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Bayesian and Classical Inference for Power-Modified Kies-Exponential Distribution

F. N. Fouad[1,*], *M. M. Nassar*[1], *S. E. Abu-Youssef*[2] *and R. M. EL-Sagheer*[2,3]

[1]Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt
[2]Mathematics Department, Faculty of Science, AL-Azhar University, Nasr City (11884), Cairo, Egypt
[3]High Institute of Computers and Management Information Systems, First Statement, New Cairo (11865), Cairo, Egypt

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Abstract: This study aims to estimate the parameters of the power-modified Kies-exponential distribution and various lifetime measures, including reliability and hazard rate functions, under progressive type-II censoring. It investigates the application of maximum likelihood estimation, two-parametric bootstrap, and Bayesian approaches to derive these parameters and characteristics. Approximate confidence intervals and highest posterior density credible intervals are constructed using the asymptotic properties of maximum likelihood estimators and the Markov chain Monte Carlo method, respectively. Furthermore, the delta method is utilized to compute variances for reliability and hazard functions, while two bootstrap techniques are employed for confidence interval estimation. Bayesian inference is developed based on squared error loss functions. Lastly, comprehensive simulation studies are carried out to evaluate the effectiveness of these estimation techniques, and a real data analysis is performed to illustrate their practical applicability.

Keywords: Power-modified Kies-exponential; Resampling methods; Bayesian MCMC; Monte Carlo experiments; Computer simulation.

1 Introduction

To strike a balance between the total duration of the experiment and the number of units utilized, the experiment needs an effective control system (censoring scheme) to help the experimenter draw reliable statistical conclusions. This approach also preserves experimental units for future use, saving both time and costs. The most traditional and widely used censoring schemes are Type-I (time-based) and Type-II (failure-based). In these schemes, units cannot be removed from the experiment until either the experiment reaches its conclusion or the number of unit failures meets a specified threshold. These schemes help identify defective items after conducting the experiment. Balakrishnan and Sandhu [1] introduced a progressive Type-II censoring (PT-IIC) scheme, which is effective in achieving the objectives of various censoring strategies. PT-IIC is a crucial method used in statistical analysis and experimental design, particularly within industry, reliability, and engineering fields. This approach involves terminating an experiment once a pre-specified number of failures, mm, has occurred, with units being progressively withdrawn at each failure instance. It helps maintain a balance between the total test duration and the number of failures observed, thereby optimizing resource use and providing reliable data under practical constraints. In industry, this method ensures that testing resources are utilized efficiently while still gathering sufficient information on product performance and durability. In reliability engineering, it aids in estimating the lifetime and failure characteristics of components, contributing to improved design and quality control. Moreover, PT-IIC allows for cost-effective testing by avoiding excessive experimentation, making it an invaluable tool for engineers and manufacturers striving to enhance product reliability and safety. In summary, consider a life test involving n independent units with m observed failure times (where $m \leq n$), arranged in a progressive sample as $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$. Additionally, there is a predefined censoring plan $R = (R_1, R_2, ..., R_m)$. When the first failure occurs at X_1, R_1 surviving units are randomly removed from the test. At the second failure X_2, R_2surviving units are similarly removed, and this process continues until the $m-th$ failure. At this point, the remaining surviving units,

* Corresponding author e-mail: fa.nasr96@gmail.com

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$R_m = n - m - \sum_{i=1}^{m-1} R_i$, are withdrawn, and the test concludes. Several authors have explored inference using PT-IIC schemes in various applications. Notable examples include works by Fu et al. [2],Chen et al. [3], Xu et al. [4], and Luo et al. [5] and EL-Sagheer et al. [6].

The Power-Modified Kies-Exponential (PMKE) distribution, which was proposed by Afify et al. [7], emerges as an exceptional choice for statistical research endeavors owing to its remarkable adaptability and effectiveness in modeling real-world data. Its unique capacity to accommodate diverse data characteristics, ranging from symmetry to skewness and tail behavior, makes it a highly versatile tool for statisticians exploring empirical datasets. One of the most compelling aspects of the PMKE distribution is its ability to capture complex data patterns with precision, thereby enhancing the accuracy and reliability of statistical models. By leveraging its parameterization, researchers can conduct sophisticated statistical analyses, including hypothesis testing, parameter estimation, and uncertainty quantification, across a wide spectrum of research domains. In essence, the PMKE distribution represents a powerful asset for statistical research, offering unparalleled flexibility, accuracy, and adaptability for modeling complex data structures and driving advancements in statistical science. However, if X is a random variable that follows the PMKE distribution, denoted by $X \backsim PMKE(\varsigma)$,where $\varsigma = (\alpha, \beta, \lambda)$ is the parameters vector with shape paramerters α, β and scale parameter λ. Then its cumulative distribution function (CDF) and probability density function (PDF) can be written, respectively, as

$$F(x;\varsigma) = 1 - e^{-\left[e^{\lambda x^{\beta}} - 1\right]^{\alpha}}, x > 0; \alpha, \beta, \lambda > 0 \quad (1)$$

and

$$f(x;\varsigma) = \alpha\beta\lambda x^{\beta-1} e^{\alpha\lambda x^{\beta}} \left[1 - e^{-\lambda x^{\beta}}\right]^{\alpha-1} e^{-\left[e^{\lambda x^{\beta}} - 1\right]^{\alpha}}, \; x > 0; \alpha, \beta, \lambda > 0. \quad (2)$$

The associated reliability characteristics of X such as reliability funcion (RF) and hazard rate function (HRF) ,at mission time t, are expressed, respectively, as follows:

$$S(t;\varsigma) = e^{-\left[e^{\lambda t^{\beta}} - 1\right]^{\alpha}}, t > 0, \quad (3)$$

and

$$h(t;\varsigma) = \alpha\beta\lambda t^{\beta-1} e^{\alpha\lambda t^{\beta}} \left[1 - e^{-\lambda t^{\beta}}\right]^{\alpha-1}. \quad (4)$$

Figures 1 and 2 illustrate the PDF and HRF of X, respectively. The plots show that the PDF of X can exhibit various shapes, including left-skewed, reverse-J-shaped, or right-skewed distributions. Additionally, the HRF of X may display different patterns, such as a bathtub shape, or show a monotonic increase or decrease.

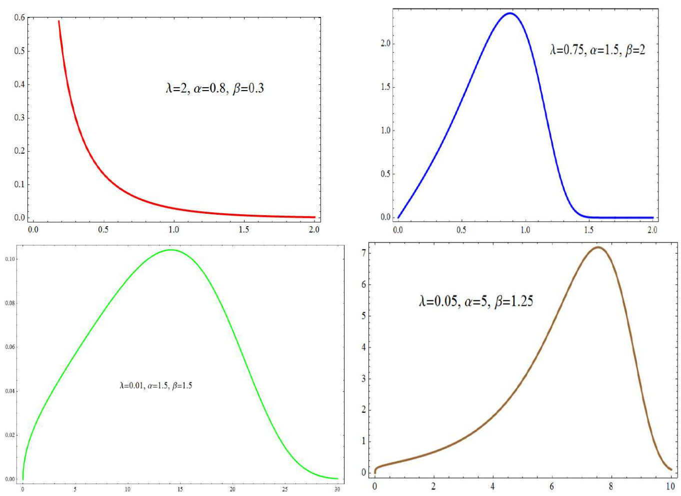

Fig. 1: PDF of the PMKE distribution

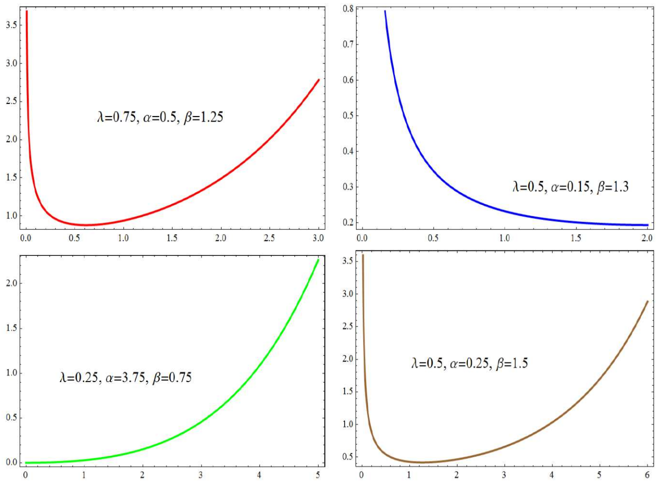

Fig. 2: HRF of the PMKE distribution

Many previous studies used PT-IIC data to consider some estimation issues for various lifetime distributions. Despite the PMKE distribution's versatility in modeling various data types, there has been no prior research, to our knowledge, on censoring mechanisms addressing parameter estimation. Therefore, this study stands out as the first attempt to investigate estimation issues for the PMKE distribution when using incomplete data collected from PT-IIC. Consequently, the main aim of this article is to estimate parameters for the PMKE distribution as well as lifetime indices reliability and hazard rate functions, under PT-IIC. It explores several estimation methods, including maximum likelihood, two-parametric bootstrap, and Bayesian techniques. Also, constructs approximate confidence intervals and highest posterior density credible intervals using the asymptotic distribution of maximum likelihood estimators and the MCMC method, respectively. It employs the delta method to calculate variances for reliability and hazard functions. Additionally, the study features extensive simulations to assess the performance of these methods and demonstrates their practical application through real data analysis. The rest of this paper is organized as follows: Section 2 addresses maximum likelihood estimation and the construction of asymptotic confidence intervals. Section 3 presents two approaches to parametric bootstrap methods. Section 4 details Bayesian estimation using the MCMC technique. In Section 5, a simulation study is conducted to evaluate and compare the effectiveness of the various estimation methods. Section 6 showcases a real-world dataset to illustrate the practical application of the proposed inference procedures. Finally, Section 7 concludes with a summary of the research findings and contributions.

2 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a method used to estimate the parameters of a statistical model by maximizing the likelihood function, which represents the probability of the observed data given certain parameter values. The ultimate goal is to find the parameter values that make the observed data most probable. MLE is widely appreciated for its desirable properties, including consistency, meaning that as the sample size increases, the estimates converge to the true parameter values. It is applicable across various models, from simple distributions to complex hierarchical models. To use MLE, one typically formulates a likelihood function based on the assumed model and observed data, then applies optimization techniques to find the parameter estimates that maximize this function. While MLE is robust and widely used, it can be sensitive to model assumptions and data quality, necessitating careful model selection and validation. In this section, we investigate MLE using the observed data provided. To improve the fit and accuracy of the data, we have expanded the PMKE distribution to include an additional parameter, resulting in a three-parameter model $\varsigma = (\alpha, \beta, \lambda)$. Furthermore, we compute the parameter estimates and approximate confidence intervals (ACIs) for both the survival function (SF) and the hazard rate function (HRF). Let $X_1 < X_2 < \cdots < X_m$ are PT-IIC sample drawn from PMKE distribution with a censoring scheme represented by $R = (R_1, R_2, ..., R_m)$. The likelihood function for this setup is specified as follows:

$$L(\varsigma|\underline{x}) = C\prod_{i=1}^{m}\alpha\beta\lambda x_i^{\beta-1}e^{\alpha\lambda x_i^{\beta}}\left[1-e^{-\lambda x^{\beta}}\right]^{\alpha-1}\left(e^{-\left[e^{\lambda x^{\beta}}-1\right]^{\alpha}}\right)^{R_i+1}, \tag{5}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-\sum_{i=1}^{m-1}(R_i+1))$ being regular constants.

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The log-likelihood function $\ell(\varsigma|\underline{x}) = \log L(\varsigma|\underline{x})$ without constant is obtained from (5) as

$$\ell(\varsigma|\underline{x}) \propto m\log\alpha + m\log\beta + m\log\lambda + (\beta-1)\sum_{i=1}^{m}\log x_i + \alpha\lambda\sum_{i=1}^{m}x_i^{\beta} + (\alpha-1)\sum_{i=1}^{m}\log\left(1-e^{-\lambda x_i^{\beta}}\right) - \sum_{i=1}^{m}\left(e^{\lambda x_i^{\beta}}-1\right)^{\alpha(R_i+1)}. \quad (6)$$

Taking the first derivatives of Equation (6) with respect to (α,β,λ) and setting each of them equal to zero, we obtain

$$\frac{m}{\alpha} + \lambda\sum_{i=1}^{m}x_i^{\beta} + \sum_{i=1}^{m}\log\left(1-e^{-\lambda x_i^{\beta}}\right) - \sum_{i=1}^{m}(R_i+1)\left(e^{\lambda x_i^{\beta}}-1\right)^{\alpha(R_i+1)}\log\left(e^{\lambda x_i^{\beta}}-1\right) = 0, \quad (7)$$

$$\frac{m}{\beta} + \sum_{i=1}^{m}\log x_i + \alpha\lambda\sum_{i=1}^{m}x_i^{\beta}\log x_i + (\alpha-1)\sum_{i=1}^{m}\frac{\lambda x_i^{\beta}e^{-\lambda x_i^{\beta}}\log x_i}{1-e^{-\lambda x_i^{\beta}}} - \sum_{i=1}^{m}\alpha\lambda\,(R_i+1)x_i^{\beta}\left(e^{\lambda x_i^{\beta}}-1\right)^{\alpha(R_i+1)}\log x_i = 0, \quad (8)$$

$$\frac{m}{\lambda} + \alpha\sum_{i=1}^{m}x_i^{\beta} + (\alpha-1)\sum_{i=1}^{m}\frac{x_i^{\beta}e^{-\lambda x_i^{\beta}}}{1-e^{-\lambda x_i^{\beta}}} - \sum_{i=1}^{m}\alpha\,(R_i+1)x_i^{\beta}\left(e^{\lambda x_i^{\beta}}-1\right)^{\alpha(R_i+1)} = 0. \quad (9)$$

Since the MLE of α, β and λ cannot be solved analytically. The Newton-Raphson iteration method has been used to get the estimates of the parameters. The algorithm is described as follows:

1. Start with initial guesses for the parameters $\left(\alpha^{(0)},\beta^{(0)},\lambda^{(0)}\right)$ and set the iteration counter $k=0$.
2. Calculate the gradient vector $\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda}\right)_{(\alpha_k,\beta_k,\lambda_k)}$ and the observed Fisher Information matrix $I^{-1}(\alpha,\beta,\lambda)$, as detailed in Subsection (2.1).
3. Update the parameter estimates using:

$$(\alpha_{k+1},\beta_{k+1},\lambda_{k+1}) = (\alpha_k,\beta_k,\lambda_k) + \left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda}\right)_{(\alpha_k,\beta_k,\lambda_k)} \times I^{-1}(\alpha,\beta,\lambda).$$

4. Increment the iteration counter $k=k+1$ and return to Step 2.
5. Repeat the iterative steps until the change in parameters $|(\alpha_{k+1},\beta_{k+1},\lambda_{k+1})-(\alpha_k,\beta_k,\lambda_k)|$ is smaller than a predefined threshold. The final estimates of α, β, and λ are the MLE of the parameters, denoted as $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$. Moreover, we can get the MLEs of $S(t;\varsigma)$ and $h(t;\varsigma)$ after replacing by their $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$ as follows:

$$\hat{S}(t;\varsigma) = e^{-\left[e^{\hat{\lambda}t^{\hat{\beta}}}-1\right]^{\hat{\alpha}}}, t>0, \quad (10)$$

and

$$\hat{h}(t;\varsigma) = \hat{\alpha}\hat{\beta}\hat{\lambda}t^{\hat{\beta}-1}e^{\hat{\alpha}\hat{\lambda}t^{\hat{\beta}}}\left[1-e^{-\hat{\lambda}t^{\hat{\beta}}}\right]^{\hat{\alpha}-1}. \quad (11)$$

2.1 Approximate confidence interval

Approximate confidence intervals (ACIs), using the Fisher information matrix (FIM), provide a reliable statistical method for estimating parameter uncertainties. By employing the second derivative of the log-likelihood function, the FIM framework facilitates efficient calculation of ACIs. These intervals are particularly useful when exact solutions are challenging to obtain, offering dependable estimates with manageable computational complexity. ACIs derived from the FIM play a crucial role in enhancing decision-making by quantifying the precision of parameter estimates in statistical inference. This method is broadly utilized across various fields due to its versatility and reliability in measuring uncertainty. Given the asymptotic normality of MLEs, the ACIs for parameters $\varsigma=(\alpha,\beta,\lambda)$ can be determined using asymptotic variances obtained from the inverse of the FIM, $I^{-1}(\varsigma)$.In practice, $I^{-1}(\hat{\varsigma})$ is commonly used as an estimate. Additionally, applying the following approximation offers a more straightforward and valid approach $\hat{\varsigma}\sim N\left(\varsigma,I^{-1}(\hat{\varsigma})\right)$. Hence,the asymptotic variance-covariance matrix is obtained as follows:

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$$I^{-1}(\varsigma)=\begin{pmatrix}-\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha^2} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha\partial\beta} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha\partial\lambda}\\ -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta^2} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta\partial\lambda}\\ -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda\partial\beta} & -\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda^2}\end{pmatrix}^{-1}_{(\varsigma=\hat{\varsigma})}=\begin{pmatrix}Var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\beta})\\ cov(\hat{\beta},\hat{\alpha}) & Var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda})\\ cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\beta},\hat{\lambda}) & Var(\hat{\lambda})\end{pmatrix}$$

where

$$\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha^2}=-\frac{m}{\alpha^2}-\sum_{i=1}^{m}(R_i+1)^2\,(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\log\left(e^{\lambda x_i^\beta}-1\right)^2, \tag{12}$$

$$\begin{aligned}\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta^2}=&-\frac{m}{\beta^2}+\alpha\lambda\sum_{i=1}^{m}x_i^\beta\log(x_i)^2\\ &+(\alpha-1)\sum_{i=1}^{m}\left(\frac{e^{-\lambda x_i^\beta}\lambda\log(x_i)^2(x_i^\beta)}{1-e^{-\lambda x_i^\beta}}-\frac{e^{-2\lambda x_i^\beta}\lambda^2\log(x_i)^2(x_i^{2\beta})}{(1-e^{-\lambda x_i^\beta})^2}-\frac{e^{-\lambda x_i^\beta}\lambda^2\log(x_i)^2(x_i^{2\beta})}{1-e^{(-\lambda x_i^\beta)}}\right)\\ &-\sum_{i=1}^{m}\left(\alpha\lambda\,(R_i+1)x_i^\beta(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\log(x_i)^2+(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\alpha^2\lambda^2x_i^{2\beta}\,(R_i+1)^2\log(x_i)^2\right),\end{aligned} \tag{13}$$

$$\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda^2}=-\frac{m}{\lambda^2}-(\alpha-1)\sum_{i=1}^{m}\left(-\frac{e^{-2\lambda x_i^\beta}\,(x_i^{2\beta})}{(1-e^{(-\lambda x_i^\beta)})^2}-\frac{e^{(-\lambda x_i^\beta)}(x_i^{2\beta})}{1-e^{(-\lambda x_i^\beta)}}\right)-\sum_{i=1}^{m}\alpha^2\,(R_i+1)^2\,(x_i^{2\beta})(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}, \tag{14}$$

$$\begin{aligned}\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha\partial\beta}=\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta\partial\alpha}=&\,\lambda\sum_{i=1}^{m}(x_i^\beta)\log(x_i)+\sum_{i=1}^{m}\frac{\lambda x_i^\beta e^{-\lambda x_i^\beta}\log x_i}{1-e^{-\lambda x_i^\beta}}+\sum_{i=1}^{m}\lambda e^{\lambda x_i^\beta}(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\log(x_i)\,(1+R_i)\,(x_i^\beta)\\ &+\sum_{i=1}^{m}e^{\lambda x_i^\beta}(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\alpha\lambda(e^{\lambda x_i^\beta}-1)\,(1+R_i)^2\log(x_i)(x_i^\beta),\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\alpha\partial\lambda}=\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda\partial\alpha}=&\sum_{i=1}^{m}(x_i^\beta)+\sum_{i=1}^{m}\frac{x_i^\beta e^{-\lambda x_i^\beta}}{1-e^{-\lambda x_i^\beta}}-\sum_{i=1}^{m}e^{\lambda x_i^\beta}(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\log(x_i)\,(1+R_i)\,(x_i^\beta)\\ &+\sum_{i=1}^{m}e^{\lambda x_i^\beta}(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\alpha\log(e^{\lambda x_i^\beta}-1)\,(1+R_i)^2\,(x_i^\beta).\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\beta\partial\lambda}=\frac{\partial^2\ell(\varsigma|\underline{x})}{\partial\lambda\partial\beta}=&\,\alpha\sum_{i=1}^{m}(x_i^\beta)\log(x_i)+(\alpha-1)\sum_{i=1}^{m}\left(\frac{e^{-\lambda x_i^\beta}\lambda\log(x_i)(x_i^\beta)}{1-e^{-\lambda x_i^\beta}}-\frac{e^{-2\lambda x_i^\beta}\lambda\log(x_i)(x_i^{2\beta})}{(1-e^{-\lambda x_i^\beta})^2}-\frac{e^{-\lambda x_i^\beta}\lambda\log(x_i)(x_i^{2\beta})}{1-e^{(-\lambda x_i^\beta)}}\right)\\ &-\sum_{i=1}^{m}\alpha(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\log(x_i)\,(1+R_i)\,(x_i^\beta)+\sum_{i=1}^{m}e^{\lambda x_i^\beta}(e^{\lambda x_i^\beta}-1)^{\alpha(R_i+1)}\alpha^2\log(x_i)\,(1+R_i)^2\,(x_i^{2\beta})\end{aligned} \tag{17}$$

Hence, The asymptotic normality of the MLEs can be used to compute the ACIs for parameters α, β and λ. The $(1-\eta)100\%$ ACIs for parameters α, β and λ are given repectively, as follows:

$$\left(\hat{\alpha}\mp Z_{\frac{\eta}{2}}\sqrt{Var(\hat{\alpha})}\right),\qquad\left(\hat{\beta}\mp Z_{\frac{\eta}{2}}\sqrt{Var(\hat{\beta})}\right),\qquad\left(\lambda\mp Z_{\frac{\eta}{2}}\sqrt{Var(\lambda)}\right), \tag{18}$$

where $Z_{\frac{\eta}{2}}$ is the percentile of the standard normal distribution with the right-tail probability $\frac{\eta}{2}$.

Moreover, to construct the ACIs of $S(t)$ and $h(t)$, which are functions of the parameters α,β and λ we need to find the variance to them. In order to find the approximation estimates of the variance of $\hat{S}(t)$ and $\hat{h}(t)$, we use the delta method, which is a technique for approximating the variance of a function of a random variable, particularly useful when the function is non-linear. It involves a Taylor series expansion to linearize the function around the mean of the estimator. When applied to reliability and hazard rate functions, the Delta method allows us to estimate their variances based on the variance of the underlying parameter estimator. As a result of this method, the variance of $\hat{S}(t)$ and $\hat{h}(t)$ respectively, are given by

$$\hat{\sigma}^2_{S(t)}=[\Delta\hat{S}(t)]^T[\hat{V}][\Delta\hat{S}(t)],\qquad\hat{\sigma}^2_{h(t)}=[\Delta\hat{h}(t)]^T[\hat{V}][\Delta\hat{h}(t)],$$

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where $\Delta\hat{S}(t)$ and $\Delta\hat{h}(t)$ are the gradient of $\hat{S}(t)$ and $\hat{h}(t)$ respectively, with respect α, β and λ and $\hat{V} = I^{-1}(\hat{\varsigma})$, $\hat{\varsigma} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$.

Thus, the $(1-\eta)100\%$ ACIs for $S(t)$ and $h(t)$ are obtained as

$$\left(\hat{S}(t) \mp Z_{\frac{\eta}{2}}\sqrt{\hat{\sigma}^2_{S(t)}}\right), \qquad \left(\hat{h}(t) \mp Z_{\frac{\eta}{2}}\sqrt{\hat{\sigma}^2_{h(t)}}\right).$$

3 Parametric Bootstrap Methods

In scenarios where sample sizes are small, relying on the normal approximation is deemed inappropriate due to its potential inaccuracies. To address this issue,we propose employing a resampling technique known as the bootstrap procedure. This method involves repeatedly sampling from the observed data with replacement to generate a large number of pseudo-samples. Through this process, we can approximate the sampling distribution of a statistic of interest and construct confidence intervals that are more robust and reliable, even with limited data. Within the bootstrap framework, we explore two parametric bootstrap procedures aimed at constructing confidence intervals, leveraging the flexibility and adaptability of this approach to accommodate various statistical scenarios and requirements. The percentile bootstrap (boot-p) and bootstrap-t (boot-t) confidence intervals follow similar steps, as outlined in DiCiccio and Efron [8] and Hall [9] , respectively. These procedures involve resampling from the observed data to create bootstrap samples, computing the statistic of interest for each resampled dataset, and then determining the appropriate confidence interval based on the distribution of these statistics. Several researchers have investigated and discussed these two types of bootstrap methods,including Reiser et al. [10] and Besseris [11].

3.1 Percentile Bootstrap

(1)From the original data $\underline{x} = x^R_{1:m:n:k}, x^R_{2:m:n:k}, ..., x^R_{m:m:n:k}$,compute the MLEs of the unknown parameters α, β and λ by solving the nonliner Equations $(7)-(10)$.

(2)Use $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ to generate PFFC sample $\underline{x}^*$with the same values of $R_i, m; (i = 1, 2, ..., m)$and compute the bootstrap estimator θ^*.

(3)Repeat Step 2 N times; then, we have $\theta^*_1, \theta^*_2, ..., \theta^*_N$.

(4)Arrange all components in ascending order; the bootstrap estimates are $\theta^*_{(1)}, \theta^*_{(2)}, ..., \theta^*_{(N)}$.

(5)Let $\phi(x) = P\left(\hat{\theta}^* \le x\right)$ be the CDF of $\hat{\theta}^*$Define $\hat{\theta}_{boot-p}(x) = \phi^{-1}(x)$ for given x. Then, two-side $100(1-\gamma)\%$ percentile bootstrap confidence intervals of θ given by

$$\left[\hat{\theta}_{boot-p}(\frac{\gamma}{2}), \hat{\theta}_{boot-p}(1-\frac{\gamma}{2})\right]$$

3.2 Bootstrap-t

(1)-(2) The same as the percentile bootstrap.

(3)Compute the t-statistics $T^*_\theta = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{var(\hat{\theta}^*)}}$,where $var(\hat{\theta}^*)$obtained using FIM forα, β and λmethod for$R(t)$ and $h(t)$.

(4)Repeat Steps 2, 3, 4 N times; then, we have $T^{*(1)}_\theta, T^{*(2)}_\theta, ..., T^{*(N)}_\theta$.

(5)Let $\psi(x) = P\left(\hat{\theta}^* \le x\right)$ be the CDF of T^*_θDefine$\hat{\theta}_{boot-t}(x) = \hat{\theta} + \psi^{-1}(x)\sqrt{var(\hat{\theta}^*)}$ for given x. Then,two-side $100(1-\gamma)\%$ bootstrap-t confidence intervals of θ given by

$$\left[\hat{\theta}_{boot-t}(\frac{\gamma}{2}), \hat{\theta}_{boot-t}(1-\frac{\gamma}{2})\right]$$

4 Bayes Estimation

Bayesian estimation differs notably from MLE and bootstrap methods by incorporating both observed sample data and prior information, alongside symmetric and asymmetric loss functions. This comprehensive approach allows for a more rational and reasoned characterization of problems. In this section, a Bayesian inference procedure utilizing MCMC

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technique is proposed to estimate parameters such as α, β and λ , as well as $S(t)$ and $h(t)$ under both SE and LINEX loss functions. Additionally, corresponding Credible Interval (CRI) constructions are implemented under the MCMC technique. The flexibility of the gamma distribution family is acknowledged, as it can accommodate a wide range of prior beliefs of the experimenter. Hence, the joint prior density can formulated as follows

$$\pi(\alpha,\beta,\lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-b_1\alpha-b_2\beta-b_3\lambda}. \tag{19}$$

where the hyper-parameters a_i and $b_i, i = 1,2,3$ are assumed to be known and non-negative.

The gamma distribution is often chosen as a prior distribution in Bayesian analysis due to several distinctive properties:

1. The gamma distribution is defined over the positive real numbers, making it suitable for modeling parameters that are inherently non-negative, such as rates or scales.
2. The gamma distribution serves as a conjugate prior for several common likelihood functions, including the Poisson, exponential, and normal distributions. This conjugacy simplifies the mathematical derivation of the posterior distribution, as the posterior remains in the same family as the prior.
3. The gamma distribution can take on various shapes, allowing it to represent different levels of prior knowledge or uncertainty about the parameter being estimated.

Consequently, from (5)and(20),the joint posterior density can be expressed as follows

$$\begin{aligned}\pi^*(\alpha,\beta,\lambda|\underline{x}) &= \frac{L(\underline{x};\alpha,\beta,\lambda)\,\pi(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\underline{x};\alpha,\beta,\lambda)\,\pi(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}\\ &\propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-b_1\alpha-b_2\beta-b_3\lambda}\alpha^m\beta^m\lambda^m e^{(\beta-1)\sum_{i=1}^m\log(x_i)+\lambda\alpha\sum_{i=1}^m x_i^\beta}\\ &\times e^{(\alpha-1)\sum_{i=1}^m\log(1-e^{-\lambda x_i^\beta})}\prod_i^m e^{-(e^{-\lambda x_i^\beta}-1)^{\alpha(1+R_i)}}.\end{aligned} \tag{20}$$

Bayesian estimators, rooted in the SE function, inherently rely on integral ratios devoid of closed-form solutions. Hence, numerical methods are pivotal for approximating these integrals. In this context, we leverage the MCMC method to derive Bayesian estimates for parameters like$\alpha, \beta$and $\lambda$and to construct credible intervals. MCMC presents an adaptable alternative to traditional techniques, offering probability intervals and accommodating various scenarios.

4.1 MCMC method

MCMC methods are fundamental to Bayesian estimation, providing effective tools for approximating complex posterior distributions through iterative sampling. MCMC, which combines Markov chains with Monte Carlo methods, has transformed statistical inference by allowing practitioners to address high-dimensional problems that are challenging for traditional analytical approaches. Essentially, MCMC constructs a Markov chain that produces a series of correlated samples from the target distribution, with its equilibrium distribution aligning with the desired posterior. Thanks to the chain's ergodicity, sufficient iterations lead to samples that accurately reflect the true posterior distribution, thus overcoming the dimensionality challenges common in Bayesian inference. Various MCMC algorithms have been developed to meet different needs in Bayesian estimation. The widely-used Metropolis-Hastings (M-H) algorithm proposes new states based on an acceptance criterion, while its extension, the Gibbs sampler, simplifies multivariate distributions by iteratively sampling from conditional distributions. These methods illustrate MCMC's flexibility in handling diverse problem structures and data types, as discussed by Geman and Geman[12], Metropolis et al.[13], and Hastings[14]. Innovations such as Hamiltonian Monte Carlo (HMC) enhance sampling efficiency in high-dimensional spaces by utilizing gradient information. Sequential Monte Carlo (SMC) methods offer alternatives for dynamic models or scenarios with changing data streams, providing robustness and adaptability in Bayesian analysis.In summary, MCMC techniques are crucial in Bayesian statistics, offering a rigorous approach to exploring and summarizing complex posterior distributions. Their ongoing development continues to advance the field, helping researchers and practitioners derive valuable insights from increasingly complex datasets and models.

We construct the CRIs based on the generated posterior samples. from (18) we get the full conditional posterior function of α as follows:

$$\pi_1^*(\alpha|\beta,\lambda,\underline{x}) \propto \alpha^{m+a_1-1}e^{-\alpha\left\{b_1-\lambda\sum_{i=1}^m x_i^\beta-\sum_{i=1}^m\log\left(1-e^{\lambda x_i^\beta}\right)\right\}}. \tag{21}$$

Also, the conditional posterior density function of α, β and λ can be written as

$$\pi_2^*(\beta|\alpha,\lambda,\underline{x}) \propto \beta^{m+a_2-1}\left[\prod_{i=1}^m x_i^{-\beta\lambda}e^{-\beta\lambda\alpha x_i^\beta}\left(1-e^{-\lambda x_i^\beta}\right)^{-\beta(\alpha-1)}\right], \tag{22}$$

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and

$$\pi_3^*(\lambda|\alpha,\beta,\underline{x}) \propto \lambda^{m+a_3-1}\left[\prod_{i=1}^{m} e^{-(1+\lambda)\lambda\alpha x_i^{\beta}}\left(1-e^{-\lambda x_i^{\beta}}\right)^{-\lambda(\alpha-1)}\right]. \tag{23}$$

Additionally, the full conditional posterior density function of α,β and λ cannot be analytically reduced to well-known distributions. Consequently, direct sampling using standard methods can pose challenges. However, as illustrated in Figure 3, these distributions display similarities to the normal distribution.

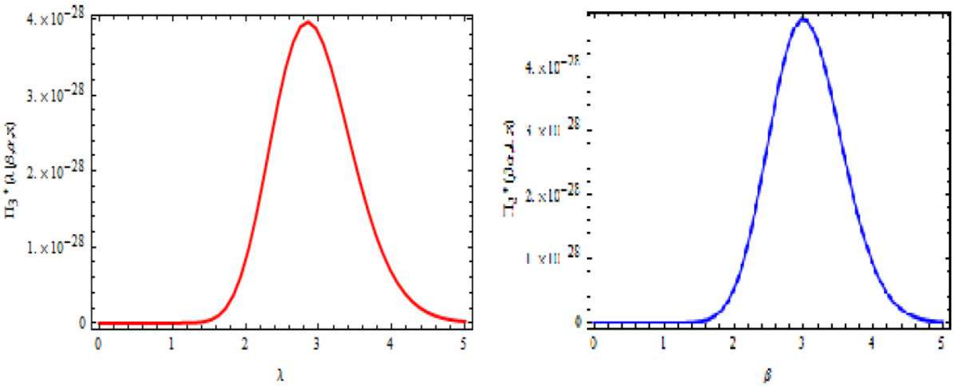

Fig. 3: Posterior density function for β and λ.

4.2 The Metropolis–Hastings algorithm within Gibbs sampling

Metropolis et al. [13] pioneered the Metropolis–Hastings (M–H) algorithm, a cornerstone in MCMC methods, subsequently expanded upon by Hastings [14]. This algorithm allows for the generation of random samples from intricate target distributions of any dimension, given knowledge up to a normalizing constant.Gibbs sampling, a variant of MCMC,emerges as a method to generate sequences of samples from the full conditional probability distributions of multiple random variables. It entails breaking down the joint posterior distribution into full conditional distributions for each parameter and iteratively sampling from them. We advocate for the application of Gibbs sampling to procure a sample from the posterior density function $\pi^*(\alpha|\beta,\lambda,\underline{x})$, facilitating the computation of Bayesian estimates and the construction of credible intervals. Furthermore, the conditional posterior distributions of α, β and λ as described in Equations (20), (21) and (22) resist analytic reduction to well-known distributions. Consequently, direct sampling by standard methods becomes impractical. However, visual inspection reveals their resemblance to normal distributions. Therefore,to generate random numbers from these distributions, we employ the M-H algorithm within the Gibbs Sampling framework, utilizing a normal proposal distribution as follows:

1.Use $\hat{\alpha}^{(0)},\hat{\beta}^{(0)}$ and $\hat{\lambda}^{(0)}$ as the initial values.
2.Set $j=1$.
3.Generate $\alpha^{(j)}$ from Gamma$\left(m+a_1,\left[b_1-\lambda\sum_{i=1}^{m}x_i^{\beta}-\sum_{i=1}^{m}\log\left(1-e^{\lambda x_i^{\beta}}\right)\right]\right)$.
4.Using M-H algorithm, generate $\beta^{(i)}$ and $\lambda^{(i)}$ from $\pi_2^*(\beta^{(j-1)}|\alpha^{(j-1)},\lambda^{(j)},\underline{x})$ and $\pi_3^*(\lambda^{(j-1)}|\alpha^{(j-1)},\beta^{(j)},\underline{x})$ with $N(\beta^{(j-1)},Var(\widehat{\beta}))$ and $N(\lambda^{(j-1)},Var(\hat{\lambda}))$ respectively.
(a)Generate β^* from $N\left(\beta^{(j-1)},var\left(\hat{\beta}\right)\right)$ and λ^* from $N(\lambda^{(j-1)},Var(\hat{\lambda}))$.
(b)Evaluate the probabilities

$$Q_\beta = \min\left[1,\frac{\pi_2^*(\beta^*|\alpha^{(j)},\lambda^{(j)},\underline{x})}{\pi_2^*(\beta^{(j-1)}|\alpha^{(j)},\lambda^{(j)},\underline{x})}\right],$$

$$Q_\lambda = \min\left[1,\frac{\pi_3^*(\lambda^*|\alpha^{(j)},\beta^{(j)},\underline{x})}{\pi_3^*(\lambda^{(j-1)}|\alpha^{(j)},\beta^{(j)},\underline{x})}\right],$$

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(c)Generate a ρ_1 and ρ_2 from a Uniform $(0,1)$.
(d)If $\rho_1 < Q_\beta$ accept the proposal and set $\beta^* = \beta^{(j)}$, else set $\beta^{(j)} = \beta^{(j-1)}$.
(e)If $\rho_2 < Q_\lambda$ accept the proposal and set $\lambda^* = \lambda^{(j)}$, else set $\lambda^{(j)} = \lambda^{(j-1)}$.

5.Compute SF and HF as

$$S^{(j)}(t) = \exp\left\{-\left(\exp\left[\lambda^{(j)} t^{\beta^{(i)}} - 1\right]^{\alpha^{(j)}}\right)\right\},$$

and

$$h^{(j)}(t) = \alpha^{(j)}\beta^{(j)}\lambda^{(j)} t^{\beta^{(j)}-1} \exp\left\{\alpha^{(j)}\lambda^{(j)} t^{\beta^{(j)}}\right\}\left[1-\exp\left\{-\lambda^{(j)} t^{\beta^{(j)}}\right\}\right]^{\alpha^{(j)}-1}.$$

6.Set $j = j+1$.
7.Repeat steps $2-5$ N times.

Obtain the Bayes estimates of ψ_j where $\psi_1 = \alpha$, $\psi_2 = \beta$, $\psi_3 = \lambda$, $\psi_4 = S(t)$, and $\psi_5 = h(t)$ for $j = 1,2,3,4$ and 5 with respect to the SELF as

$$E(\psi_j|data) = \frac{1}{N-M}\sum_{i=M+1}^{N} \psi_j^i ,$$

where M is the burn-in period.

To establish the CRIs of ψ_j order $\psi_j^{(M+1)}$, $\psi_j^{(M+2)}$,...,$\psi_j^{(N)}$ and as $\psi_{j(1)} < \psi_{j(2)} < ... < \psi_{j(N-M)}$. Hence. The $100(1-2\gamma)\%$ CRIs of ψ_j can be constructed as

$$\left(\psi_{j(\gamma(N-M))}, \psi_{j((1-\gamma)(N-M))}\right).$$

4.3 Applications

In this section, we provide a hands-on demonstration of the proposed methodologies using a real dataset, as outlined in Table1.Originating from a physics study and reanalyzed following its provision by GomMak et al. [15], this dataset comprises 83 observed times. Our objective is to showcase the practical utility of the suggested techniques in real-world scenarios. Through this empirical analysis, we aim to highlight how these methodologies can be effectively applied to actual datasets, emphasizing their importance and relevance in modern data analysis practices

Table 1: Physics data set

0.08	0.08	0.18	0.30	0.47	0.48	0.51	0.59	0.64	0.66	0.67	0.69	0.76	0.77	0.82
0.95	1.01	1.12	2.04	1.20	1.27	1.39	1.40	1.50	1.54	1.73	1.77	1.79	1.94	1.97
2.11	2.27	2.38	2.45	2.45	2.57	2.72	2.86	2.97	3.18	3.19	3.20	3.22	3.25	2.38
3.31	3.39	3.65	3.72	3.77	3.94	3.95	4.16	4.16	4.25	4.45	4.63	4.95	4.99	5.02
5.07	5.16	5.25	5.39	5.52	5.52	5.58	5.72	5.90	5.97	6.23	6.28	6.79	7.16	7.37
7.38	7.55	7.61	8.40	8.57	9.87	2.92	6.58							

To assess the goodness of fit of the PMKE distribution to the data,we computed the Kolmogorov-Smirnov (KS) statistic and its associated p-value.The calculated KS statistic is 0.0385, and the corresponding p-value is 0.9997. With such a high p-value close to 1, we can conclude that the PMKE distribution provides an excellent fit to the data.Additionally, Figure 4 illustrates how closely the empirical values and fitted PMKE distribution align,further supporting our conclusion regarding the goodness of fit.The visual comparison indicates a high degree of agreement between the observed data and the fitted PMKE distribution.In summary, based on both the statistical analysis and visual inspection, it is evident that the PMKE distribution serves as an excellent model for fitting the given dataset.

Based on the datasets provided, we have constructed the PT-IIC sample using a censoring scheme. For this scenario, let us consider $m = 40$, $n = 83$.We have specified the values of R ={3, 1, 2, 2, 0, 2, 0, 1, 0, 1, 0, 1, 1, 0, 2, 0, 1, 0, 2, 0, 2, 1, 2, 2, 0, 2, 0, 2, 0, 1, 0, 2, 2, 0, 2, 0, 1, 0, 2, 2, 2, 1, 0} is given as follows: $x^{(1)}$ ={0.08, 0.08, 0.3, 0.47, 0.48, 0.51, 0.59, 0.64, 0.66, 0.67, 0.69, 0.76,0.77,0.82,1.01,1.12,1.2,1.39, 1.94,1.97,2.11, 2.04, 2.27, 2.38, 2.45, 2.45, 2.57, 2.72, 2.86, 2.92, 2.97,3.18, 3.31, 3.39, 3.65, 3.72,3.77,3.94, 4.16, 4.16, 4.25, 5.39}. Using the prior information we aimed to estimate the unknown parameters of the PT-IIC model. This involves calculating MLEs, BP, BT and Baysian estimates using the

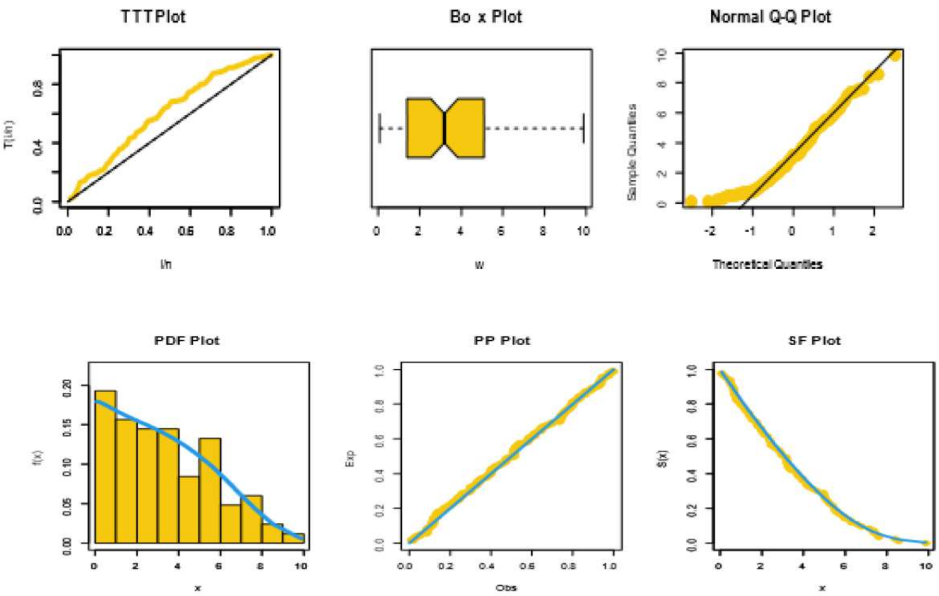

Fig. 4: Empirical and fitted survival functions

Table 2: Point estimates of α, β, λ, $S(t)$ and $h(t)$

Parameter	MLE	Bootstrap		Bayes
		BP	BT	MCMC
α	1.0592	0.9742	0.9892	0.9097
β	2.6158	3.2284	3.2805	2.0657
λ	0.4974	0.6271	0.5313	0.4823
$S(t)$	0.9621	0.9588	0.9600	0.9458
$h(t)$	0.2735	0.3087	0.2995	0.2513

Table 3: 95% ACIs and CRIs of α, β, λ, $S(t)$ and $h(t)$ baed on $x^{(1)}$.

Method	α		β		λ	
	[U,L]	Length	[U,L]	Length	[U,L]	Length
ACI	$[0.3401, 1.7782]$	1.4382	$[2.2539, 2.9778]$	0.7239	$[-0.0975, 1.0924]$	1.1899
BPCI	$[0.5182, 1.1855]$	0.6673	$[2.2856, 5.2485]$	2.9628	$[0.3591, 0.8894]$	0.5303
BTCI	$[0.5171, 1.2085]$	0.6914	$[2.2779, 6.5883]$	4.3104	$[0.3820, 0.9623]$	0.5802
MCMC	$[0.5925, 0.9960]$	0.4035	$[1.5104, 2.6901]$	1.1796	$[0.3250, 0.8574]$	0.5324
	$S(t)$		$h(t)$			
	[U,L]	Length	[U,L]	Length		
ACI	$[0.7257, 1.1985]$	0.4728	$[-0.1034, 0.6505]$	0.7539		
BPCI	$[0.9061, 0.9834]$	0.0773	$[0.1781, 0.5596]$	0.3815		
BTCI	$[0.8987, 0.9862]$	0.0875	$[0.1634, 0.6242]$	0.4608		
MCMC	$[0.8865, 0.9903]$	0.1038	$[0.1946, 0.4753]$	0.2807		

MCMC method. To compute the MLEs, we obtained the estimates for the parameters;and The results displayed in Tables 2. In addition, we calculated the 95% ACIs, with the results presented in Tables 2. .Based on the M-H technique within Gibbs sampling, we generate 12000 MCMC samples and discarded the initial 2000 values as a 'burn-in' period to ensure reliable results. We employed non-informative gamma prior with hyperparameters set as $a_i = 0$ and $b_i = 0$, $i = 1, 2, 3$, the results are reported in Tables 2also computed the 95% CRIs, the results are given in Table 3.

Additionally, a comparison of CIs for MLEs, BP, BT and BEs in Table 2 shows that the BEs have consistently narrower CIs than the MLEs for all three parameters (α, β, λ). Table 3 presents the 95% ACIs for the MLEs and CRIs for the MCMC-based Bayes estimators. Figure 5 displays the Trace plots, histogram and autocorrelation coefficient diagram of

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a real dataset, as outlined in Table1 in the MCMC sampling process through Figure 5. It can be seen that the posterior distribution presents obvious normal distribution characteristics. The applicability and effectiveness of PMKE model in MCMC sampling are also illustrated. In Tables 2 and 3, it is evident that the BEs perform better than the MLEs under PT-II censoring samples. Furthermore, the Bayes CRIs exhibit shorter interval lengths than the approximate CIs, emphasizing their efficiency. This approach can support effective failure analysis for the Physics dataset, given that the PMKE distribution aligns well with the data.

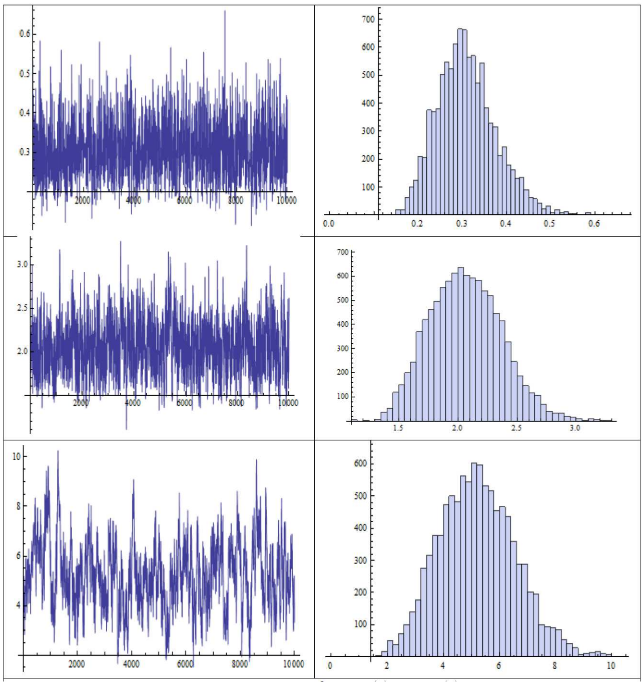

Fig. 5: Trace plots and histogram of α, β, λ, $S(t)$ and $h(t)$ of MCMC approach

5 Simulation Study

In this section, we conduct a Monte Carlo simulation study to assess the performance of the proposed Bayes estimators in comparison with MLEs and bootstrap methods, using various combinations of (n,m) and different values for the censoring scheme R (different $R_i$values). Following the algorithm developed by Balakrishnan and Sandhu [1] to generate PT-IIC samples from the PMKE distribution with parameters $(\alpha,\beta,\lambda)=(1.5,2.5,0.75)$. The true values of $S(t)$ and $h(t)$ at $t=0.4$ are calculated as 0.9781 and 0.21555, respectively. The performance of the estimators is evaluated in terms of mean square error (MSE), defined as $MSE=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\omega}_j^i-\omega_j\right)^2$, where $N=1000, j=1,2,3,...,5$, $\omega_1=\alpha$, $\omega_2=\beta$, $\omega_3=\lambda$, $\omega_4=S(t)$, and $\omega_5=h(t)$ for point estimates. We also assess interval estimates (asymptotic, bootstrap, and HPD) in terms of AILs and CPs. CRIs are computed using 12000 MCMC samples, discarding the first 2000 values as "burn-in." We assume informative Gamma priors for α, β, and λ with hyperparameters $(a_i,b_i)=\{(2,2),(3,2),(2,1)\}$. Additionally, 95% CRIs are calculated for each simulated sample and consider the following censoring schemes:

SC F:$R_1=n-m$, $R_i=0$ for $i\neq 1$.

SC M:$\begin{cases} R_{(m+1)/2}=n-m, R_i=0 \text{ for } i\neq (m+1)/2 \text{ if } m \text{ odd},\\ R_{m/2}=n-m, R_i=0 \text{ for } i\neq m/2 \text{ if } m \text{ even}.\end{cases}$

SC L:$R_m=n-m$, $R_i=0$ for $i\neq m$.

Tables $4-6$ show the results of the estimate parameters and their MSEs, while Table $7-11$ displays the values of the AIL and CP of CIs.

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Table 4: MSEs of estimates for the parameters α and β.

		α				β			
(n,m)	CSs	ML	BP	BT	MCMC	ML	BP	BT	MCMC
$(30,15)$	F	0.4755	0.4462	0.3963	0.3162	0.7861	0.6547	0.5427	0.4836
	M	0.5247	0.4913	0.4236	0.3457	0.8342	0.7154	0.5966	0.5267
	L	0.5627	0.5328	0.4678	0.3819	0.8891	0.7653	0.6491	0.5764
$(30,20)$	F	0.3947	0.3549	0.2847	0.2346	0.6892	0.5763	0.4867	0.4169
	M	0.4355	0.3962	0.3247	0.2634	0.7254	0.6155	0.5321	0.4633
	L	0.4793	0.4435	0.3764	0.3059	0.7765	0.6835	0.5799	0.5109
$(60,30)$	F	0.3146	0.2783	0.2246	0.1938	0.5236	0.4437	0.3972	0.3249
	M	0.3465	0.3099	0.2594	0.2198	0.5691	0.4836	0.4264	0.3692
	L	0.3856	0.3547	0.2937	0.2456	0.6251	0.5364	0.4793	0.3993
$(60,40)$	F	0.2774	0.2366	0.1892	0.1347	0.4257	0.3769	0.3214	0.2794
	M	0.3147	0.2655	0.2197	0.1544	0.4699	0.4157	0.3695	0.2997
	L	0.3462	0.2999	0.2468	0.1877	0.5124	0.4596	0.3998	0.3364
$(90,45)$	F	0.2163	0.1789	0.1467	0.1194	0.3327	0.2837	0.2314	0.1902
	M	0.2469	0.2147	0.1892	0.1397	0.3945	0.3547	0.2768	0.2219
	L	0.2947	0.2643	0.2134	0.1609	0.4327	0.3952	0.2995	0.2601
$(90,65)$	F	0.1893	0.1579	0.1234	0.0999	0.2563	0.2139	0.1853	0.1359
	M	0.2254	0.2064	0.1637	0.1197	0.2836	0.2469	0.2145	0.1578
	L	0.2561	0.2346	0.1954	0.1321	0.3165	0.2797	0.2547	0.1864

Table 5: MSE of estimates for the parameter λ and $S(t=0.4)$.

		λ				$S(t=0.4)$			
(n,m)	CSs	ML	BP	BT	MCMC	ML	BP	BT	MCMC
$(30,15)$	F	0.0286	0.0224	0.0176	0.0115	0.0432	0.0364	0.0315	0.0297
	M	0.0337	0.0269	0.0217	0.0146	0.0479	0.0396	0.0364	0.0327
	L	0.0379	0.0318	0.0264	0.0179	0.0513	0.0446	0.0399	0.0356
$(30,20)$	F	0.0236	0.0189	0.0137	0.0098	0.0314	0.0285	0.0254	0.0198
	M	0.0279	0.0226	0.0187	0.0121	0.0348	0.0314	0.0279	0.0226
	L	0.0317	0.0264	0.0219	0.0150	0.0374	0.0336	0.0305	0.0257
$(60,30)$	F	0.0185	0.0156	0.0123	0.0083	0.0265	0.0224	0.0189	0.0156
	M	0.0227	0.0190	0.0154	0.0096	0.0296	0.0257	0.0219	0.0173
	L	0.0266	0.0231	0.0187	0.0125	0.0313	0.0275	0.0246	0.0194
$(60,40)$	F	0.0135	0.0116	0.0097	0.0069	0.0218	0.0179	0.0148	0.0119
	M	0.0191	0.0167	0.0122	0.0075	0.0245	0.0198	0.0176	0.0132
	L	0.0234	0.0202	0.0156	0.0086	0.0269	0.0221	0.0202	0.0164
$(90,45)$	F	0.0115	0.0092	0.0083	0.0057	0.0176	0.0135	0.0117	0.0092
	M	0.0147	0.0119	0.0099	0.0066	0.0189	0.0156	0.0137	0.0119
	L	0.0172	0.0146	0.0119	0.0078	0.0211	0.0172	0.0161	0.0141
$(90,65)$	F	0.0094	0.0088	0.0074	0.0049	0.0129	0.0105	0.0088	0.0075
	M	0.0117	0.0101	0.0086	0.0058	0.0142	0.0131	0.0094	0.0083
	L	0.0145	0.0129	0.0093	0.0069	0.0167	0.0154	0.0119	0.0097

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Table 6: MSE of estimates for $h(t = 0.4)$.

(n,m)	CSs	ML	BP	BT	MCMC
$(30,15)$	F	0.0088	0.0082	0.0073	0.0061
	M	0.0093	0.0089	0.0078	0.0067
	L	0.0099	0.0096	0.0084	0.0072
$(30,20)$	F	0.0067	0.0058	0.0046	0.0039
	M	0.0073	0.0065	0.0052	0.0044
	L	0.0079	0.0071	0.0059	0.0051
$(60,30)$	F	0.0048	0.0039	0.0031	0.0025
	M	0.0055	0.0046	0.0037	0.0029
	L	0.0061	0.0052	0.0043	0.0034
$(60,40)$	F	0.0039	0.0028	0.0022	0.0019
	M	0.0044	0.0037	0.0029	0.0024
	L	0.0053	0.0045	0.0036	0.0029
$(90,45)$	F	0.0028	0.0021	0.0016	0.0011
	M	0.0035	0.0026	0.0019	0.0015
	L	0.0042	0.0033	0.0024	0.0018
$(90,65)$	F	0.0022	0.0017	0.0012	0.0009
	M	0.0027	0.0022	0.0014	0.0011
	L	0.0032	0.0027	0.0018	0.0014

Table 7: AILs and CPs of estimates for the parameter α.

(n,m)	CSs	MLE		BP		BT		MCMC	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(30,15)$	F	3.4507	0.924	3.1519	0.917	2.8509	0.939	2.2509	0.939
	M	3.5481	0.934	3.2479	0.928	2.9641	0.928	2.4651	0.941
	L	3.7654	0.931	3.3942	0.919	3.0584	0.937	2.6642	0.941
$(30,20)$	F	3.1357	0.941	2.8844	0.938	2.4456	0.928	1.8321	0.936
	M	3.2362	0.939	2.9946	0.927	2.5993	0.917	2.0457	0.942
	L	3.5124	0.926	3.2213	0.947	2.7963	0.941	2.3146	0.951
$(60,30)$	F	2.7845	0.917	2.5473	0.939	2.1465	0.936	1.6148	0.943
	M	2.9632	0.936	2.7452	0.951	2.3699	0.927	1.8543	0.934
	L	3.2543	0.928	2.9945	0.955	2.6278	0.945	2.0995	0.953
$(60,40)$	F	2.3658	0.941	2.1457	0.942	1.7948	0.951	1.3625	0.929
	M	2.5628	0.937	2.3643	0.937	1.9631	0.942	1.5687	0.951
	L	2.9146	0.942	2.6399	0.947	2.1945	0.951	1.8362	0.947
$(90,45)$	F	1.9968	0.945	1.7658	0.928	1.4472	0.943	1.0699	0.951
	M	2.2473	0.946	1.9999	0.939	1.6478	0.919	1.1998	0.943
	L	2.4358	0.939	2.2875	0.945	1.8692	0.928	1.3694	0.934
$(90,65)$	F	1.6475	0.928	1.4758	0.937	1.2479	0.939	0.8997	0.956
	M	1.8694	0.937	1.7014	0.945	1.5462	0.944	1.2369	0.947
	L	2.1245	0.941	1.9946	0.937	1.7638	0.945	1.4996	0.951

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Table 8: AILs and CPs of estimates for the parameter β.

(n,m)	CSs	MLE		BP		BT		MCMC	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(30,15)$	F	5.9335	0.939	5.8276	0.938	4.7335	0.939	4.2427	0.947
	M	6.2347	0.928	6.1249	0.941	5.2362	0.928	4.6538	0.951
	L	6.5423	0.927	6.4562	0.925	5.6021	0.938	4.9365	0.949
$(30,20)$	F	5.6532	0.941	5.4736	0.945	4.3652	0.919	3.9523	0.946
	M	5.9364	0.944	5.8674	0.951	4.7698	0.927	4.2369	0.955
	L	6.2562	0.943	6.1995	0.944	5.1658	0.947	4.4635	0.959
$(60,30)$	F	4.9655	0.918	4.7694	0.939	3.8793	0.938	3.2654	0.961
	M	5.2366	0.924	5.0657	0.942	4.2652	0.927	3.6457	0.951
	L	5.6242	0.919	5.3945	0.947	4.7864	0.938	3.9962	0.947
$(60,40)$	F	4.3582	0.928	4.1366	0.951	3.2156	0.941	2.7358	0.949
	M	4.6581	0.943	4.5572	0.943	3.6499	0.937	2.9991	0.942
	L	4.9658	0.927	4.8623	0.955	3.9975	0.942	3.3562	0.950
$(90,45)$	F	3.8371	0.948	3.6987	0.947	2.7965	0.943	2.3145	0.952
	M	4.0526	0.958	3.8825	0.936	3.0147	0.944	2.5634	0.947
	L	4.3672	0.947	4.1986	0.951	3.3478	0.951	2.8045	0.955
$(90,65)$	F	3.2415	0.953	3.1054	0.949	2.2472	0.948	1.9578	0.953
	M	3.5621	0.949	3.4924	0.953	2.6127	0.953	2.3689	0.949
	L	3.8416	0.951	3.7652	0.947	2.9956	0.949	2.6574	0.956

Table 9: ILs and CPs of estimates for the parameter λ.

(n,m)	CSs	MLE		BP		BT		MCMC	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(30,15)$	F	1.4468	0.928	1.4499	0.931	1.1469	0.941	0.9967	0.951
	M	1.6485	0.931	1.5432	0.942	1.3567	0.943	1.1369	0.949
	L	1.8369	0.919	1.7563	0.928	1.5364	0.937	1.2357	0.947
$(30,20)$	F	1.1357	0.925	1.0549	0.937	0.9678	0.947	0.9125	0.961
	M	1.3642	0.937	1.2974	0.942	1.1324	0.932	0.9736	0.942
	L	1.5367	0.935	1.4968	0.940	1.2875	0.929	1.1556	0.944
$(60,30)$	F	0.9993	0.941	0.9998	0.951	0.8997	0.951	0.7865	0.947
	M	1.1963	0.935	1.1864	0.947	0.9473	0.945	0.8534	0.953
	L	1.3112	0.937	1.2947	0.936	1.0099	0.939	0.9266	0.956
$(60,40)$	F	0.8767	0.926	0.8564	0.932	0.7345	0.945	0.6699	0.949
	M	0.9254	0.947	0.9058	0.922	0.7835	0.955	0.7163	0.957
	L	1.0967	0.936	0.9863	0.947	0.8632	0.948	0.7769	0.954
$(90,45)$	F	0.6957	0.934	0.6734	0.953	0.6134	0.956	0.5763	0.949
	M	0.7781	0.951	0.7692	0.947	0.6637	0.948	0.6201	0.961
	L	0.8467	0.947	0.8279	0.939	0.7125	0.947	0.6874	0.958
$(90,65)$	F	0.6155	0.953	0.5993	0.951	0.4879	0.953	0.4361	0.951
	M	0.6498	0.948	0.6488	0.939	0.5137	0.948	0.4833	0.953
	L	0.6972	0.943	0.6855	0.941	0.5546	0.950	0.5142	0.952

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Table 10: AILs and CPs of estimates for $S(t = 0.4)$.

(n,m)	CSs	MLE		BP		BT		MCMC	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
(30,15)	F	0.4235	0.932	0.3957	0.931	0.3362	0.942	0.2696	0.955
	M	0.4436	0.928	0.4256	0.928	0.3545	0.938	0.2935	0.947
	L	0.4703	0.919	0.4534	0.917	0.3716	0.929	0.3254	0.952
(30,20)	F	0.3567	0.918	0.2976	0.928	0.2564	0.947	0.1863	0.953
	M	0.3845	0.917	0.3345	0.945	0.2738	0.942	0.2245	0.949
	L	0.3994	0.921	0.3643	0.934	0.3002	0.937	0.2597	0.955
(60,30)	F	0.2967	0.923	0.2342	0.944	0.1967	0.936	0.1434	0.947
	M	0.3259	0.928	0.2761	0.925	0.2365	0.933	0.1753	0.938
	L	0.3623	0.911	0.3202	0.919	0.2694	0.928	0.2045	0.947
(60,40)	F	0.2465	0.917	0.1964	0.930	0.1536	0.931	0.1197	0.946
	M	0.2762	0.925	0.2435	0.928	0.1897	0.940	0.1398	0.951
	L	0.2997	0.923	0.2687	0.919	0.2134	0.928	0.1637	0.954
(90,45)	F	0.1999	0.924	0.1768	0.944	0.1366	0.934	0.0999	0.948
	M	0.2234	0.931	0.2049	0.938	0.1604	0.941	0.1094	0.953
	L	0.2573	0.929	0.2363	0.923	0.1915	0.929	0.1293	0.949
(90,65)	F	0.1535	0.941	0.1295	0.925	0.1101	0.951	0.0935	0.956
	M	0.1814	0.933	0.1604	0.941	0.1382	0.939	0.0974	0.951
	L	0.2142	0.927	0.2093	0.939	0.1715	0.942	0.1162	0.948

Table 11: AILs and CPs of estimates for $h(t = 0.4)$.

(n,m)	CSs	MLE		BP		BT		MCMC	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
(30,15)	F	0.1965	0.920	0.1768	0.925	0.1657	0.939	0.1269	0.949
	M	0.2146	0.919	0.1964	0.919	0.1803	0.925	0.1396	0.939
	L	0.2436	0.917	0.2147	0.917	0.1993	0.934	0.1524	0.951
(30,20)	F	0.1568	0.924	0.1469	0.928	0.1238	0.941	0.0976	0.952
	M	0.1694	0.917	0.1552	0.931	0.1398	0.936	0.1099	0.947
	L	0.1967	0.916	0.1874	0.927	0.1562	0.933	0.1236	0.948
(60,30)	F	0.1278	0.923	0.1149	0.926	0.0957	0.942	0.0859	0.950
	M	0.1459	0.931	0.1297	0.924	0.1119	0.931	0.0915	0.942
	L	0.1775	0.928	0.1569	0.918	0.1318	0.951	0.1089	0.947
(60,40)	F	0.1066	0.941	0.1002	0.931	0.0817	0.947	0.0799	0.936
	M	0.1236	0.936	0.1196	0.932	0.0936	0.952	0.0842	0.941
	L	0.1598	0.934	0.1399	0.922	0.1098	0.941	0.0897	0.951
(90,45)	F	0.0997	0.928	0.0936	0.941	0.0779	0.938	0.0689	0.955
	M	0.1128	0.937	0.1100	0.938	0.0865	0.936	0.0748	0.948
	L	0.1366	0.925	0.1297	0.934	0.1097	0.942	0.0849	0.953
(90,65)	F	0.0854	0.942	0.0814	0.942	0.0699	0.939	0.0629	0.961
	M	0.1027	0.937	0.0975	0.941	0.0765	0.945	0.0685	0.957
	L	0.1235	0.933	0.1167	0.939	0.0967	0.949	0.0749	0.952

6 Conclusion

The primary objective of this paper was to develop various methods for estimating the unknown parameters α, β, λ, $S(t)$ and $h(t)$,of the PMKE Distribution within a PT-IIC scheme. MLEs were calculated, along with ACIs based on asymptotic distributions. The delta method was applied to obtain confidence intervals for the reliability and hazard functions. Additionally, two parametric bootstrap procedures, BP and BT, were explored to provide widely used confidence intervals. Upon examining the Bayesian estimates, it became clear that the posterior distribution equations of the unknown quantities are complex and not easily simplified into familiar forms. To address this, we employed the MCMC method to compute Bayes estimators. For illustration, we used the Physics dataset to evaluate and compare the proposed methods, a simulation study was conducted using various sample sizes (n,m) and different censoring schemes (F, M, L). From the results, we note the following:

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1. As expected from Tables $2-11$, as sample sizes (n,m) increase, both the MSEs and AILs decrease.
2. Bayes estimates yield the smallest MSEs and AILs for the unknown parameters α, β, λ, $S(t)$ and $h(t)$, outperforming both MLEs and bootstrap methods.
3. Among the methods, bootstrap performs better than MLE in terms of MSEs and AILs. Additionally, the BT method outperforms BP in both MSEs and AILs.
4. For fixed sample sizes and observed failures, Scheme F proves optimal, offering smaller MSEs and AILs.
5. The estimates from MLE, BP, BT, and Bayesian methods are closely aligned, with their ACIs exhibiting high CPs around 0.95. Moreover, Bayesian CRIs demonstrate the highest CPs.

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