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Poly-Frobenius-Genocchi Polynomials: A Probabilistic Perspective with Applications

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Abstract: In the paper, we introduce probabilistic extensions of poly-Frobenius-Genocchi polynomials and modified probabilistic Genocchi-polynomials. By making use of their generating functions, we derive explicit identities and a symmetric relation. In special cases, the obtained results reduce to classical one. Additionally, by choosing appropriate random variable, we obtain new identities including Stirling numbers of the first kind, Frobenius-Euler numbers, Frobenius-Genocchi numbers and Bernoulli numbers of negative order.

Keywords: Probabilistic Frobenius-Genocchi polynomials; Modified probabilistic Frobenius-Genocchi polynomials; Random variables

1 Introduction

Special functions and polynomials appear as solutions to ordinary and partial differential equations. Their importance properties are frequently used in approximation, computation, and numerical analysis. The various properties and connections of special functions and polynomials have been studied and investigated by many authors. For example, in [1], Choi *et al.* derived identities for Frobenius-Euler numbers and polynomials by using the fermionic p-adic q-integral equation on $\mathbb{Z}_p$. In [2], Duran *et al.* considered a new class of Frobenius-Genocchi polynomials, called type 2 poly-Frobenius-Genocchi polynomials, through the polyexponential function. In [3], Khan and Srivastava also worked on introducing a new class of generalized Apostol-type Frobenius-Genocchi polynomials and explored their properties and relations. In [4], Kim and Kim discussed higher-order Frobenius-Euler polynomials associated with poly-Bernoulli polynomials, which have been derived from the polylogarithmic function. In [5], Kim *et al.* derived new identities for poly-Genocchi polynomials. In [6], Wani *et al.* focused on introducing Gould–Hopper-based Frobenius-Genocchi polynomials and developed their properties. Yasar and Ozarslan [7] derived differential equations for Frobenius-Euler polynomials using the quasi-monomiality principle. They also introduced Frobenius-Genocchi polynomials and obtained some recurrence relations and differential equations. In [8], Araci and Acikgoz studied Bernstein polynomials and Frobenius-Euler numbers and polynomials. They also applied the method of generating functions and fermionic p-adic integral representation on $\mathbb{Z}_p$, which has been used to derive further classes of Bernstein polynomials and Frobenius-Euler numbers and polynomials. In [9], Adell introduced a new generalization of the Stirling numbers of the second kind and obtained new interesting identities. In [10], Karagenc *et al.* introduced probabilistic Bernstein polynomials and has derived new and interesting correlations among various special functions and special number sequences, such as Euler polynomials, higher-order Bernoulli polynomials, higher-order Frobenius-Euler polynomials, Stirling numbers of the second kind, and Bell polynomials. We now begin with the following notations:

$$\mathbb{N} := \{1,2,3,...\}, \mathbb{N}_0 := \{0,1,2,3,...\} = \mathbb{N} \cup \{0\}. \quad (1)$$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}^+$ will respectively, be denoted integers, rational numbers, and positive real numbers.

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Let Y be chosen as a random variable satisfying the moment conditions by

$$\mathrm{E}\left[|Y|^n\right] < \infty \text{ and } \lim_{n\to\infty} \frac{|t|^n \mathrm{E}\left[|Y|^n\right]}{n!} = 0, (|t| < r; r > 0) \quad (2)$$

where E means mathematical expectation. From here, one may write that

$$\mathrm{E}\left[e^{tY}\right] = \sum_{n=0}^{\infty} \mathrm{E}\left[Y^n\right]^n \frac{t^n}{n!}, \left(|t| < r, r \in \mathbb{R}^+\right) \quad (3)$$

or equivalently,

$$\mathrm{E}\left[|Y|^n\right] < \infty, \ \left(|t| < r, r \in \mathbb{R}^+\right). \quad (4)$$

$\{Y_j\}_{j=1}^k$ is a sequence of mutually independent copies of the Y with $S_k = Y_1 + Y_2 + ... + Y_k$, $(k \in \mathbb{N})$ with $S_0 = 0$. (see [9,10]).
Several distributions that will be used in deriving the results of this paper are given below:

1. ***Poisson distribution.*** $Y \sim Poisson(\alpha)$ with the parameter with yielding moment generating function (mgf) as

$$\mathrm{E}\left[e^{tY}\right] = e^{\alpha(e^t - 1)}. \quad (5)$$

2. ***Gamma distribution.*** Let $Y \sim \Gamma(1,1)$ be random variable with mgf as

$$\mathrm{E}\left[e^{tY}\right] = \frac{1}{1-t}, \ t < 1. \quad (6)$$

3. ***Exponential distribution.*** $Y \sim E(\alpha)$ be random variable with mgf as

$$\mathrm{E}\left[e^{tY}\right] = \frac{1}{1-\alpha t}, \ 0 < \alpha. \quad (7)$$

(see [9,10]).
We now reminder polylogarithms $\mathrm{Li}_k(t)$, Genocchi polynomials $\mathscr{G}_n(x)$, Frobenius-Genocchi polynomials $G_n^F(x|u)$, Euler polynomials $E_n(x)$, Frobenius Euler polynomials $H_n(x|u)$, Bernoulli polynomials $\beta_n(x)$, Bernoulli polynomials $B_n^{(-\alpha)}(x)$ of negative order, Stirling numbers of the first kind $S_1(n,k)$, Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, and Bell polynomials $\phi_n(x)$. Also we recall the probabilistic Stirling numbers of second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y$, probabilistic Bernoulli polynomials $\beta_n^Y(x)$, probabilistic Euler polynomials $\varepsilon_n^Y(x)$, probabilistic Frobenius-Euler polynomials $H_n^Y(x|u)$, modified probabilistic Bernoulli polynomials $B_n^Y(x)$ and probabilistic Lah numbers $L_n^Y(n,k)$.
Motivated by the above, we introduce probabilistic poly-Frobenius-Genocchi numbers $G_n^{(k,Y)}(u)$ associated with Y, modified probabilistic Frobenius-Genocchi numbers $G_n^Y(u)$ associated with Y.

In [2,5], it is well known that the polylogarithms are defined by

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}, (|t| < 1, k \in \mathbb{Z}), \quad (8)$$

which have the following derivative property:

$$\frac{d}{dt}\mathrm{Li}_k(f(t)) = \frac{f'(t)}{f(t)}\mathrm{Li}_{k-1}(f(t)). \quad (9)$$

In [13], the Euler polynomials, $E_n(x)$, are described by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, (|t| < \pi) \quad (10)$$

where provided that $x = 0$, we have $E_n(0) := E_n$ that denotes Euler numbers.
In [13], Frobenius-Euler polynomials are defined as

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x,u) \frac{t^n}{n!}, (u \in \mathbb{C} - \{1\}) \quad (11)$$

where $H_n(x;-1) := H_n(x)$.
In [2,12], the Genocchi polynomials, $\mathscr{G}_n(x)$, are described by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} \mathscr{G}_n(x) \frac{t^n}{n!}, (|t| < \pi) \quad (12)$$

where provided that $x = 0$, we have $\mathscr{G}_n(0) := \mathscr{G}_n$ that means Genocchi numbers.
In [2,3], Frobenius-Genocchi polynomials $G_n^F(x,u)$ are given by

$$\frac{(1-u)t}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} G_n^F(x,u) \frac{t^n}{n!}, (u \in \mathbb{C} - \{1\}), \quad (13)$$

satisfying $G_n^F(x,-1) := \mathscr{G}_n(x)$ and

$$\frac{G_{n+1}^F(x,u)}{n+1} = H_n(x,u). \quad (14)$$

In [14], the each Bernoulli polynomials of degree n, $\beta_n(x)$, can be found by means of the following generating function:

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} \beta_n(x) \frac{t^n}{n!}, (|t| < 2\pi) \ . \quad (15)$$

Let α be a non-negative integer. Bernoulli polynomials of negative order, $B_n^{(-\alpha)}(x)$, are defined as follows:

$$\left(\frac{e^t-1}{t}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(-\alpha)}(x) \frac{t^n}{n!}, (\text{see } [20]). \quad (16)$$

In [8], the Stirling numbers of the first kind, $S_1(n,k)$, are defined by means of the following generating function:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{17}$$

Throughout of this paper, we will assume as follows: If $1+t$ becomes a complex number, the value of $\log(1+t)$ is then defined by

$$\log(1+t) = \log|1+t| + i\arg(1+t), \tag{18}$$

where

1. $\log|1+t|$ is the real-valued logarithm of the modulus $|1+t|$.
2. $\arg(1+t)$ is the angle of $1+t$, typically restricted to the principal branch $(-\pi,\pi]$.

In [9], it is well known that the Stirling numbers of the second kind are defined as

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!}. \tag{19}$$

In [21], Bell polynomials $\phi_n(x)$ is defined thanks to

$$e^{x(e^t-1)} = \sum_{n=k}^{\infty} \phi_n(x)\frac{t^n}{n!}, \tag{20}$$

which aligns with the moment generating function of Poisson distribution having the mean x.
Adell and Lekuona ([9,11]) gave the probabilistic Stirling numbers of second kind are defined by

$$\frac{(\mathrm{E}[e^{tY}]-1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_Y \frac{t^n}{n!}. \tag{21}$$

Kim and Kim ([16]) introduced new families of Bernoulli and Euler polynomials subject to a random variable Y as follows, respectively:

$$\frac{t}{\mathrm{E}[e^{tY}]-1}\left(\mathrm{E}[e^{tY}]\right)^x = \sum_{n=0}^{\infty} \beta_n^Y(x)\frac{t^n}{n!}, (n \geq 0) \tag{22}$$

and

$$\frac{2}{\mathrm{E}[e^{tY}]+1}\left(\mathrm{E}[e^{tY}]\right)^x = \sum_{n=0}^{\infty} E_n^Y(x)\frac{t^n}{n!} \tag{23}$$

where $\beta_n^Y(x)$ and $E_n^Y(x)$ are called probabilistic Bernoulli and Euler polynomials. In the case when $Y=1$, $\beta_n^Y(x)$=$\beta_n(x)$ and $E_n^Y(x)$=$E_n(x)$ turn out to be well known (classical or ordinary) Bernoulli and Euler polynomials. Also, at the value of $x=0$, $\beta_n^Y(0)=\beta_n^Y$ and $E_n^Y(0)=E_n$ are called the probabilistic Bernoulli and Euler numbers.
In [18], modified probabilistic Bernoulli polynomials are known as

$$\frac{\log(\mathrm{E}[e^{tY}])}{\mathrm{E}[e^{tY}]-1}\left(\mathrm{E}[e^{tY}]\right)^x = \sum_{n=0}^{\infty} B_n^Y(x)\frac{t^n}{n!}. \tag{24}$$

In [18], probabilistic Frobenius-Euler polynomials are defined as $(u \in \mathbb{C}-\{1\})$

$$\frac{1-u}{\mathrm{E}[e^{tY}]-u}\left(\mathrm{E}[e^{tY}]\right)^x = \sum_{n=0}^{\infty} H_n^Y(x;u)\frac{t^n}{n!}. \tag{25}$$

In [19], the probabilistic Lah numbers are defined by

$$\frac{1}{k!}\left(\mathrm{E}\left[\left(\frac{1}{1-t}\right)^Y - 1\right]\right)^k = \sum_{n=k}^{\infty} L^Y(n,k)\frac{t^n}{n!}, (k \in \mathbb{N}_0). \tag{26}$$

In the next section, we introduce the generating function for the probabilistic poly-Frobenius-Genocchi polynomials. Utilizing this generating function, we derive explicit identities. Furthermore, by selecting appropriate random variables, we establish connections between the aforementioned polynomial and other special functions and polynomials.

2 Main Results

We are now in a position to state the following theorem, which serves as the main definition of this paper for deriving new identities, relations, and properties.

Definition 1. *Let Y be a random variable and $k \in \mathbb{Z}, u \in \mathbb{C}, u \neq 1$. The probabilistic poly-Frobenius-Genocchi polynomials $G_n^{(k,Y)}(x,u)$ associated with Y by*

$$\sum_{n=0}^{\infty} G_n^{(k,Y)}(x,u)\frac{t^n}{n!} = \frac{(1-u)\mathrm{Li}_k\left(1-\mathrm{E}\left[e^{-tY}\right]\right)}{u-\mathrm{E}\left[e^{-tY}\right]}(\mathrm{E}[e^{-tY}])^x. \tag{27}$$

For $x=0, G_n^{(k,Y)}(0,u) := G_n^{(k,Y)}(u)$ are called probabilistic poly-Frobenius-Genocchi numbers.

Definition 2. *Let Y be a random variable and $k \in \mathbb{Z}, u \in \mathbb{C}, u \neq 1$. The modifed probabilistic Frobenius-Genocchi polynomials $G_n^Y(x,u)$ associated with Y by*

$$\sum_{n=0}^{\infty} G_n^Y(x,u)\frac{t^n}{n!} = \frac{(1-u)\log(\mathrm{E}\left[e^{tY}\right])}{\mathrm{E}\left[e^{tY}\right]-u}(\mathrm{E}[e^{tY}])^x. \tag{28}$$

For $x=0, G_n^Y(0,u) := G_n^Y(u)$ are called modified probabilistic Frobenius-Genocchi numbers.

Theorem 1. *The probabilistic poly-Frobenius-Genocchi numbers associated with Y can be expressed as*

$$\frac{(-1)^n G_n^{(k,Y)}(u)}{n!} = \sum_{j=1}^{n}\sum_{l=1}^{j} \left\{ {j \atop l} \right\}_Y \frac{H_{n-j}^Y(u)(-1)^{l-1}l!}{(n-j)!j!l^k}. \tag{29}$$

Proof. By (27), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(k,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)\,\mathrm{Li}_k\left(1-\mathbb{E}[e^{-tY}]\right)}{u-\mathbb{E}[e^{-tY}]}\\ &= \frac{1-u}{u-\mathbb{E}[e^{-tY}]}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{(l+1)^k}\frac{\left(\mathbb{E}[e^{-tY}]-1\right)^{l+1}}{(l+1)!}\cdot(l+1)!\\ &= \left(\sum_{n=0}^{\infty}(-1)^{n+1}H_n^Y(u)\frac{t^n}{n!}\right)\times\\ &\quad \left(\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{(l+1)^k}\sum_{n=l}^{\infty}\begin{Bmatrix} n+1\\ l+1\end{Bmatrix}_Y\right.\\ &\quad \left.(-1)^{n+1}\frac{t^{n+1}}{(n+1)!}(l+1)!\right).\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above, we see that

$$\begin{aligned}G_n^{(k,Y)}(u) &= \sum_{j=0}^{n-1}\sum_{l=0}^{j}\binom{n}{j+1}\begin{Bmatrix} j+1\\ l+1\end{Bmatrix}_Y\frac{(-1)^{n+l}}{(l+1)^k}(l+1)!H_{n-1-j}^Y(u)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{j}\binom{n}{j}\begin{Bmatrix} j\\ l\end{Bmatrix}_Y\frac{(-1)^{n+l-1}}{l^k}l!H_{n-j}^Y(u)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{j}\frac{n!(-1)^{n+l-1}l!}{(n-j)!j!l^k}\begin{Bmatrix} j\\ l\end{Bmatrix}_Y H_{n-j}^Y(u),\end{aligned}$$

by straight forward manipulation over sum, we get the desired result.

Theorem 2. *Let Y be a random variable, we have*

$$\begin{aligned}\sum_{l=0}^{n} G_l^{(k,Y)}(u)S_1(n,l)(-1)^n &= \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{m=0}^{n-1-j}\binom{n}{j}\frac{(m+1)!(-1)^{j+l+1}}{(m+1)^k}\\ &\quad\times S_1(j,l)H_l^Y(u)L^Y(n-j,m+1).\end{aligned}$$

Proof. Since

$$\sum_{n=0}^{\infty} G_n^{(k,Y)}(u)\frac{t^n}{n!} = \frac{(1-u)\mathrm{Li}_k\left(1-\mathrm{E}\left[e^{-tY}\right]\right)}{u-\mathrm{E}\left[e^{-tY}\right]},$$

and replacing t by $\log(1-t)$ in (27) in this case, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} G_l^{(k,Y)}(u)\frac{(\log(1-t))^l}{l!} &= \sum_{l=0}^{\infty} G_l^{(k,Y)}(u)\sum_{n=l}^{\infty}S_1(n,l)(-1)^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} G_l^{(k,Y)}(u)S_1(n,l)(-1)^n\frac{t^n}{n!}.\end{aligned}$$

Then, by (27), we get

$$\begin{aligned}&= \frac{1-u}{u-\mathrm{E}\left[e^{-\log(1-t)Y}\right]}\mathrm{Li}_k\left(1-\mathrm{E}\left[\left(\frac{1}{1-t}\right)^Y\right]\right)\\ &= \left(\sum_{l=0}^{\infty}H_l^Y(u)(-1)^{l+1}\frac{(\log(1-t))^l}{l!}\right)\times\\ &\quad\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\left(1-\mathrm{E}\left[\left(\frac{1}{1-t}\right)^Y\right]\right)^{m+1}\right)\\ &= \left(\sum_{l=0}^{\infty}H_l^Y(u)(-1)^{l+1}\sum_{n=l}^{\infty}S_1(n,l)(-1)^n\frac{t^n}{n!}\right)\times\\ &\quad\left(\sum_{m=0}^{\infty}\frac{(m+1)!}{(m+1)^k}\sum_{n=m}^{\infty}L^Y(n+1,m+1)\frac{t^{n+1}}{(n+1)!}\right)\\ &= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}H_l^Y(u)(-1)^{n+l+1}S_1(n,l)\frac{t^n}{n!}\right)\times\\ &\quad\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(m+1)!}{(m+1)^k}L^Y(n+1,m+1)\frac{t^{n+1}}{(n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{m=0}^{n-j}\binom{n+1}{j}\frac{(m+1)!(-1)^{j+l+1}}{(m+1)^k}\right.\\ &\quad\left.\times S_1(j,l)H_l^Y(u)L^Y(n-j+1,m+1)\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.

Theorem 3. *Let Y be a random variable, we have*

$$G_n^{(1,Y)}(u) = (-1)^n G_n^Y(u). \tag{30}$$

Proof. By (27) for $k=1$, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(1,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Li}_1\left(1-\mathrm{E}\left[e^{-tY}\right]\right)}{u-\mathrm{E}\left[e^{-tY}\right]}\\ &= \frac{(1-u)\log\left(\mathrm{E}\left[e^{-tY}\right]\right)}{\mathrm{E}\left[e^{-tY}\right]-u}\\ &= \sum_{n=0}^{\infty}(-1)^n G_n^Y(u)\frac{t^n}{n!},\end{aligned}$$

which means the claimed equality.

Theorem 4. *Let Y be a random variable. Then we have the explicit identity:*

$$G_n^{(1,Y)}(u) = \sum_{j=0}^{n-1}\sum_{l=0}^{j}\binom{n}{j+1}\begin{Bmatrix} j+1\\ l+1\end{Bmatrix}_Y(-1)^{n+l-1}l!H_{n-1-j}^Y(u). \tag{31}$$

*Proof.*By (27) for $k=1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(1,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Li}_1\left(1-\mathbb{E}[e^{-tY}]\right)}{u-\mathbb{E}[e^{-tY}]}\\ &= \sum_{l=0}^{\infty}\frac{(1-u)\left(1-\mathbb{E}[e^{-tY}]\right)^{l+1}}{(l+1)\left(u-\mathbb{E}[e^{-tY}]\right)}\\ &= \sum_{l=0}^{\infty} l!\sum_{n=l}^{\infty}\left\{\begin{matrix} n+1\\ l+1\end{matrix}\right\}_Y(-1)^{n+l+1}\frac{t^{n+1}}{(n+1)!}\\ &\quad\times\sum_{m=0}^{\infty}(-1)^{m+1}H_m^Y(u)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\left\{\begin{matrix} n+1\\ l+1\end{matrix}\right\}_Y(-1)^{n+l+1}l!\right)\\ &\quad\times\frac{t^{n+1}}{(n+1)!}\sum_{m=0}^{\infty}(-1)^{m+1}H_m^Y(u)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n+1}{j+1}\left\{\begin{matrix} j+1\\ l+1\end{matrix}\right\}_Y\right.\\ &\quad\left.\times(-1)^{n+l}l!H_{n-j}^Y(u)\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain the desired identity.

Theorem 5. *Let Y be a random variable. Then the following equality holds true:*

$$G_n^{(2,Y)}(u) = (-1)^{n-1}\sum_{i=0}^{n-1}\sum_{j=0}^{i}\binom{i}{j}\binom{n}{i+1} \times\mathbb{E}[Y^{i-j+1}]H_{n-1-i}^Y(u)B_j^Y. \tag{32}$$

*Proof.*By (27) for $k=2$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(2,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Li}_2\left(1-\mathbb{E}[e^{-tY}]\right)}{u-\mathbb{E}[e^{-tY}]}\\ &= -\frac{1-u}{u-\mathbb{E}[e^{-tY}]}\int_0^t\mathbb{E}[Ye^{-xY}]\\ &\quad\times\left(\frac{\log\mathbb{E}[e^{-xY}]}{\mathbb{E}[e^{-xY}]-1}\right)dx\\ &= -\frac{1-u}{u-\mathbb{E}[e^{-tY}]}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n\mathbb{E}[Y^{n+1}]}{n!}\right.\\ &\quad\left.\times\sum_{m=0}^{\infty}B_m^Y\frac{(-1)^m x^m}{m!}\right)dx\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\binom{n+1}{i+1}(-1)^n\right.\\ &\quad\left.\times\mathbb{E}[Y^{i-j+1}]H_{n-i}^Y(u)B_j^Y\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.

Theorem 6.*Let Y be a random variable, we get*

$$G_n^Y(u) = \sum_{l=0}^{n}\sum_{k=0}^{l}\left\{\begin{matrix} n\\ l\end{matrix}\right\}_Y G_k^F(u)S_1(l,k). \tag{33}$$

*Proof.*By (28), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^Y(u)\frac{t^n}{n!} &= \frac{(1-u)\log\left(\mathbb{E}\left[e^{tY}\right]\right)}{\mathbb{E}\left[e^{tY}\right]-u}\\ &= \frac{(1-u)\log\left(\mathbb{E}[e^{tY}]\right)}{e^{\log(\mathbb{E}[e^{tY}])}-u}\\ &= \sum_{k=0}^{\infty} G_k^F(u)\frac{\left(\log(\mathbb{E}[e^{tY}]-1+1)\right)^k}{k!}\\ &= \sum_{k=0}^{\infty} G_k^F(u)\sum_{l=k}^{\infty}S_1(l,k)\frac{(\mathbb{E}[e^{tY}]-1)^l}{l!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}G_k^F(u)S_1(l,k)\left\{\begin{matrix} n\\ l\end{matrix}\right\}_Y\right)\frac{t^n}{n!}.\end{aligned}$$

which means the claimed equality.

Theorem 7.*Let Y be a random variable, we have*

$$G_{n+1}^Y(u) = \sum_{j=0}^{n}\sum_{k=0}^{n-j}\left\{\begin{matrix} n+1-j\\ k+1\end{matrix}\right\}_Y\binom{n+1}{j}(-1)^k k!H_j^Y(u). \tag{34}$$

Proof. Since

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n+1}^Y(u)\frac{t^{n+1}}{(n+1)!} &= \frac{(1-u)\log\left(\mathbb{E}[e^{tY}]\right)}{\mathbb{E}[e^{tY}]-u}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(1-u)\left(\mathbb{E}[e^{tY}]-1\right)^{k+1}}{(k+1)(\mathbb{E}[e^{tY}]-u)}\\ &= \sum_{k=0}^{\infty}(-1)^k k!\sum_{n=k}^{\infty}\left\{\begin{matrix} n+1\\ k+1\end{matrix}\right\}_Y\frac{t^{n+1}}{(n+1)!}\\ &\quad\times\sum_{m=0}^{\infty}H_m^Y(u)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^k k!H_j^Y(u)\right.\\ &\quad\left.\times\left\{\begin{matrix} n-j+1\\ k+1\end{matrix}\right\}_Y\binom{n+1}{j}\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

Note that $G_0^Y(u)=0$. Thus, by comparing coefficients $\frac{t^{n+1}}{(n+1)!}$ on both sides of the above, we arrive at the desired result.

Theorem 8.*Let Y be a random variable, then we have the explicit identity*

$$G_n^Y(u) = \sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{k=0}^{n-1-j}\sum_{m=0}^{j}\binom{n}{i}\left\{\begin{matrix} n-i\\ k+1\end{matrix}\right\}_Y\left\{\begin{matrix} i\\ j\end{matrix}\right\}_Y \times(-1)^k k!H_m(u)S_1(j,m). \tag{35}$$

*Proof.*By (28) we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_n^Y(u)\frac{t^n}{n!} &= \frac{(1-u)\log\left(\mathbb{E}[e^{tY}]\right)}{e^{\log(\mathbb{E}[e^{tY}])}-u} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(1-u)\left(\mathbb{E}[e^{tY}]-1\right)^{k+1}}{(k+1)\left(\mathbb{E}[e^{tY}]-u\right)} \\
&= \sum_{k=0}^{\infty}(-1)^k k! \sum_{m=0}^{\infty} H_m(u)\frac{\left(\log\mathbb{E}[e^{tY}]\right)^m}{m!} \\
&\quad \times \sum_{n=k}^{\infty} \left\{ {n+1 \atop k+1} \right\}_Y \frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k k! \left\{ {n+1 \atop k+1} \right\}_Y \frac{t^{n+1}}{(n+1)!}\right) \\
&\quad \times \sum_{m=0}^{\infty} H_m(u)\frac{\left(\log(\mathbb{E}[e^{tY}])\right)^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k k! \left\{ {n+1 \atop k+1} \right\}_Y \frac{t^{n+1}}{(n+1)!}\right) \\
&\quad \times \sum_{m=0}^{\infty} H_m(u)\sum_{j=m}^{\infty} S_1(j,m)\frac{(\mathbb{E}[e^{tY}]-1)^j}{j!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k k! \left\{ {n+1 \atop k+1} \right\}_Y \frac{t^{n+1}}{(n+1)!}\right) \\
&\quad \times \sum_{j=0}^{\infty}\left(\sum_{m=0}^{j} H_m(u)S_1(j,m)\right)\sum_{l=j}^{\infty} \left\{ {l \atop j} \right\}_Y \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k k! \left\{ {n+1 \atop k+1} \right\}_Y \frac{t^{n+1}}{(n+1)!}\right) \\
&\quad \times \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}\sum_{m=0}^{j} H_m(u)S_1(j,m)\left\{ {l \atop j} \right\}_Y\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i}\sum_{j=0}^{i}\sum_{m=0}^{j}(-1)^k k!\binom{n+1}{i}\left\{ {n-i+1 \atop k+1} \right\}_Y\right. \\
&\quad \left. \times \left\{ {i \atop j} \right\}_Y H_m(u)S_1(j,m)\right)\frac{t^{n+1}}{(n+1)!}.
\end{aligned}
$$

Thus, by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.

Theorem 9.*Let $Y \sim \Gamma(1,1)$, we get*

$$
G_n^{(k,Y)}(u) = \sum_{m=0}^{n-1}\sum_{l=0}^{m}\sum_{j=0}^{n-1-m}\binom{n-1-m}{j}\frac{(-1)^{n-m-j+l}n!}{m!(j+1)^k} \times S_1(m,l)H_l(u). \tag{36}
$$

*Proof.*By (6) and (27), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(k,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)\,\mathrm{Li}_k\left(1-\mathbb{E}[e^{-tY}]\right)}{u-\mathbb{E}[e^{-tY}]} \\
&= \frac{1-u}{u-\frac{1}{1+t}}\mathrm{Li}_k\left(1-\frac{1}{1+t}\right) \\
&= \frac{1-u}{u-e^{\log\frac{1}{1+t}}}\sum_{j=0}^{\infty}\left(\frac{t}{1+t}\right)^{j+1}\frac{1}{(j+1)^k} \\
&= \sum_{l=0}^{\infty}(-1)^{l+1}H_l(u)\frac{(\log(1+t))^l}{l!}\sum_{j=0}^{\infty}\frac{t^{j+1}}{(j+1)^k} \\
&\quad \times \sum_{n=0}^{\infty}\binom{n+j}{j}(-1)^n t^n \\
&= \sum_{l=0}^{\infty}(-1)^{l+1}H_l(u)\sum_{n=l}^{\infty}S_1(n,l)\frac{t^n}{n!} \\
&\quad \times \sum_{j=0}^{\infty}\frac{t^{j+1}}{(j+1)^k}\sum_{n=j}^{\infty}\binom{n}{j}(-1)^{n-j}t^{n-j} \\
&= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^{l+1}H_l(u)S_1(n,l)\frac{t^n}{n!}\right) \\
&\quad \times \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{n-j}}{(j+1)^k}t^{n+1}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{j=0}^{n-m}\binom{n-m}{j}\frac{(n+1)!(-1)^{n-m-j+l+1}}{(j+1)^k m!} \\
&\quad \times H_l(u)S_1(m,l)\frac{t^{n+1}}{(n+1)!}.
\end{aligned}
$$

By comparing coefficients t^n on both sides of the above, we arrive at the desired result.

Theorem 10.*Let $Y \sim \Gamma(1,1)$, we get*

$$
G_n^Y(u) = \sum_{k=0}^{n} S_1(n,k)G_k^F(u)(-1)^{n+k}. \tag{37}
$$

*Proof.*Since,

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_n^Y(u)\frac{t^n}{n!} &= \frac{(1-u)\log(\mathrm{E}[e^{tY}])}{\mathrm{E}[e^{tY}]-u} \\
&= \frac{(1-u)\log(\frac{1}{1-t})}{\frac{1}{1-t}-u} \\
&= \frac{(1-u)(-\log(1-t))}{e^{-\log(1-t)}-u} \\
&= \sum_{k=0}^{\infty} G_k^F(u)(-1)^k\frac{(\log(1-t))^k}{k!} \\
&= \sum_{k=0}^{\infty} G_k^F(u)(-1)^k\sum_{n=k}^{\infty}S_1(n,k)(-1)^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}G_k^F(u)(-1)^{n+k}S_1(n,k)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Thus, by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.

Theorem 11. *Let $Y \sim E(\alpha)$, we get*

$$G_n^{(k,Y)}(u) = \sum_{m=0}^{n-1}\sum_{l=0}^{m}\sum_{j=0}^{n-1-m}\binom{n-1-m}{j}\frac{\alpha^n(-1)^{n-1-m-j+l}(n+1)!}{m!(j+1)^k}S_1(m,l)H_l(u). \tag{38}$$

Proof. We omit the proof because it is same as the proof of Theorem 9.

Theorem 12. *Let $Y \sim E(\alpha)$, we get*

$$G_n^Y(u) = \sum_{k=0}^{n}\alpha^n S_1(n,k)G_k^F(u)(-1)^{n+k}. \tag{39}$$

Proof. We omit the proof because it is same as the proof of Theorem 10.

Theorem 13. *Let $Y \sim Poisson(\alpha)$, we get*

$$G_n^Y(u) = \sum_{k=0}^{\infty}G_k^F(u)\alpha^k\left\{\begin{matrix} n \\ k \end{matrix}\right\}. \tag{40}$$

Proof. By (5) and (28) we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_n^F(u)\frac{t^n}{n!} &= \frac{(1-u)\log(\mathrm{E}[e^{tY}])}{\mathrm{E}[e^{tY}]-u}\\ &= \frac{(1-u)\alpha(e^t-1)}{e^{\alpha(e^t-1)}-u}\\ &= \sum_{k=0}^{\infty}G_k^F(u)\alpha^k\frac{(e^t-1)^k}{k!}\\ &= \sum_{k=0}^{\infty}G_k^F(u)\alpha^k\sum_{n=k}^{\infty}\left\{\begin{matrix} n \\ k \end{matrix}\right\}\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}\left(\sum_{k=0}^{\infty}G_k^F(u)\alpha^k\left\{\begin{matrix} n \\ k \end{matrix}\right\}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.

Theorem 14. *Let $Y \sim Poisson(\alpha)$, we get*

$$G_n^Y(u) = \sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n-1}{j}\left\{\begin{matrix} j \\ k \end{matrix}\right\}B_{n-1-j}^{(-1)}H_k(u)\alpha^{k+1}. \tag{41}$$

Proof. By (5) and (28) we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_n^Y(u)\frac{t^n}{n!} &= \frac{(1-u)\log(\mathbb{E}[e^{tY}])}{\mathbb{E}[e^{tY}]-u}\\ &= \alpha\frac{e^t-1}{t}t\frac{1-u}{e^{\alpha(e^t-1)}-u}\\ &= \sum_{n=0}^{\infty}B_n^{(-1)}\frac{t^{n+1}}{n!}\sum_{k=0}^{\infty}H_k(u)\alpha^{k+1}\frac{(e^t-1)^k}{k!}\\ &= \sum_{n=0}^{\infty}B_n^{(-1)}\frac{t^{n+1}}{n!}\sum_{k=0}^{\infty}H_k(u)\alpha^{k+1}\sum_{m=k}^{\infty}\left\{\begin{matrix} m \\ k \end{matrix}\right\}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}B_n^{(-1)}\frac{t^{n+1}}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left\{\begin{matrix} m \\ k \end{matrix}\right\}H_k(u)\alpha^{k+1}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\left\{\begin{matrix} j \\ k \end{matrix}\right\}B_{n-j}^{(-1)}H_k(u)\alpha^{k+1}\\ &\qquad\times(n+1)\frac{t^{n+1}}{(n+1)!},\end{aligned}$$

which means the claimed equality.

3 Conclusion

In the paper, we have explored extensions of probabilistic poly-Frobenius-Genocchi polynomials, and introduced probabilistic poly-Frobenius Genocchi polynomials associated with Y. We have also derived explicit formulas, established various related identities, and obtained a symmetric relation for these polynomials. Furthermore, we have investigated explicit expressions for the modified probabilistic Genocchi polynomials associated with Y, which differ slightly from the probabilistic Frobenius-Genocchi polynomials associated with Y. As special cases, we consider applications in case that Y is poisson, gamma, and exponential distributions.

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References

[1] Choi, J, Kim, D. S., Kim, T, Kim, Y. H. *A note on some identities of Frobenius-Euler numbers and polynomials.* International Journal of Mathematics and Mathematical Sciences, 2012, 861797, 9 pages, 2012.

[2] Duran, U., Acikgoz, M., Araci, S. *Construction of the type 2 poly-Frobenius–Genocchi polynomials with their certain applications.* Adv Differ Equ 2020, 432 (2020).

[3] Khan, W. A., Srivastava, D. *On the generalized Apostol-type Frobenius-Genocchi polynomials.* Filomat, 33(7), 1967-1977, (2019).

[4] Kim, D. S., Kim, T. *Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials.* Adv Differ Equ 2013, 251 (2013).

[5] Kim, T., San Kim, D., Jang, G. W., Kwon, J. *Poly-Genocchi polynomials with umbral calculus viewpoint.* J. Comput. Anal. Appl, v.26, no.6, pp 1014 - 1031.

[6] Wani, S. A., Khan, S., Nahid, T. *Gould-Hopper based Frobenius–Genocchi polynomials and their generalized form.* Afr. Math. Afr. Mat. 31, 1397-1408 (2020).

[7] Yasar, B. Y., Ozarslan, M.A. *Frobenius–Euler and Frobenius–Genocchi polynomials and their differential equations.*New Trends Math. Sci. 3(2), 172–180, (2015).

[8] Araci, S., Acikgoz, M. *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials.* Adv. Stud. Contemp. Math., Volume 22, Issue 3, Pages 399-406, 2012.

[9] Adell, J. A. *Probabilistic Stirling numbers of the second kind and applications.* Journal of Theoretical Probability, 35(1), 636-652, (2022).

[10] Karagenc, A., Acikgoz, M., Araci S. *Exploring probabilistic Bernstein polynomials: identities and applications.* Applied Mathematics in Science and Engineering, 2024, Vol. 32, No. 1, 2398591.

[11] Adell, J. A., Lekuona, A. *A probabilistic generalization of the Stirling numbers of the second kind.* Journal of Number Theory, 194, 335-355, (2019).

[12] Araci, S. *Novel identities for q-Genocchi numbers and polynomials.* J. Funct. Spaces Appl. (2012), Article ID 214961.

[13] Araci, S., Acikgoz, M., *Construction of Fourier expansion of Apostol Frobenius–Euler polynomials and itsapplications.* Adv Differ Equ 2018, 67 (2018).

[14] Karagenc, A., Acikgoz, M., Araci, S. *A new family of q-Bernstein polynomials: Probabilistic viewpoint*. Arab Journal of Basic and Applied Sciences, 32(1), 42-50, 2025.

[15] Kim, T., Kim, D. S. *Probabilistic degenerate Bell polynomials associated with random variables.* Russ. J. Math. Phys. 30, 528-542 (2023).

[16] Kim, T., Kim, D. S. *Probabilistic Bernoulli and Euler polynomials.* Russ. J. Math. Phys. 31, 94-105 (2024).

[17] Kim, T., Kim, D. S., Kwon, J. *Probabilistic degenerate Stirling polynomials of the second kind and their applications.* Mathematical and Computer Modelling of Dynamical Systems, 30(1), 16-30, (2024).

[18] Liu, W., Ma, Y., Kim, T., Kim, D.S. Probabilistic poly-Bernoulli numbers. Mathematical and Computer Modelling of Dynamical Systems 2024, Vol. 30, No. 1, 840-856.

[19] Ma, Yuankui, Taekyun Kim, and Dae San Kim. *Probabilistic Lah numbers and Lah-Bell polynomials.* arXiv preprint arXiv:2406.01200 (2024).

[20] Schikhof, W. H. Ultrametric Calculus: An introduction to p-adic analysis, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press Cambridge, 1984.

[21] Yasar, B. Y., Ozarslan, M.A. *Frobenius–Euler and Frobenius-Genocchi polynomials and their differential equations.*New Trends Math. Sci. 3(2), 172–180, (2015).

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