, we require the second partial derivatives of ℓ concerning σ_l and λ_{jl} for $l = 1, 2, \ldots, k$. These derivatives are represented as $\frac{\partial^2 \ell}{\partial \Phi_l \partial \Phi_m}$, where $l, r = 1, 2, \ldots, k$.

3.4 The exponential distribution case

We assume that T_{ji} follow an exponential distribution with unknown parameters, α_j, say $E(\alpha_j)$, for $i = 1, 2, ..., N$ and $j = 1, 2, ..., k$. That is, T_{ji} has the survival rate function is

$$S_{0j}(t) = e^{-\alpha_j t}, \ \alpha_j > 0, \ t > 0, \tag{19}$$

and the hazard rate function is

$$h_{0j}(t) = \alpha_j, \tag{20}$$

where α_j is the shape parameter. The PDF and hazard rate function plots for the exponential model have been included in Figure 3 to provide a clearer visual representation of their behavior and facilitate model comparison.

Substituting from Eqs. (19) and (20) into Eq. (8), then the log-likelihood function is

$$\ell = \sum_{i=1}^{n} \sum_{j=1}^{k} \left[I(\delta_i = j) \left(\ln \alpha_j + \lambda_j X_i \right) - \alpha_i \, t_i \, e^{\lambda_j X_i} \right], \tag{21}$$

The maximum likelihood estimate of α is obtained by maximizing the log-likelihood function. Then, by differentiating the log-likelihood function ℓ with respect to α_l and λ_l, $l = 1, 2, ..., k$. Thus we have the likelihood equation for α_l and λ_l, $l = 1, 2, ..., k$ respectively, as

$$\frac{\partial \ell}{\partial \alpha_l} = \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{lj} \left[I(\delta_i = j) \left(\frac{1}{\alpha_j} \right) - t_i \, e^{\lambda_j X_i} \right], \tag{22}$$

and

$$\frac{\partial \ell}{\partial \lambda_{jl}} = \sum_{i=1}^{n} \sum_{j=1}^{k} \delta_{lj} \, X_i \left[I(\delta_i = j) - \alpha_i \, t_i \, e^{\lambda_j X_i} \right], \tag{23}$$

whereabouts $\delta_{lj}, l, j = 1, 2, ..., k$ is the Kronecker delta function.

To get the maximum likelihood point estimates for the parameters, we set Eqs. (22) and (23) equal to zero and solve it with respect to α_j and λ_{jl}, $j = 1, 2, ..., k$.

An exact solution is not easy to obtain. So, we should use a numerical technique to find an approximate solution. The obtained solutions act as the maximum likelihood point estimators of the unknown parameters. Also, we propose to use the asymptotic normality of $\Phi = (\alpha_1, \alpha_2, ..., \alpha_k, \lambda_{j1}, ..., \lambda_{jm}), \ j = 1, 2, ..., k$.

To obtain the information matrix, we need the second partial derivatives of ℓ with respect to α_l and $\lambda_{jl}, l = 1, 2, ..., k$, which are $\frac{\partial^2 \ell}{\partial \Phi_l \partial \Phi_m}$, where $l, r = 1, 2, ..., k$.

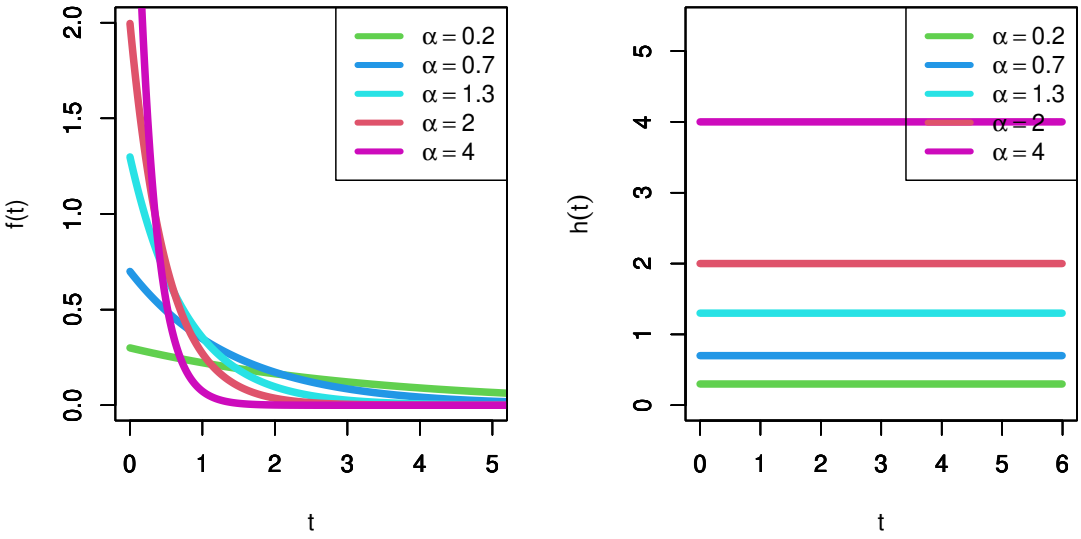


Fig. 3: Density, hazard curves for exponential distribution

3.5 Confidence intervals

In this section, we discuss approximate confidence intervals for the parameters. The asymptotic standard error for an estimator is obtained from the inverse of the Fisher information matrix.

The asymptotic confidence intervals for the vector of the unknown parameters $\Phi = (\theta_1, \theta_2, ..., \theta_k, \lambda_{j1}, ..., \lambda_{jm}), \quad j = 1,2,..,k,$ are discussed. Even though the maximum likelihood estimators for the parameters do not have analytical forms and their exact distributions cannot be determined, the asymptotic distribution of the maximum likelihood estimator can still be used to calculate confidence intervals for $\Phi = (\theta_1, \theta_2, \ldots, \theta_k, \lambda_{j1}, \ldots, \lambda_{jm}), \quad j = 1,2,\ldots,k,$ as shown below

$$\left(\hat{\Phi} - \Phi\right) \rightarrow \left(N_{(2+m)k}(0, V(\hat{\Phi}))\right), \tag{24}$$

where $\hat{\Phi} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k, \hat{\lambda}_{j1}, ..., \hat{\lambda}_{jm}), \quad j = 1,2,..,k$ are the point estimates by the maximum likelihood method, $N_{(2+m)k}$ denotes $(2+m)k$-multidimensional normal distribution and $V(\hat{\Phi})$ is the variance-covariance matrix that can be obtained as the inverse of the information matrix of $\hat{\Phi}$. The components of the information matrix are the second partial derivatives of ℓ. That is $I(\hat{\Phi}) = I_{ij}(\hat{\Phi}), i, j = 1,2,...,k,$

$$I_{ij}(\hat{\Phi}) = -\left[\frac{\partial^2 \ell(\Phi)}{\partial \Phi_i \partial \Phi_j}\right]_{\Phi=\hat{\Phi}}, i, j = 1,2,...,k, \tag{25}$$

Then the $100(1-\gamma)\%$ confidence interval for γ is given by

$$\hat{\Phi} \pm Z_{\gamma/2}\sqrt{V}, \tag{26}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$ and V is the variance-covariance matrix of the maximum likelihood estimates.

4 The Relative Risks

Relative risk [28] is one of the important characteristics in the competing risk models. It determines the probability of death due to a specific risk in the presence of all other risks. The cumulative incidence function for risk j gives the probability of failure due to a risk j at time t in the presence of all other risks. That is,

$$F_{C_j}(t;\theta|X) = \int_0^t h_j(u;\phi_j|X)\, S(u;\theta|X) du, \; j = 1,2,\ldots,k. \tag{27}$$

Case 1: For the Akshaya model discussed in this paper,

$$F_{C_j}\quad(t;\theta|X)=\int_0^t \frac{\theta_j^4(1+u)^3}{A2}\times$$
$$\cdot\prod_{l=1}^{k}\left(\left[1+\frac{C1}{\theta_j^3+3\theta_j^2+6\theta_j+6}\right]e^{-\theta_j u}\right)^{e^{\lambda_l' x}}du,$$

where $j=1,2,\cdots,k$, and $A2=\theta_j^3u^3+3\theta_j^2(\theta_j+1)u^2+3\theta_j(\theta_j^2+2\theta_j+2)u+(\theta_j^3+3\theta_j^2+6\theta_j+6)$, $C1=\theta_j^3u^3+3\theta_j^2(\theta_j+1)u^2+3\theta_j(\theta_j^2+2\theta_j+2)u$. The relative risk of causing j is the limit of the incidence function when time goes to infinity. That is,

$$F_{C_j}(t;\theta|X)=\int_0^t h_j(u;\phi_j|X)\,S(u;\theta|X)du,\ j=1,2,\ldots,k. \tag{28}$$

For the Akshaya model,

$$\pi_j(t;\theta|X)=\int_0^\infty \frac{\theta_j^4(1+t)^3}{A1}$$
$$\cdot\prod_{l=1}^{k}\left(\left[1+\frac{C2}{\theta_j^3+3\theta_j^2+6\theta_j+6}\right]e^{-\theta_j t}\right)^{e^{\lambda_l' x}} \tag{29}$$

where $C2=\theta_j^3t^3+3\theta_j^2(\theta_j+1)t^2+3\theta_j(\theta_j^2+2\theta_j+2)t$. The integral in (29) has no analytic solution.

Case 2: For the Rayleigh model discussed in this paper,

$$F_{C_j}(t;\theta|X)=\int_0^t \frac{u}{\sigma_j}e^{\lambda_j' x}\prod_{l=1}^{k}\left(e^{-\frac{u^2}{2\sigma_l}}\right)^{e^{\lambda_l' x}}dt,\quad j=1,2,\cdots,k. \tag{30}$$

The relative risk of cause j is the limit of the incidence function when time goes to infinity. That is,

$$\pi_j(\theta|X)=\int_0^\infty \frac{u}{\sigma_j}e^{\lambda_j' x}\prod_{l=1}^{k}\left(e^{-\frac{u^2}{2\sigma_l}}\right)^{e^{\lambda_l' x}}cdt,\quad j=1,2,\cdots,k. \tag{31}$$

The integral in (31) has no analytic solution in the general case.

Case 3: For the exponential model discussed in this paper,

$$F_{C_j}(t;\theta|X)=\int_0^t \alpha_j e^{\lambda_j' x}\prod_{l=1}^{k}\left(e^{-\alpha_l u}\right)^{e^{\lambda_l' x}}dt,\quad j=1,2,\cdots,k, \tag{32}$$

the relative risk of cause j is the limit of the incidence function when time goes to infinity. That is,

$$\pi_j(\theta|X)=\int_0^\infty \alpha_j e^{\lambda_j' x}\prod_{l=1}^{k}\left(e^{-\alpha_l u}\right)^{e^{\lambda_l' x}}dt,\quad j=1,2,\cdots,k, \tag{33}$$

the integral in (33) has no analytic solution in the general case.

Using the invariant property of the maximum likelihood method, we can get the maximum likelihood estimate for $F_{C_j}(t;\theta|X)$ at a given time t or $\pi_j(\theta|X)$ or both. We should use a numerical integral technique to estimate. $F_{C_j}(t;\theta|X)$. For Bayesian estimation, we use the random draws obtained from the joint distribution along with the formulae above to get random draws from the posterior distribution $F_{C_j}(t;\theta|X)$ at a given time, t or $\pi_j(x)$, then use it to perform any Bayesian analysis we wish.

5 Bayesian Estimation

The Bayesian approach treats parameters as random variables and expresses uncertainties about these parameters using a joint prior distribution, established before any failure data is collected. The ability of this method to integrate prior knowledge into the analysis is particularly advantageous for the regression competing risk model, as a major challenge with this model is the limited data availability.

5.1 The Akshaya distribution case

In this subsection, we present the posterior densities of the unknown parameters $\Phi=(\theta_1,\theta_2,...,\theta_k,\lambda_{j1},...,\lambda_{jm})$,

$j=1,2,..,k$,and then obtain the corresponding Bayes estimators of these parameters.

Assume that θ_j and λ_{jl}, $j=1,2,..,k;l=1,2,..,m$ are independent. Also θ_j has gamma prior densities $g(\theta_j)$ with different known and nonnegative hyperparameters a_j and b_j. Also $g(\lambda_{jl})$, $l=1,2,..,m$ have the normal prior densities, $g(\lambda_{jl})$, with means μ_{jl} and variances σ_{jl}^2.

Therefore, the joint prior density function of $\Phi=(\theta_1,\theta_2,...,\theta_k,\lambda_{j1},...,\lambda_{jm})$, $j=1,2,..,k$, up to a constant, is

$$g(\Phi)=\prod_{j=1}^{k}\prod_{l=1}^{m}g(\theta_j)\,g(\lambda_{jl})$$
$$=\prod_{j=1}^{k}\prod_{l=1}^{m}\frac{b_j^{a_j}}{\sigma_{jl}\sqrt{2\pi}\Gamma(a_j)}\theta_j^{a_j-1}e^{-\left(b_j\theta_j+\frac{1}{2\sigma_{jl}^2}(\lambda_{jl}-\mu_{jl})^2\right)}. \tag{34}$$

From the likelihood function in Eq. (3.4) and the joint prior density Eq. (5.1) the joint posterior density function of Φ, up to a constant, can be obtained as

$$g(\Phi \mid \text{data}) \propto \prod_{j=1}^{k}\prod_{l=1}^{m}\left[\frac{b_j^{a_j}}{\sigma_{jl}\sqrt{2\pi}\Gamma(a_j)}\theta_j^{a_j-1} \times e^{-\left(b_j\ \theta_j+\frac{1}{2\sigma_{jl}^2}(\lambda_{jl}-\mu_{jl})^2\right)}\right] \times \prod_{i=1}^{n}\prod_{j=1}^{k}\left\{\begin{array}{c}\left[\frac{\theta_j^4(1+t_i)^3}{A1}\ e^{\lambda_j X_i}\right]^{I(\delta_i=j)} \\ \left[1+\frac{}{\theta_j^3+3\theta_j^2+6\theta_j+6}e^{-\theta_j t_i}\right]^{e^{\lambda_j X_i}}\end{array}\right\}, \quad (35)$$

where $D3 = \theta_j^3 t_i^3 + 3\theta_j^2(\theta_j+1)t_i^2 + 3\theta_j(\theta_j^2+2\theta_j+2)t_i$.

Under the quadratic loss function, the Bayesian estimate of any function of the vector of unknown parameters Φ, say $\nu(\Phi)$, is the posterior mean of that function. That is,

$$\hat{\nu} = E_{\Phi|.}(\nu(\Phi)) = \int_0^{\infty} \nu(\Phi)\ g(\Phi \mid .)\ d\ \Phi. \quad (36)$$

The integral in Eq. (36) and the normalization constant in Eq. (35) do not have analytical solutions. Consequently, numerical methods are needed to perform a Bayesian analysis of the underlying model. Out of several available methods, we will use the Markov Chain Monte Carlo (MCMC) simulation technique for the analysis. The MCMC algorithm allows us to generate random samples from the posterior distribution, as described in Eq. (35), without needing to compute the normalization constant. These random samples can then be used to analyze the model parameters and characteristics as desired.

5.2 The Rayleigh distribution case

In this subsection, we present the posterior densities of the unknown parameters $\Phi = (\sigma_1, \sigma_2, ..., \sigma_k, \lambda_{j1}, ..., \lambda_{jm}, \lambda_{j1}, ..., \lambda_{jm})$, $j = 1,2,..,k$, and then obtain the corresponding Bayes estimators of these parameters.

Assume that σ_j and λ_{jl}, $j = 1,2,..,k; l = 1,2,..,m$ are independent. Also σ_j has gamma prior densities $g(\sigma_j)$ with different known and nonnegative hyperparameters a_j and b_j. Also $g(\lambda_{jl})$, $l = 1,2,..,m$ have the normal prior densities, $g(\lambda_{jl})$, with means μ_{jl} and variances δ_{jl}^2.

Therefore, the joint prior density function of $\Phi = (\sigma_1, \sigma_2, ..., \sigma_k, \lambda_{j1}, ..., \lambda_{jm})$, $j = 1,2,..,k$, up to a constant is

$$g(\Phi) = \prod_{j=1}^{k}\prod_{l=1}^{m} g(\sigma_j)\ g(\lambda_{jl}) = \prod_{j=1}^{k}\prod_{l=1}^{m}\frac{b_j^{a_j}}{\delta_{jl}\sqrt{2\pi}\Gamma(a_j)}\sigma_j^{a_j-1}e^{-\left(b_j\ \sigma_j+\frac{1}{2\delta_{jl}^2}(\lambda_{jl}-\mu_{jl})^2\right)}. \quad (37)$$

From the likelihood function Eq. (37) and the joint prior density, the joint posterior density function of Φ, up to a constant, can be obtained as

$$g(\Phi \mid \text{data}) \propto \prod_{j=1}^{k}\prod_{l=1}^{m}\left[\frac{b_j^{a_j}}{\sigma_{jl}\sqrt{2\pi}\Gamma(a_j)}\theta_j^{a_j-1}e^{-\left(b_j\theta_j+\frac{1}{2\sigma^2}(\lambda_{jl}-\mu_{jl})^2\right)}\right] \times \prod_{i=1}^{n}\prod_{j=1}^{k}\left\{\left[\frac{t_i}{\sigma_j}\ e^{\lambda_j X_i}\right]^{I(\delta_i=j)}\left[e^{-\frac{t_i^2}{2\sigma_j}}\right]^{e^{\lambda_j X_i}}\right\}. \quad (38)$$

With the quadratic loss function, the Bayesian estimate for any function of the unknown parameter vectorΦ, denoted as $\nu(\Phi)$ is the posterior mean of that function. That is,

$$\hat{\nu} = E_{\Phi|.}(\nu(\Phi)) = \int_0^{\infty} \nu(\Phi)\ g(\Phi \mid .)\ d\ \Phi. \quad (39)$$

The integral in Eq. (39) and the normalization constant in Eq. (38) do not have analytical solutions. As a result, numerical methods are necessary for performing a Bayesian analysis of the underlying model. Among the various methods available, we will use the Markov Chain Monte Carlo (MCMC) simulation technique for this analysis. The MCMC algorithm enables us to generate random samples from the posterior distribution, as specified in Eq. (38), without needing to compute the normalization constant. These random samples can then be utilized to conduct any analysis of the model parameters and characteristics that we desire.

5.3 The Exponential distribution case

In this subsection, we present the posterior densities of the unknown parameters $\Phi = (\alpha_1, \alpha_2, ..., \alpha_k, \lambda_{j1}, ..., \lambda_{jm})$, $j = 1,2,..,k$, and then obtain the corresponding Bayes estimators of these parameters.

Assume that α_j and λ_{jl}, $j = 1,2,..,k; l = 1,2,..,m$ are independent. Also α_j has gamma prior densities $g(\alpha_j)$ with different known and nonnegative hyperparameters a_j and b_j. Also $g(\lambda_{jl})$, $l = 1,2,..,m$ have the normal prior densities, $g(\lambda_{jl})$, with means μ_{jl} and variances σ_{jl}^2.

Therefore, the joint prior density function of $\Phi = (\alpha_1, \alpha_2, ..., \alpha_k, \lambda_{j1}, ..., \lambda_{jm})$, $j = 1,2,..,k$, up to a constant is

$$g(\Phi) = \prod_{j=1}^{k}\prod_{l=1}^{m} g(\alpha_j)\, g(\lambda_{jl}) = \prod_{j=1}^{k}\prod_{l=1}^{m} \frac{b_j^a}{\sigma_{jl}\sqrt{2\pi}\Gamma(a_j)} \alpha_j^{a_j-1} e^{-\left(b_j\ \alpha_j + \frac{1}{2\sigma_{jl}^2}(\lambda_{jl}-\mu_{jl})^2\right)}. \quad (40)$$

From the likelihood function Eq. (40) and the joint prior density, the joint posterior density function of Φ, up to a constant, can be obtained as

$$g(\Phi \mid \text{data}) \propto \prod_{j=1}^{k}\prod_{l=1}^{m} \left[\frac{b_j^{a_j}}{\sigma_{jl}\sqrt{2\pi}\Gamma(a_j)} \alpha_j^{a_j-1} \times e^{-\left(b_j\ \alpha_j + \frac{1}{2\sigma^2}(\lambda_{jl}-\mu_{jl})^2\right)} \right] \times \prod_{i=1}^{n}\prod_{j=1}^{k} \left\{ \left[t_i\, e^{\lambda_j X_i} \right]^{I(\delta_i=j)} \left[e^{-\alpha_j t_i} \right]^{e^{\lambda_j X_i}} \right\}. \quad (41)$$

With the quadratic loss function, the Bayesian estimate for any function of the vector of unknown parametersΦ, represented as $\nu(\Phi)$ is the posterior mean of that function. In other words,

$$\hat{\nu} = E_{\Phi|.}(\nu(\Phi)) = \int_0^{\infty} \nu(\Phi)\, g(\Phi \mid .)\, d\,\Phi. \quad (42)$$

The integral in Eq. (42) and the normalization constant in Eq. (41) lack analytical solutions. Therefore, numerical methods must be applied for Bayesian analysis of the model. Out of several possible methods, we will use the Markov Chain Monte Carlo (MCMC) simulation technique for this purpose. The MCMC algorithm allows us to obtain random samples from the posterior distribution, as specified by the density in Eq. (41), without needing to compute the normalization constant. These random samples can then be utilized for any desired model parameters and characteristics analysis.

5.4 Markov chain Monte Carlo method

The Markov Chain Monte Carlo (MCMC) technique is one of the most effective methods in modern Bayesian statistics. The MCMC is an algorithm that summarizes the posterior distribution without requiring the calculation of the normalization constant. It has become one of the primary computational tools in Bayesian statistical inference due to its extensive use [29]. The Metropolis-Hastings sampler is a variation of the MCMC method. A key concept in MCMC is finding an appropriate distribution function, referred to as the "proposal," which meets two criteria: 1) it is easy to simulate from, and 2) it closely resembles the posterior distribution of interest. Once such a proposal is identified, random samples are drawn from it, and the acceptance-rejection rule is applied to obtain random samples from the target posterior distribution.

The Metropolis-Hastings algorithm is used to generate random samples from the posterior distribution $g(\Phi \mid .)$:

(1)Start with initial guess $\Phi^{(0)}$.
(2)Determine the number of trials for generating random samples, denoted as M.
(3)Repeat the following steps for $i = 1, \ldots, M$:
(i)Set $\Phi = \Phi^{(i-1)}$.
(ii)Generate a candidate Φ^* from a proposal distribution $P(\Phi^* \mid \Phi)$ and Φ^* from a proposal distribution $P(\Phi^* \mid \Phi)$.
(iii)Evaluate the acceptance probabilities

$$\eta_\Phi = \min\left[1, \frac{g(\Phi^* \mid .)\, P(\Phi^* \mid \Phi)}{g(\Phi^* \mid .)\, P(\Phi^* \mid \Phi)}\right].$$

(iv)Generate a $u_1$from a Uniform $(0,1)$ distribution,
If $u_1 < \eta_\Phi$, accept the proposal and set; else, $\Phi^{(i)} = \Phi^*$, else set $\Phi^{(i)} = \Phi^{(i-1)}$.

Under some regularity conditions on the proposal density, $P(\Phi^* \mid \Phi)$, the sequence of the simulated draws $\left\{\Phi^{(i)}\right\}_{i=1}^{M}$ and $\left\{\Phi^{(i)}\right\}_{i=1}^{M}$ will converge to random draws that follow the posterior density $g(\Phi \mid .)$.

6 Simulation

In this segment, simulation experiments are carried out to assess the effectiveness of the suggested techniques. The evaluation will involve a comparison of point estimation and interval estimation for parameters of exponential and Akshaya regression competing risk models performance from three distinct viewpoints:

–Mean square error (MSE) is calculated as the average of the squared differences between $\hat{\Phi}_i$ and Φ, where i represents the iteration number of simulation cycles. It serves as a measure to assess the extent of data variation. A smaller MSE value indicates a prediction model that provides a more accurate description of the experimental data.
–Bias is determined by the average difference between $\hat{\Phi}_i$ and Φ. This is a metric for assessing the level of data variation. When the bias is close to zero, it suggests that the prediction model provides a more accurate description of the experimental data and employs unbiased estimators.
–Regarding the length of confidence intervals (LCI), when all other variables remain constant, the average width estimated within the confidence interval at a

$100(1-\gamma)\%$ confidence level is influenced by both the sample size and the chosen confidence level γ.

Following the specification of the sample size (n) and parameters ($\alpha_1, \lambda_1, \lambda_2$ and α_2) of exponential and Akshaya regression competing risk models, the simulation process is carried out. Here in Table 3, ($\alpha_1 = 0.75 \text{and} 3, \lambda_1 = 1.2, \lambda_2 = 1.3$ and $\alpha_2 = 0.6$ and 2) & while in Table 4, ($\alpha_1 = 0.75 \text{and} 3, \lambda_1 = 0.5, \lambda_2 = 0.3$ and $\alpha_2 = 0.6$ and 2) have been taken. The results for various sample sizes and termination times have been supplied as 30, 50, 100, and 200. Tables 3 and 4 contain the bias and MSE as well as average lengths of confidence intervals (ALCIs) with a 95% confidence level. The ALCIs for MLE are designated as ALACI, and the ALCIs for Bayesian are designated as ALCCI. When it comes to Bayesian inference, we take into account the informative Gamma prior. The hyperparameters are taken by hyper-parameter elicitation techniques as follows:

$$a_j = \frac{\left[\frac{1}{I}\sum_{i=1}^{I}\hat{\Phi}_j^i\right]^2}{\frac{1}{I-1}\sum_{i=1}^{I}\left[\hat{\Phi}_j^i - \frac{1}{I}\sum_{i=1}^{I}\hat{\Phi}_j^i\right]^2}, \; j = 1,2,3,$$

$$b_j = \frac{\frac{1}{I}\sum_{i=1}^{I}\hat{\Phi}_j^i}{\frac{1}{I-1}\sum_{i=1}^{I}\left[\hat{\Phi}_j^i - \frac{1}{I}\sum_{i=1}^{I}\hat{\Phi}_j^i\right]^2},$$

where I is the number of iterations.

Every simulation's outcome is derived from 5000 replications. Based on the findings presented in Tables 1 and 2, several key observations stand out. First, it is evident that the bias and mean squared error (MSE) associated with maximum likelihood estimators (MLEs) and Bayesian estimation consistently diminish as the sample size increases. This phenomenon suggests that the estimators approach asymptotic unbiasedness, meaning that as the sample size grows significantly, these estimators converge toward the true parameter value. Furthermore, when examining the 95% ALCIs, it is clear that they contract as the sample size expands. All these findings collectively underscore the robustness and consistency of the proposed estimator. Using average weights, we note that the Bias, MSE, and ALCI decrease when the value of α_1 decreases and the value of α_2 increases. The outcomes from MLE and Bayesian estimation are both quite pleasing. Even though Bayesian estimation primarily uses an informative gamma prior, point estimation by Bayesian remains the more efficient approach. In the case of interval estimations, two methods are available, which can be chosen based on specific requirements. If a shorter estimation interval is desired, the highest posterior density (HPD) interval is the preferable option.

7 Application

In this section, the previous methods of statistical inference are applied to a real-life data set. In the study involving 329 men who have sex with men (MSM), we focus on the progression from HIV infection to AIDS, SI switch, and eventual death, paying particular attention to the cause of failure and the time of death for the patients. The data was collected from a specific period up until the introduction of combination anti-retroviral therapy in 1996. This data describes the competing risks model with two causes of failure. For further information on this data, see [30], [31], and [32].

This data set was used as an example for competing risk analysis in [33] and [34]. The data can be considered competing risk data with two different risks. Some participants in the study neither experienced an infection switch nor died, and these cases are treated as censored observations. The Aidssi2 data set expands on the original Aidssi data set in three ways: first, it accounts for events occurring after the initial one; second, it includes the entry times of individuals who joined the study after contracting HIV; and third, it adds the age at HIV infection as an additional covariate. The numbers in this set differ slightly from the original AIDS data set. Some participants were diagnosed with AIDS only upon death, while others had their last follow-up at the time of their AIDS diagnosis. To avoid recording two transitions at the same time, the time to AIDS was reduced by 0.025 years for these cases. This data set was also used as an example in multi-state analyses in [34]. In this study, we focus on two covariates from the real data: age at HIV infection and CCR5 genotype, which is categorized as "WW" (wild-type allele on both chromosomes) or "WM" (mutant allele on one chromosome). All calculations are applied by using the R language.

Table 2 presents a comparative analysis of MLE and Bayesian estimation methods for Exponential, Akshaya, and Rayleigh models. The table shows parameter estimates ($\alpha_1, \alpha_2, \lambda_1, \lambda_2$), their StEr, and the model selection criteria Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Criterion (HQIC), and Consistent Akaike Information Criterion (CAIC).

For the exponential model, both MLE and Bayesian approaches provide similar estimates for the parameters $\alpha_1, \alpha_2, \lambda_1$, and λ_2, with Bayesian estimates showing slightly lower standard errors compared to MLE. The model selection criteria (AIC, BIC, HQIC, and CAIC) show relatively low values, indicating a good fit for this model. In the Akshaya model, the Bayesian approach produces parameter estimates and standard errors relatively close to the MLE results. However, the standard errors in the Bayesian case appear larger for most parameters. Notably, the estimates for α_1 and α_2 differ slightly between the two approaches. The model selection criteria indicate moderate values compared to the exponential model. For the Rayleigh model, both MLE and Bayesian estimates differ more prominently, particularly for α_1 and α_2. The standard errors in the Bayesian approach are generally larger than those in MLE. The Rayleigh model's AIC, BIC, HQIC, and CAIC

values are higher compared to the exponential and Akshaya models, suggesting a relatively poorer fit.

Overall, the exponential model appears to outperform the other two models regarding model selection criteria, with lower AIC, BIC, HQIC, and CAIC values. Additionally, Bayesian estimates across all models exhibit slightly higher robustness, as seen through smaller standard errors compared to MLE in some cases. However, the differences between MLE and Bayesian estimates suggest the need for careful consideration of prior assumptions and model selection.

Likelihood profile plots are a valuable tool for checking the properties of MLE and understanding the behavior of the likelihood function. They can help you assess the stability and confidence intervals of the MLE for a specific parameter of interest. Figure 4 discusses the likelihood profile of competing risk parameters of exponential regression by MLE. We note the profile likelihood is a symmetric, unimodal shape with a single peak, then it is typically indicative of a well-behaved MLE.

MCMC plots are graphical representations of the output generated by MCMC algorithms, which are used for Bayesian inference and parameter estimation of competing risk parameters of exponential regression. MCMC is a powerful technique for sampling from complex probability distributions, especially when you want to estimate posterior distributions in Bayesian statistics. Here are some common types of MCMC plots that were used in Figures 5,6,7:

- A plot of trace shows the values of a parameter or variable at each iteration of the MCMC chain. It is a simple line plot where the x-axis represents the iteration number and the y-axis represents the parameter's value. A good trace plot should exhibit "mixing," where the chain explores the parameter space without getting stuck in a single region. Lack of mixing could indicate convergence problems.
- Density plots, also known as kernel density plots or histogram plots, provide an estimate of the probability density function of the values of the sampled parameters. They can help you visualize the shape of the posterior distribution.
- Convergence diagnostics plots often include diagnostic statistics like the Geweke score, effective sample size, or divergent transitions (for Hamiltonian Monte Carlo methods like Stan). These diagnostics provide information about the convergence and the quality of the MCMC sampling.

Table 1: Risk analysis for MLE and Bayesian

	Exponential		Akshaya		Rayleigh	
	MLE	Bayesian	MLE	Bayesian	MLE	Bayesian
π_1	0.5826	0.6620	0.5633	0.6811	0.41803	0.33485
π_2	0.4174	0.3380	0.4366	0.3188	0.58197	0.66515

Figures 5,6,7 refers to the MCMM plots of competing exponential regression risk parameters having convergence, and this result is well done with exponential regression competing risk parameters.

Table 1 presents the computation of relative risks and survival rates for different methods at different time points. In Figure 8, you can observe the trace and posterior density plots for the π_j draws and random draw plots for π_j with symmetric normal distribution. We then utilized these random draws to generate samples for sub-survivors and overall survivors, which were subsequently used to derive Bayesian estimators and 95% credible intervals for these metrics at various time points, as depicted in Figure 9.

can achieve which is not appropriate for the system. Overview of the analysis of survival competing risk:

- Researchers need to determine if their research objective is to address etiological questions to estimate incidence or to predict prognosis.
- The Fine-Gray subdistribution hazard model should be used to estimate incidence or predict prognosis in the presence of competing risks.
- To address etiological questions, use the cause-specific hazards model.
- In certain situations, estimating both types of regression models for each competing risk is important to fully understand how covariates influence the incidence and rate of occurrence of each outcome.

8 Conclusions

In this paper, we introduced a competing risk model with the effect of covariates and applied the Cox regression model to estimate the coefficient of the covariates. Lifetimes follow three different distributions with one parameter. The likelihood function is used to obtain the point estimates of the unknown parameters of the underlying models and obtain asymptotic confidence intervals in the presence of censoring data. The Markov chain Monte Carlo technique by the Metropolis-Hastings algorithm is applied to do the Bayesian analysis of the parameters. The MCMC gets random draws from the joint posterior distribution function. In addition, credible intervals of all parameters are obtained, as are important reliability measures such as relative risks in the general case. In Bayes analysis, a gamma-predicted distribution is used for the unknown parameters of the Weibull distribution with known hyperparameters. In addition, the prior normal distribution of the unknown coefficient parameters with known means and variances is considered. A real data set is used in the application of the underlying model. The results are also listed in a detailed discussion.

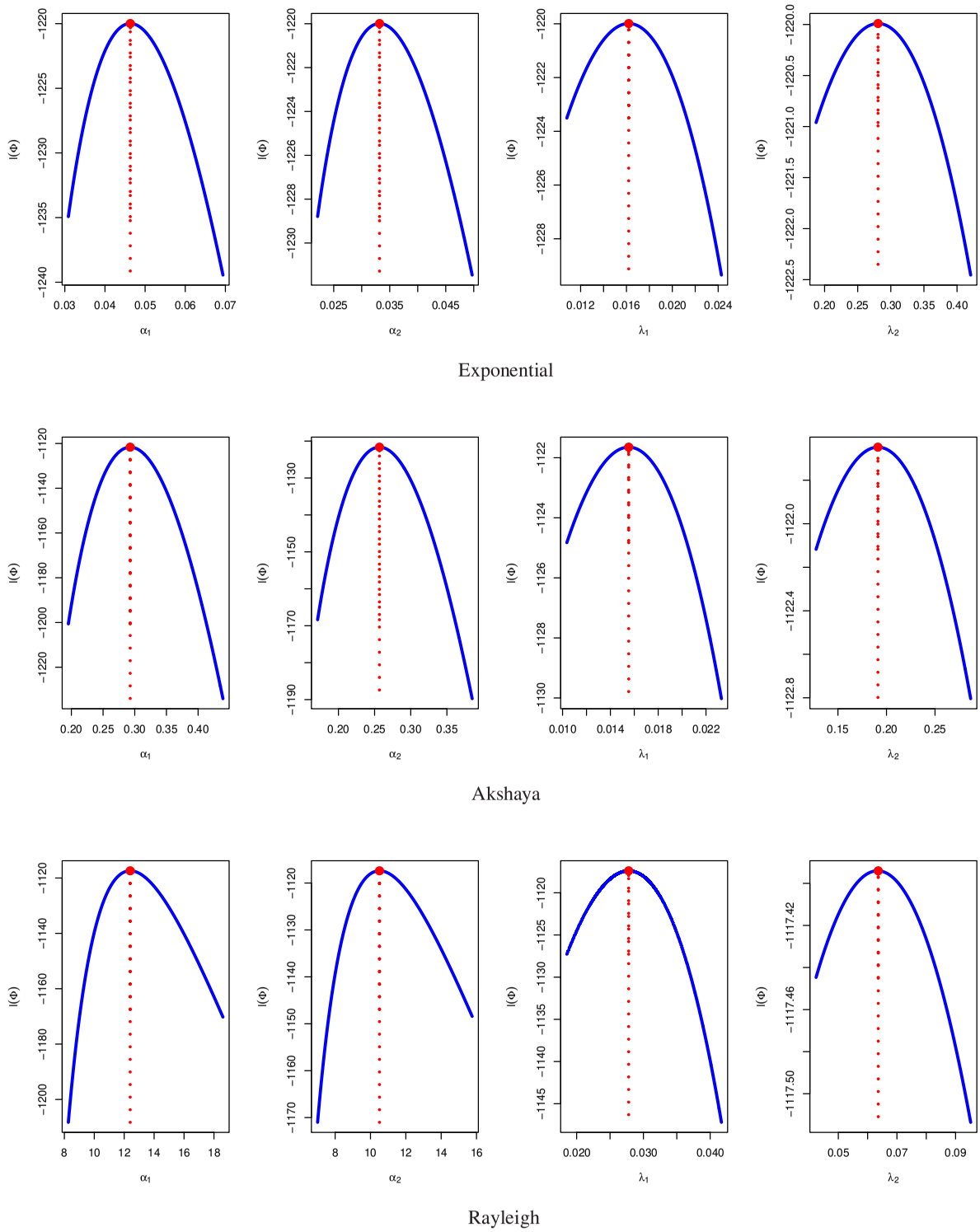


Fig. 4: Likelihood profile to check of maximum Likelihood estimation

Table 2: MLE and Bayesian

	Exponential				Akshaya				Rayleigh			
	MLE		Bayesian		MLE		Bayesian		MLE		Bayesian	
	estimates	StEr	estimates	StEr	estimates	StEr	estimates	StEr	estimates	StEr	estimates	StEr
α_1	0.0463	0.0025	0.0472	0.0013	0.2928	0.0358	0.3025	0.0286	12.4016	4.6308	13.1544	2.2906
α_2	0.0332	0.0080	0.0241	0.0055	0.2567	0.0264	0.1704	0.0175	10.5107	2.0422	9.3333	0.7359
λ_1	0.0162	0.0092	0.0168	0.0084	0.0155	0.0082	0.0133	0.0065	0.0278	0.0001	0.0300	0.0101
λ_2	0.2807	0.0382	0.4965	0.0166	0.1912	0.1739	0.8671	0.1672	0.0636	0.0387	-0.0058	0.1004
AIC	-2431.981				-2235.296				-2226.788			
BIC	-2416.797				-2220.112				-2211.604			
HQIC	-2425.923				-2229.239				-2220.731			
CAIC	-2431.857				-2235.173				-2226.665			

Table 3: MLE and Bayesian for exponential regression competing risks with bias, MSE, and LCI: $\lambda_1 = 1.2, \lambda_2 = 1.3$

α_1			0.75						3					
$\lambda_1 = 1.2, \lambda_2 = 1.3$			MLE			Bayesian			MLE			Bayesian		
α_2	n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
0.6	30	α_1	0.1757	0.4433	2.5187	-0.0936	0.0534	0.7626	0.4267	3.2357	6.8561	-0.2304	0.3543	1.9593
		α_2	0.0220	0.0213	0.5660	-0.0272	0.0189	0.4638	0.1068	1.8472	5.6070	-0.0731	0.0458	0.7180
		λ_1	-1.2351	2.4998	3.8713	-0.7331	0.6250	0.9389	-1.1743	1.9536	2.9742	-0.8413	0.7396	0.6350
		λ_2	-1.3337	2.0471	2.0316	-0.7124	0.6091	1.0804	-1.4080	3.6052	3.9643	-0.5816	0.5439	1.5528
	50	α_1	0.0810	0.1666	1.5690	-0.0908	0.0361	0.6260	0.2656	1.5439	4.7605	-0.2241	0.2468	1.6904
		α_2	0.0099	0.0189	0.5371	-0.0087	0.0155	0.3419	0.0621	0.0555	0.9240	-0.0612	0.0345	0.6079
		λ_1	-1.2118	2.3298	3.6401	-0.7241	0.6203	1.0559	-0.9211	1.8191	2.3259	-0.7281	0.7083	0.5328
		λ_2	-1.2893	1.8293	1.6028	-0.6819	0.5727	0.8105	-1.3123	2.5754	3.6228	-0.5064	0.5073	1.3998
	100	α_1	0.0397	0.0844	1.1286	-0.0291	0.0245	0.4770	0.1454	0.7459	3.3389	-0.1384	0.1135	1.0802
		α_2	0.0050	0.0076	0.3411	-0.0070	0.0069	0.2449	0.0107	0.0357	0.7395	-0.0592	0.0194	0.4044
		λ_1	-1.1841	1.8344	2.5783	-0.6824	0.6073	0.7436	-0.9020	1.6822	1.9294	-0.4981	0.6916	0.4207
		λ_2	-1.1298	1.7460	0.9673	-0.6081	0.4944	0.4694	-0.9331	2.1491	2.4078	-0.4805	0.4711	0.8410
	200	α_1	0.0214	0.0318	0.6938	-0.0170	0.0137	0.3608	0.0647	0.3527	2.3154	-0.1259	0.0644	0.7702
		α_2	0.0026	0.0042	0.2546	-0.0026	0.0041	0.1707	-0.0050	0.0176	0.5209	-0.0478	0.0118	0.3007
		λ_1	-1.0212	1.6348	1.5951	-0.4931	0.5890	0.5052	-0.8200	1.5520	1.3137	-0.3502	0.4603	0.2507
		λ_2	-0.9299	1.7231	0.7430	-0.6024	0.4060	0.3660	-0.8286	1.8130	1.5624	-0.3091	0.3858	0.6442
2	30	α_1	0.2543	1.0025	3.7990	-0.1008	0.0773	0.9095	0.4668	3.2415	6.8197	-0.2285	0.3546	2.0668
		α_2	0.1285	0.3560	2.2857	-0.1073	0.1101	1.0560	0.2387	0.1666	1.5982	-0.2082	0.1142	1.0345
		λ_1	-1.2093	3.5264	5.6357	-0.6091	0.4945	1.2193	-1.2312	2.1236	3.0578	-0.8503	0.7585	0.6138
		λ_2	-1.2998	1.9949	2.1683	-0.8434	0.7546	0.7079	-1.3176	1.9076	1.6241	-0.8447	0.7554	0.7163
	50	α_1	0.1433	0.3619	2.2916	-0.0983	0.0567	0.8292	0.2387	1.3427	4.4472	-0.1963	0.1599	1.3482
		α_2	0.0572	0.1945	1.7150	-0.0483	0.0808	0.8243	0.0118	0.1012	1.2471	-0.0109	0.0606	0.6140
		λ_1	-1.1935	2.6797	4.3940	-0.5646	0.4155	1.1910	-1.2289	1.8497	2.2848	-0.7941	0.6901	0.4181
		λ_2	-1.2093	1.8439	1.6220	-0.7940	0.6906	0.5207	-1.3017	1.7829	1.1673	-0.7966	0.6948	0.4426
	100	α_1	0.1141	0.2406	1.8709	-0.0910	0.0347	0.5989	0.0719	0.5533	2.9037	-0.0621	0.1240	1.0822
		α_2	0.0102	0.1001	1.2400	-0.0102	0.0420	0.5202	-0.0108	0.0615	0.9721	-0.0091	0.0538	0.4973
		λ_1	-0.9261	2.2974	3.2989	-0.4786	0.4067	0.5523	-1.1887	1.5881	1.6410	-0.6767	0.6678	0.3968
		λ_2	-1.1275	1.7068	1.1097	-0.6037	0.6084	0.3107	-1.2842	1.6974	0.8608	-0.6044	0.5981	0.3094
	200	α_1	0.0436	0.0804	1.0989	-0.0308	0.0176	0.3977	0.0573	0.3869	2.4290	-0.0517	0.0799	0.8515
		α_2	0.0101	0.0640	0.9859	-0.0101	0.0271	0.3709	-0.0010	0.0471	0.8510	-0.0008	0.0353	0.4123
		λ_1	-0.9022	1.8115	2.2505	-0.3875	0.3789	0.5316	-1.1901	1.5428	1.3944	-0.2993	0.5994	0.3269
		λ_2	-0.9312	1.2767	0.8392	-0.5108	0.4232	0.2117	-1.2894	1.6978	0.7364	-0.3506	0.5110	0.2925

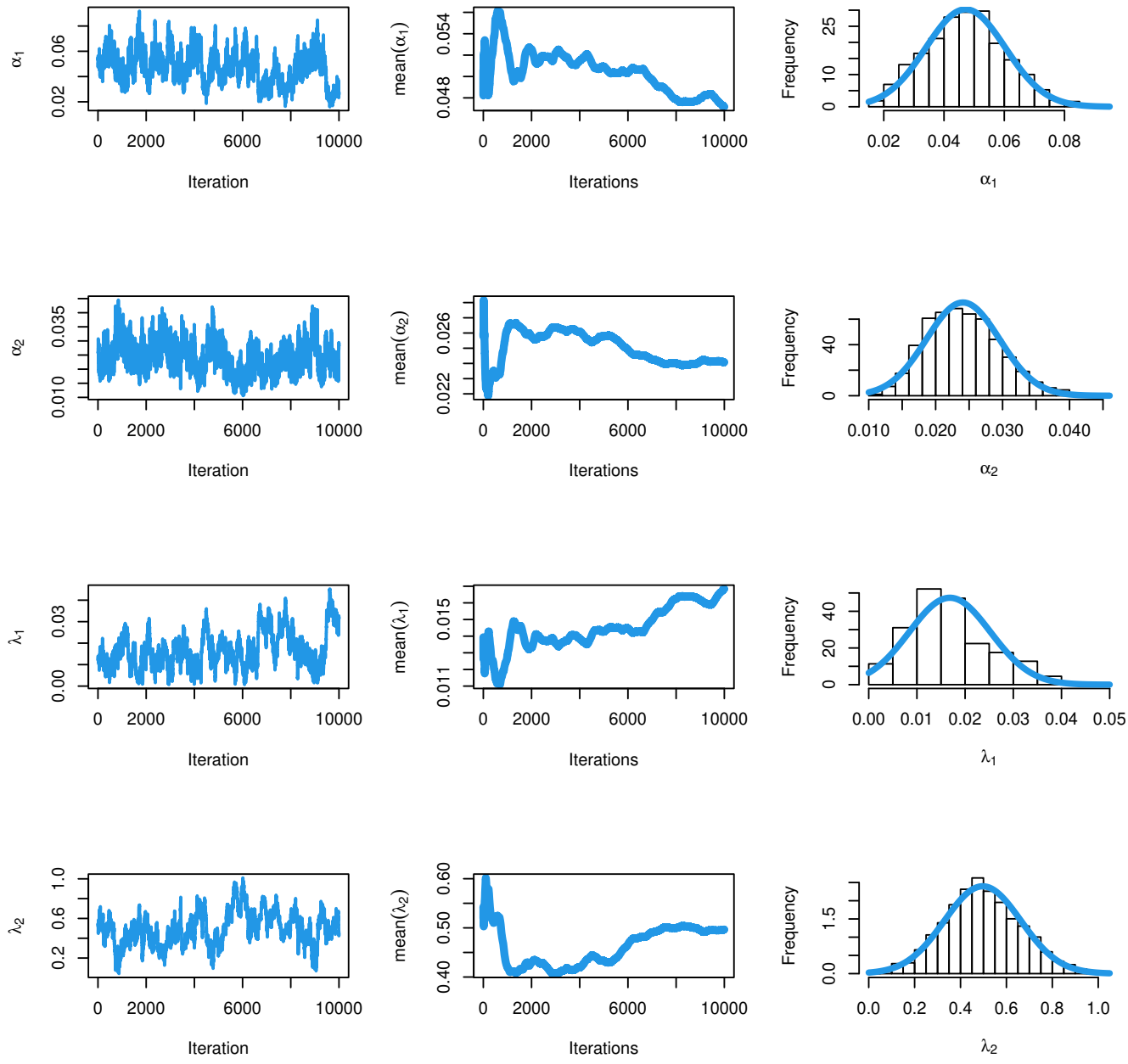


Fig. 5: MCMC plots for Exponential model parameters

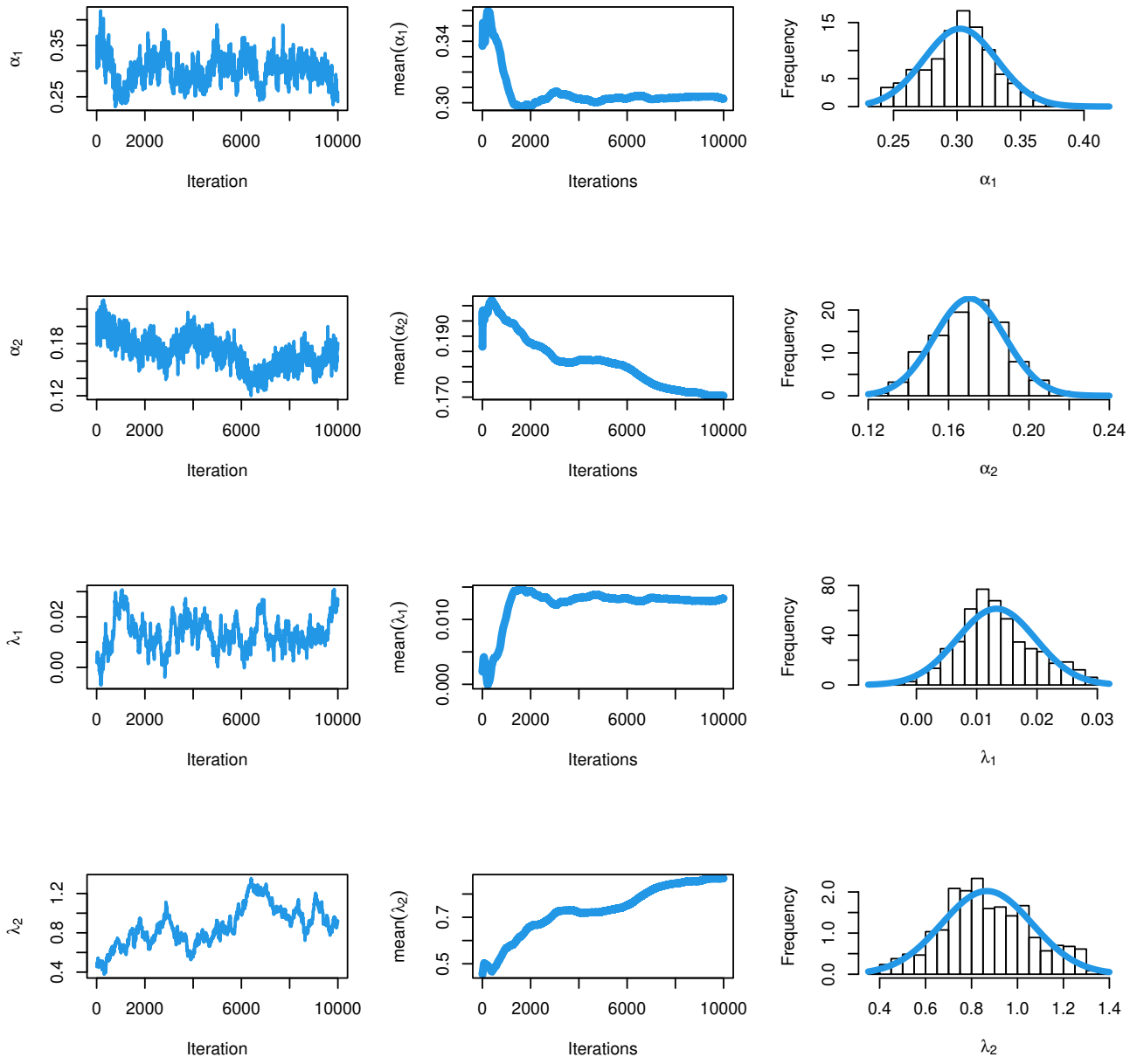


Fig. 6: MCMC plots for Akshaya model parameters

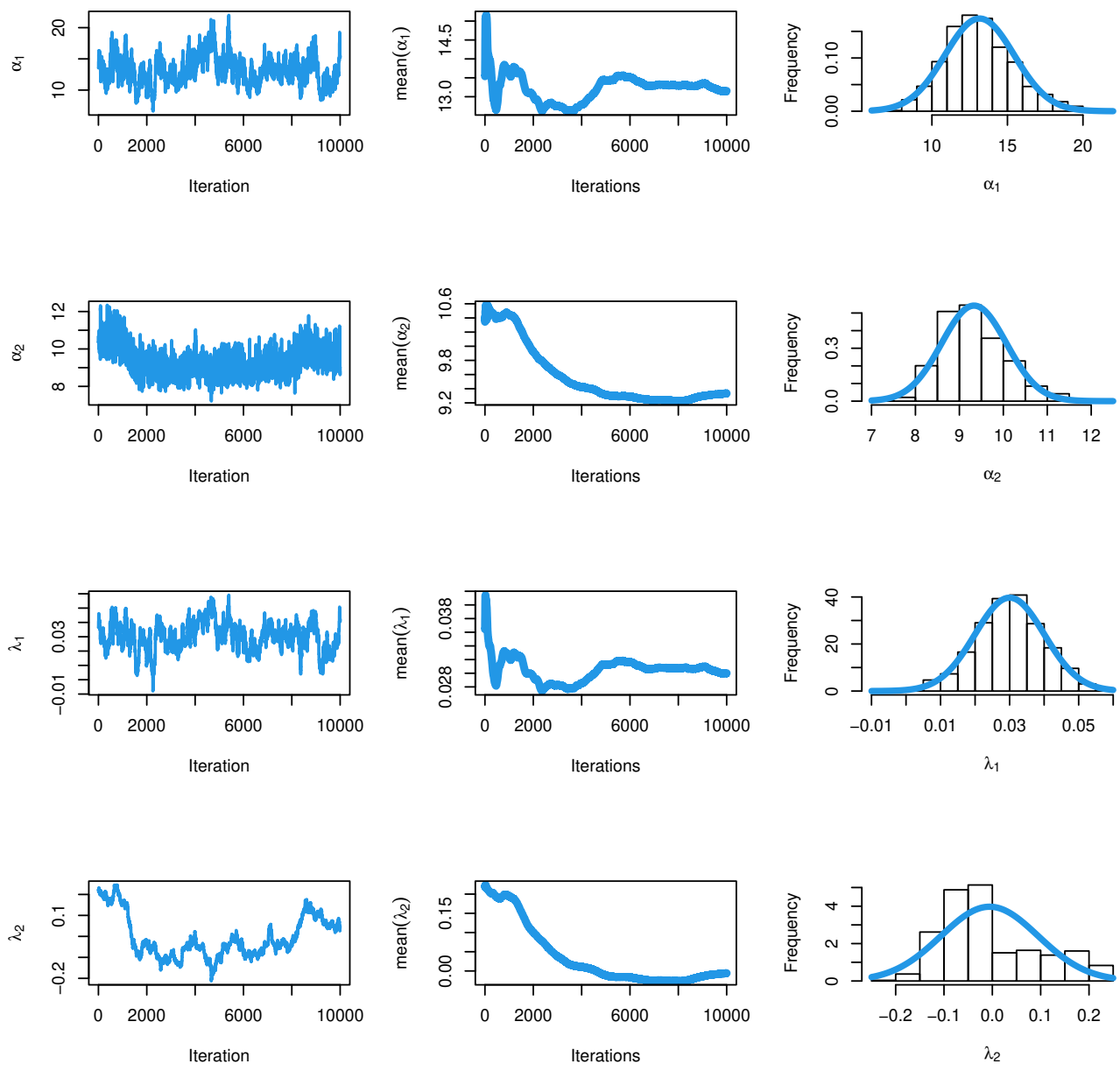
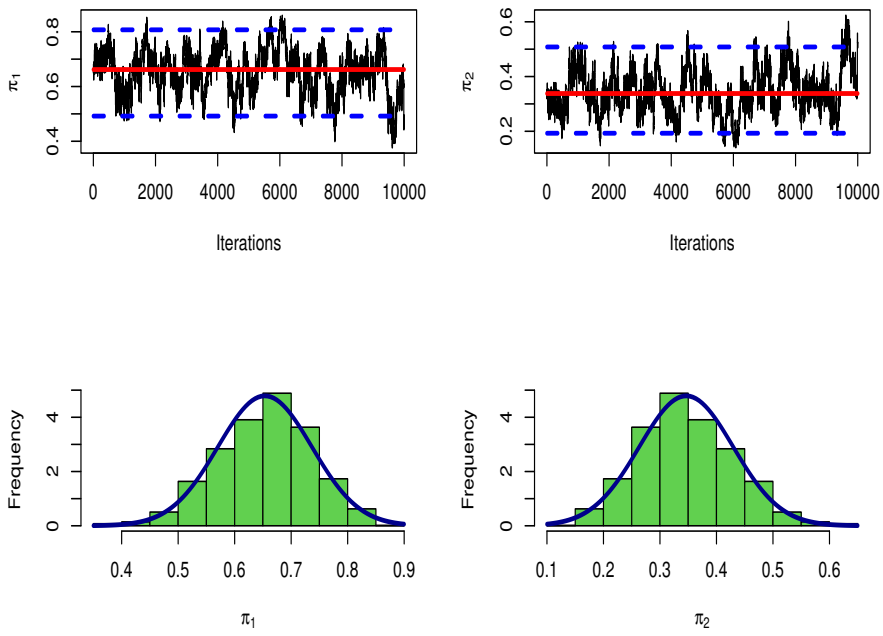
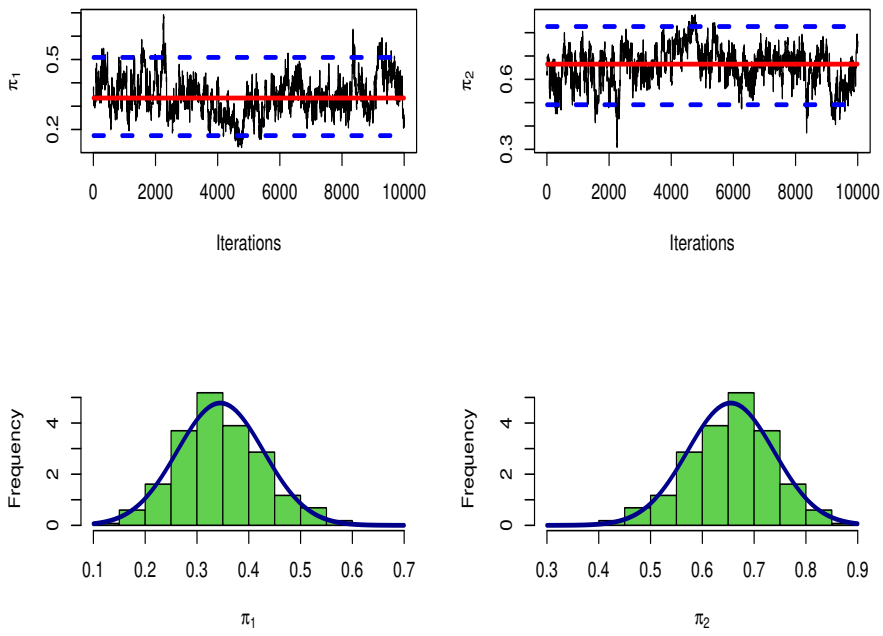


Fig. 7: MCMC plots for Rayleigh model parameters

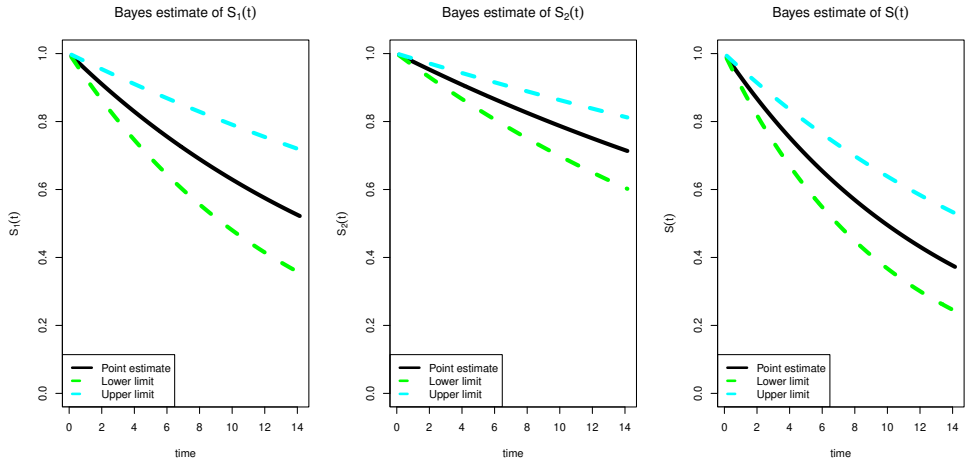


Exponential



Rayleigh

Fig. 8: MCMC plots for risk model estimate



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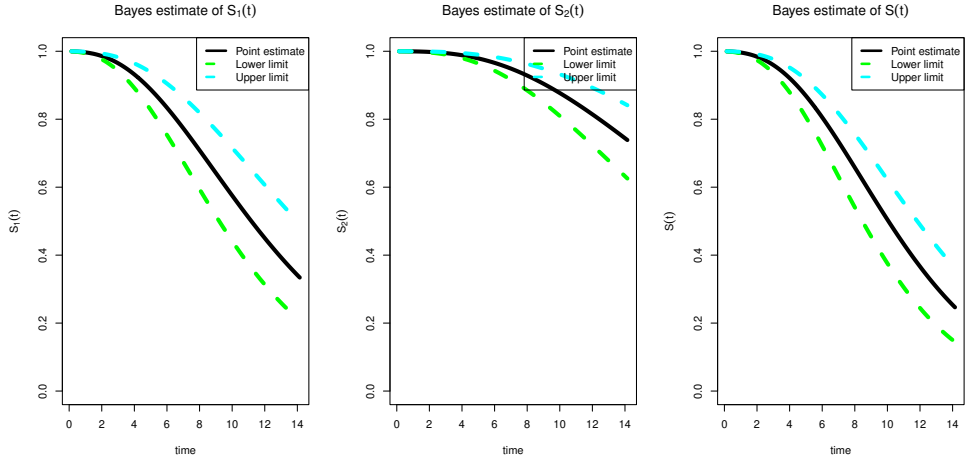
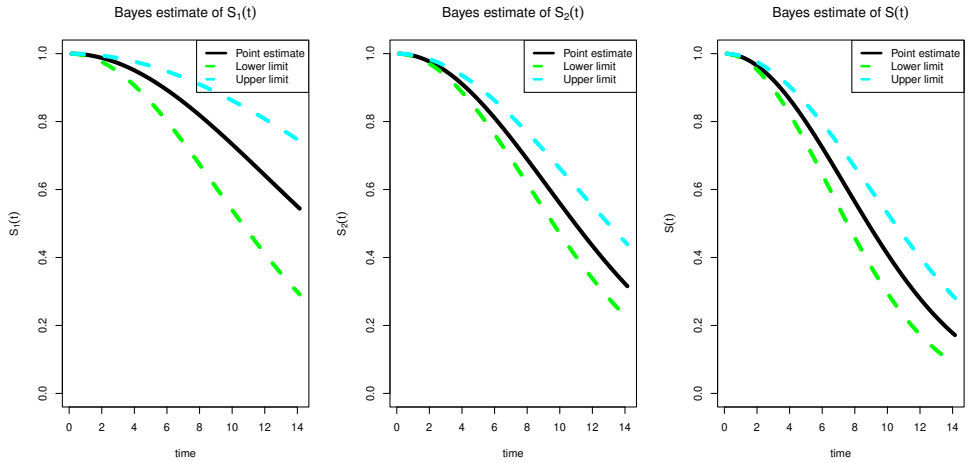


Akshaya



Rayleigh

Fig. 9: Survival plots by Bayesian estimation

Table 4: MLE and Bayesian for exponential regression competing risks with bias, MSE, and LCI: $\lambda_1 = 0.5, \lambda_2 = 0.3$

α_1			0.75						3					
$\lambda_1 = 0.5, \lambda_2 = 0.3$			MLE			Bayesian			MLE			Bayesian		
α_2	n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
0.6	30	α_1	0.2307	0.5587	2.7885	-0.0625	0.0541	0.8301	0.2570	1.6198	4.8886	-0.2427	0.2638	1.7715
		α_2	0.0845	0.0230	0.5938	-0.0721	0.0129	0.4277	0.0132	0.0576	0.9399	-0.0131	0.0246	0.5917
		λ_1	-0.5858	1.3155	3.8672	-0.2510	0.0884	0.9255	-0.5055	0.6356	2.4178	-0.2981	0.0844	0.6589
		λ_2	-0.3060	0.3900	2.1351	0.0951	0.0593	0.7905	-0.4500	2.5659	6.0293	0.0190	0.0585	0.7787
	50	α_1	0.0728	0.1612	1.5485	-0.0599	0.0298	0.5863	0.1230	0.7755	3.4200	-0.1020	0.1287	1.1295
		α_2	0.0514	0.0203	0.5555	-0.0465	0.0086	0.3198	0.0117	0.0324	0.7052	-0.0054	0.0126	0.3878
		λ_1	-0.4547	0.9371	3.3516	-0.2101	0.0869	0.9056	-0.4996	0.4511	1.7604	-0.2825	0.0958	0.4294
		λ_2	-0.3016	0.2994	1.7906	-0.0931	0.0501	0.7325	-0.3024	0.3717	2.0763	-0.0150	0.0468	0.7420
	100	α_1	0.0461	0.0722	1.0383	-0.0415	0.0163	0.4300	0.0825	0.4268	2.5417	-0.0815	0.0651	0.8162
		α_2	0.0104	0.0085	0.3595	-0.0461	0.0058	0.2374	-0.0074	0.0169	0.5095	-0.0055	0.0084	0.2856
		λ_1	-0.4019	0.6099	2.3467	-0.2026	0.0804	0.6697	-0.4901	0.3700	1.4128	-0.2326	0.1146	0.3229
		λ_2	-0.2931	0.1723	1.0851	-0.0485	0.0339	0.5882	-0.2843	0.2362	1.5462	-0.0137	0.0388	0.6530
	200	α_1	0.0157	0.0390	0.7719	-0.0106	0.0104	0.3204	0.0015	0.0202	0.5571	-0.0014	0.0171	0.3165
		α_2	0.0053	0.0045	0.2631	-0.0134	0.0043	0.1778	0.0016	0.0012	0.1348	-0.0015	0.0010	0.1328
		λ_1	-0.3761	0.4325	1.7792	-0.0913	0.0801	0.5367	-0.4099	0.2559	0.3162	-0.2143	0.1083	0.1111
		λ_2	-0.2921	0.1218	0.7498	-0.0033	0.0271	0.4560	-0.2530	0.1016	0.3893	-0.0123	0.0269	0.3671
2	30	α_1	0.2647	1.0678	3.9296	-0.0833	0.0862	0.9913	0.4928	5.2647	8.7888	-0.2056	0.4562	2.3849
		α_2	0.1438	0.3611	2.2955	-0.0539	0.0916	1.1740	0.0697	0.3081	2.1597	-0.0384	0.1064	1.2788
		λ_1	-0.4291	2.5582	6.0616	-0.0962	0.1017	1.0236	-0.4464	1.1452	3.8143	-0.1308	0.0822	0.8105
		λ_2	-0.3093	0.3853	2.1173	-0.0430	0.0484	0.7847	-0.3440	0.4666	2.3143	-0.0078	0.0479	0.6843
	50	α_1	0.1333	0.3696	2.3263	-0.0737	0.0543	0.8088	0.4124	3.4732	7.1280	-0.1997	0.2263	1.6522
		α_2	0.0497	0.1892	1.6947	-0.0412	0.0760	0.9441	0.0376	0.2481	1.9479	-0.0111	0.0658	0.9113
		λ_1	-0.4153	1.4669	4.2748	-0.0835	0.1017	0.9934	-0.4450	0.8795	3.1041	-0.1209	0.0752	0.5991
		λ_2	-0.2938	0.2647	1.6563	-0.0384	0.0360	0.6250	-0.2853	0.2928	1.8033	-0.0061	0.0479	0.7122
	100	α_1	0.0768	0.1486	1.4818	-0.0676	0.0288	0.5967	0.1943	1.2122	4.2503	-0.1570	0.1156	1.1427
		α_2	0.0253	0.1063	1.2750	-0.0280	0.0427	0.6242	0.0024	0.0966	1.2188	-0.0012	0.0391	0.6189
		λ_1	-0.4076	0.8650	3.0566	-0.0714	0.0822	0.8273	-0.4151	0.5635	2.1666	-0.1027	0.0692	0.4668
		λ_2	-0.2894	0.1637	1.1091	-0.0281	0.0273	0.4849	-0.2191	0.1503	1.0060	-0.0058	0.0271	0.4829
	200	α_1	0.0412	0.0672	1.0040	-0.0407	0.0163	0.4112	0.0635	0.5235	2.8268	-0.0514	0.0591	0.7693
		α_2	0.0199	0.0550	0.9164	-0.0120	0.0275	0.4403	0.0014	0.0577	0.9421	-0.0011	0.0299	0.4944
		λ_1	-0.3852	0.5589	2.1028	-0.0622	0.0805	0.5672	-0.3473	0.4026	1.6580	-0.0913	0.0511	0.3662
		λ_2	-0.2730	0.1372	0.8300	-0.0122	0.0266	0.3679	-0.2094	0.1308	0.8285	-0.0049	0.0253	0.4421

Table 5: MLE and Bayesian for Akshaya regression competing risks with bias, MSE, and LCI: $\lambda_1 = 1.2, \lambda_2 = 1.3$

α_1			0.75						3					
$\lambda_1 = 1.2, \lambda_2 = 1.3$			MLE			Bayesian			MLE			Bayesian		
α_2	n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
2	30	α_1	0.9561	1.3254	4.9593	0.0852	0.0688	0.9286	1.5978	5.1813	6.3585	0.5410	0.6301	2.2428
		α_2	0.5526	0.4913	1.6910	0.2932	0.1400	0.8511	0.2024	0.1839	1.4831	0.0855	0.0796	1.1019
		λ_1	-2.3659	6.0840	7.5962	-0.2885	0.5582	2.2745	-1.3184	2.4754	3.3676	-0.7023	0.6041	1.0973
		λ_2	-1.1311	1.6553	2.4043	-0.7231	0.6249	1.0909	-0.8720	1.5953	3.5834	-0.4621	0.4582	1.7016
	50	α_1	0.6723	0.9657	2.9415	0.0710	0.0504	0.7695	1.4224	3.3458	4.5102	0.5117	0.4400	1.5855
		α_2	0.4799	0.3807	1.5209	0.2388	0.0886	0.6707	0.0376	0.1242	1.3743	0.0246	0.0446	0.8502
		λ_1	-1.8450	5.0739	5.6734	-0.2441	0.5511	1.8038	-1.1383	2.2626	3.3110	-0.7048	0.5893	1.0385
		λ_2	-1.1211	1.4848	1.8730	-0.6834	0.6175	0.7668	-0.6122	0.7393	2.3680	-0.4587	0.4466	1.3951
	100	α_1	0.5414	0.4093	1.3371	0.0613	0.0412	0.5944	1.3819	2.5157	3.0530	0.5031	0.3595	1.0758
		α_2	0.4592	0.3178	1.0769	0.2272	0.0819	0.5313	0.0348	0.0607	0.9479	0.0235	0.0244	0.5712
		λ_1	-1.5346	4.9918	4.7854	-0.2158	0.5387	1.3677	-0.9413	2.2687	2.0430	-0.6843	0.5236	0.6683
		λ_2	-1.0176	1.4798	1.3259	-0.5917	0.6088	0.6016	-0.5106	0.5890	1.5938	-0.4166	0.4051	0.9945
	200	α_1	0.5262	0.3826	1.0161	0.0550	0.0345	0.4003	1.3386	2.0874	2.1325	0.4728	0.3269	0.7659
		α_2	0.4278	0.2615	0.7156	0.2095	0.0820	0.3687	0.0290	0.0299	0.6681	0.0215	0.0157	0.4379
		λ_1	-1.4462	4.7642	3.4644	-0.2057	0.5063	0.8790	-0.9241	2.1148	1.4665	-0.6491	0.4856	0.5116
		λ_2	-0.9167	1.4113	0.8822	-0.5196	0.5943	0.4740	-0.5016	0.4534	1.1460	-0.3662	0.3914	0.8495
0.6	30	α_1	0.5380	0.3912	1.2509	0.2668	0.0966	0.5686	0.7351	1.3316	3.4941	0.1173	0.1327	1.3577
		α_2	0.0862	0.0155	0.3533	0.0735	0.0134	0.3630	-0.2495	0.2559	1.7286	-0.0191	0.0382	0.8075
		λ_1	-1.5665	3.1879	3.3602	-0.7375	0.6624	0.9574	-1.1872	1.7238	2.2023	-0.5940	0.5039	1.0250
		λ_2	-0.4950	0.6591	2.5238	-0.3599	0.4623	1.9331	-1.1589	6.5545	8.4255	-0.0934	0.7171	2.9575
	50	α_1	0.5269	0.3590	1.1353	0.2570	0.0883	0.4843	0.6786	0.9262	2.6764	0.1106	0.1040	1.0976
		α_2	0.0318	0.0068	0.2979	0.0546	0.0051	0.2920	-0.1622	0.2042	1.6540	-0.0140	0.0228	0.6153
		λ_1	-1.3954	3.0397	2.9884	-0.6837	0.6556	0.7598	-1.0212	1.7139	1.9404	-0.5677	0.4608	1.0052
		λ_2	-0.1967	0.3295	2.1151	-0.1835	0.3197	1.8127	-1.1295	5.0804	6.0816	-0.0745	0.6629	2.6956
	100	α_1	0.4562	0.3507	0.7363	0.2302	0.0797	0.3150	0.6483	0.7204	2.1486	0.1027	0.0878	0.7484
		α_2	0.0306	0.0042	0.2257	0.0305	0.0036	0.2040	-0.1226	0.1104	1.2112	-0.0136	0.0166	0.4514
		λ_1	-1.1846	2.7782	2.3869	-0.5922	0.6088	0.5322	-1.0218	1.6285	1.4919	-0.4358	0.4179	1.0007
		λ_2	-0.1831	0.2395	1.4793	-0.1429	0.2352	1.4650	0.2874	4.3503	5.5072	-0.0616	0.6598	2.6105
	200	α_1	0.4536	0.3038	0.4991	0.2149	0.0610	0.2322	0.6234	0.4956	1.2833	0.0902	0.0810	0.5432
		α_2	0.0247	0.0037	0.1517	0.0214	0.0035	0.1470	-0.0939	0.0405	0.6976	-0.0126	0.0113	0.3317
		λ_1	-0.9772	2.2841	1.4944	-0.5030	0.6068	0.2821	-1.0231	1.5878	1.0618	-0.3955	0.3929	0.3816
		λ_2	-0.1437	0.2251	0.9627	-0.1349	0.2033	0.9119	0.0189	2.5159	4.0111	-0.0528	0.5269	2.2182

Table 6: MLE and Bayesian for Akshaya regression competing risks with bias, MSE, and LCI: $\lambda_1 = 0.5, \lambda_2 = 0.3$

α_1			0.75						3					
$\lambda_1 = 0.5, \lambda_2 = 0.3$			MLE			Bayesian			MLE			Bayesian		
α_2	n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
0.6	30	α_1	0.5640	0.4312	1.3194	0.2807	0.1029	0.6026	0.7110	1.1038	3.0336	0.2317	0.1649	1.3216
		α_2	0.0853	0.0163	0.3730	0.1413	0.0152	0.2558	-0.1481	0.2159	1.7272	-0.0098	0.0407	0.7518
		λ_1	-0.9725	1.7178	3.4462	-0.1463	0.0962	0.9272	-0.5189	0.6665	2.4719	-0.1218	0.0800	0.8471
		λ_2	0.5074	0.7223	2.6739	0.0451	0.0719	0.8706	-1.8174	3.7401	6.7198	0.0336	0.0636	0.8419
	50	α_1	0.5406	0.3567	0.9959	0.2788	0.0979	0.4599	0.6648	0.7832	2.2912	0.1857	0.0859	0.8604
		α_2	0.0950	0.0075	0.3368	0.1229	0.0068	0.2164	-0.0732	0.1527	1.3741	0.0068	0.0157	0.5075
		λ_1	-0.9059	1.6790	3.2081	-0.1446	0.0936	0.8292	-0.4525	0.5582	2.0832	-0.1029	0.0620	0.6846
		λ_2	0.4832	0.6804	2.2978	0.0448	0.0671	0.8283	1.4331	2.2434	4.1708	0.0106	0.0523	0.7476
	100	α_1	0.5337	0.3132	0.6594	0.2698	0.0963	0.3313	0.6316	0.5696	1.6201	0.1694	0.0710	0.6505
		α_2	0.0402	0.0047	0.2186	0.0204	0.0042	0.1651	-0.0682	0.0530	0.8446	0.0062	0.0084	0.3303
		λ_1	-0.8164	1.4434	2.0508	-0.1255	0.0939	0.5632	-0.4054	0.4570	1.5954	-0.0982	0.0617	0.6094
		λ_2	0.3731	0.6692	1.5616	0.0437	0.0607	0.8049	0.9138	1.8733	3.6947	0.0091	0.0505	0.7058
	200	α_1	0.5302	0.2954	0.4690	0.2318	0.0910	0.2335	0.6044	0.4353	1.0372	0.1304	0.0611	0.4891
		α_2	0.0363	0.0028	0.1515	0.0201	0.0024	0.1398	-0.0582	0.0248	0.5279	0.0051	0.0041	0.2530
		λ_1	-0.8078	1.2999	1.4560	-0.1133	0.0912	0.3550	-0.3916	0.3581	1.0182	-0.0826	0.0608	0.4623
		λ_2	0.3572	0.6007	1.0999	0.0411	0.0602	0.7844	0.8360	1.0396	3.1124	0.0081	0.0457	0.6251
2	30	α_1	1.4422	4.3357	6.1153	0.1481	0.0862	0.9655	1.5326	4.3263	5.5152	0.5811	0.6458	2.0011
		α_2	0.5447	0.4667	1.6174	0.3927	0.2152	0.9026	0.3004	0.2108	1.3615	0.3277	0.1572	0.8886
		λ_1	-1.7760	4.0257	5.0148	-0.0208	0.1438	1.2761	-0.7734	1.4735	3.6693	-0.1078	0.1008	0.9757
		λ_2	-0.1238	0.4093	2.4615	0.0164	0.0592	0.8125	0.1051	0.4640	2.6396	0.0406	0.0684	0.8408
	50	α_1	0.5307	0.6085	2.2423	0.1460	0.0656	0.7690	1.4430	3.4257	4.5456	0.5506	0.4650	1.5254
		α_2	0.5106	0.4105	1.5180	0.3300	0.1457	0.7340	0.1691	0.1231	1.2058	0.2581	0.0961	0.6594
		λ_1	-1.5890	3.3352	4.4515	-0.0100	0.1251	1.2141	-0.7085	1.3229	3.2971	-0.0912	0.0932	0.9358
		λ_2	-0.1156	0.2718	1.9514	0.0082	0.0522	0.7230	0.1028	0.3235	1.9488	0.0402	0.0634	0.8147
	100	α_1	0.5031	0.3659	1.3177	0.1446	0.0458	0.5633	1.2135	1.8853	2.5193	0.5418	0.3652	1.0304
		α_2	0.4716	0.2965	1.0678	0.3248	0.1263	0.5629	0.1166	0.0597	0.8420	0.2336	0.0474	0.5422
		λ_1	-1.5467	3.0889	3.7987	-0.0091	0.1081	0.9781	-0.6744	0.8587	2.1648	-0.0816	0.0785	0.7803
		λ_2	-0.1043	0.1409	1.3621	-0.0072	0.0349	0.6300	0.1003	0.2353	1.3723	0.0399	0.0603	0.7686
	200	α_1	0.4528	0.3383	0.9574	0.1377	0.0414	0.4060	1.1284	1.5898	1.9606	0.5377	0.3272	0.7860
		α_2	0.4626	0.2497	0.7411	0.3188	0.1158	0.4544	0.1036	0.0417	0.5980	0.2137	0.0406	0.4504
		λ_1	-1.4686	2.6454	3.5164	-0.0082	0.0938	0.8715	-0.6079	0.7881	1.5807	-0.0723	0.0760	0.5628
		λ_2	-0.0915	0.0815	0.9578	-0.0061	0.0289	0.5608	0.0928	0.1386	0.9510	0.0372	0.0547	0.7552

The results for all competing causes and the cause-specific and sub-distribution hazard functions were presented. This approach provides a more thorough understanding not only of the effects of prognostic factors but also of the absolute risks associated with different outcomes in the study sample. Decision-makers often find it challenging to account for all hazards when making clinical decisions. With the availability of advanced software, analyzing the cumulative incidence function has become more popular and commonly reported in recent years. Biases can arise when the Kaplan-Meier estimator is used to estimate the cumulative incidence of the event of interest or when a proportional hazards model is used to assess the effects of covariates on the cumulative incidence function for the cause-specific hazard function. Misclassifying competing events as censoring events can have practical implications in these analyses. Generally, the more competing events there are, the greater the likelihood that they will be incorrectly treated as censoring events. When the proportion of competing events exceeds 10%, it is crucial to carefully consider the scientific goals of the analysis, as well as the appropriate choice of endpoint and analysis method. In future studies, we will study the regression of competing risk models with dependent causes of failures. Also, we will study a two-stage estimation procedure for a copula-based model with competing risks data and semi-competing risks data

Data Availability Statement: The data used to support the findings of this study are included within the article.

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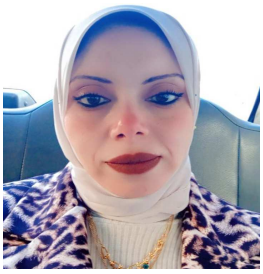

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