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Weighted Least Squares Estimator for ARPD(1) Model: Methodology and Properties

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Abstract: Autoregressive models are fundamental tools in time series and panel data analysis, enabling the modeling of a variable based on its past values to predict future outcomes. These models become particularly useful in panel data contexts, where observations are collected across multiple entities over time. The Autoregressive Panel Data (ARPD) model is a prominent variant, offering insights into both time-dependent and cross-sectional variations. Specifically, the ARPD model of order one, denoted as ARPD(1), is a first-order model where the current value of the dependent variable is influenced by its immediate past value. The importance of the ARPD(1) model lies in its ability to capture the dynamic behavior of the data while accounting for individual-specific effects. This paper focuses on estimating parameters in a fixed-effect conditional ARPD(1) model using the Weighted Least Squares (WLS) method with different weights. The study delves into the properties of this estimator, demonstrating its linearity, unbiasedness, and variance. Furthermore, the performance of the WLS estimator is compared with alternative methods under the ARPD(1) framework. A Monte Carlo simulation is conducted to evaluate the effectiveness of the WLS method versus the Ordinary Least Squares (OLS) method, using Mean Squared Error (MSE), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) as benchmarks. The results from the simulation highlight the superiority of the WLS estimator over OLS, making it the preferred choice for parameter estimation in ARPD(1) models. Moreover, empirical estimation using real ARPD(1) data is performed, further reinforcing the advantages of the WLS approach over traditional methods, particularly in terms of providing more accurate and reliable estimates.

Keywords: Conditional autoregressive panel; ARPD(1) model; fixed effect model; weighted least squares; Monte Carlo simulation.

1 Introduction

Time series analysis involves the statistical examination of time series data, which represents observations collected at specific intervals or time periods. Numerous models are employed for representing and analyzing time series, with a focus in this paper on conditional autoregressive panel data models (ARPD). Anderson and Hsiao [1] presented various estimation methods for regression models with autoregressive covariance structures of order one. Their emphasis was on maximum likelihood estimation for the stationary autoregressive panel data model. Levin and Lin [11] proposed an autoregressive model to include individual fixed effects and a time trend in their model. The model with individual fixed effects can be written:

$$y_{it} = \alpha_i + \varphi y_{i(t-1)} + \varepsilon_{it}, \quad i = 1,2,3,\ldots,N \quad \text{and} \quad t = 1,2,\ldots,T \tag{1}$$

where y_{it} is the dependent variable for individual i at time t, $y_{i(t-1)}$ is the lagged dependent variable, α_i represents the individual fixed effects for individual i, φ is the autoregressive coefficient, and ε_{it} is the error term for individual i at time t with mean 0 and variance σ^2_ε. This model allows for more accurate and individualized analysis by considering both the

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time-specific trends and individual-specific effects. Also, they consider the ordinary least squares estimator of $\hat{\varphi}_{OLS}$ which is defined as:

$$\hat{\varphi}_{OLS} = \left[\sum_{i=1}^{N}\sum_{t=2}^{T}(y_{it}-\bar{y}_i)^2\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=2}^{T}(y_{it}-\bar{y}_i)(y_{i(t-1)}-\bar{y}_i)\right] \tag{2}$$

Quah [13] derived the estimator of the ARPD(1) model panel data without a constant based on OLS as follows:

$$k_{it} = \varphi k_{i(t-1)} + v_{it}, \quad i = 1,2,3,\dots,N \quad \text{and} \quad t = 1,2,\dots,T$$

where k_{i0} is the initial value, which is a given random variable with mean μ, variance σ^2, and v_{it} are independent identically distributed with mean zero, variance σ_v^2. The term k_{it} represents the deviations of values around the mean, specifically $k_{it} = y_{it} - \bar{y}_i$ and $k_{i(t-1)} = y_{i(t-1)} - \bar{y}_{i(-1)}$, where $\bar{y}_i = \frac{1}{T}\sum_{t=2}^{T} y_{it}$ and $\bar{y}_{i(-1)} = \frac{1}{T}\sum_{t=2}^{T} y_{i(t-1)}$. The cross-section dimension N and the time dimension T are assumed to be of the same order of magnitude, that is, $N = N(T) = O(T)$.

For the previous model, the ordinary least squares estimator $\hat{\varphi}_{OLS}$ is considered, that is:

$$\hat{\varphi}_{OLS} = \frac{\sum_{i=1}^{N}\sum_{t=1}^{T} k_{it}k_{i(t-1)}}{\sum_{i=1}^{N}\sum_{t=1}^{T} k_{i(t-1)}^2} \tag{3}$$

Issa et al. [10] employed the weighted symmetric method (WS) to estimate the parameters of a transformed second-order autoregressive panel data model, ARPD(2), ensuring that no additional observations were lost. This study extends the approach of Park and Fuller [12] by applying the weighted symmetric method (WS) to estimate parameters for the ARPD(2) model with fixed effects. The model is specified as follows:

$$k_{it} = \varphi_1 k_{i(t-1)} + \varphi_2 k_{i(t-2)} + v_{it}, \quad i = 1,2,3,\dots,N \quad \text{and} \quad t = 1,2,\dots,T$$

Hsiao et al. [7] suggested a transformed likelihood approach to estimate a fixed effects ARPD(1) model. They proposed conditions on the data generating process of exogenous variables to address the issue of ”incidental parameters,” showing both are consistent and asymptotically normally distributed. Monte Carlo studies were conducted to evaluate MLE, MDE, instrumental variable (IV), and generalized method of moments (GMM) estimators, showing the likelihood approach to outperform GMM in terms of bias, root mean square error, and test statistics performance.

El-Sayed et al. [5] estimated the parameters of a second-order autoregressive panel data model. They used the ordinary least squares (OLS) method to obtain the least squares estimators for these parameters. Additionally, they proved that these estimators were linear, unbiased, and converged in probability.

Youssef et al. [14] investigated dynamic panel models, specifically applying the generalized method of moments (GMM) to the ARPD(1) model, which had been widely used for its efficient estimators. They found that the efficiency was influenced by the choice of the initial weight matrix. While it was common to use the inverse of the moment matrix, the optimal initial weight matrix remained unknown, especially in system GMM estimation. The study introduced an optimal weight matrix for the level GMM estimator and suboptimal ones for the system GMM, significantly enhancing efficiency, particularly when the variance of individual effects was high compared to error variance.

Issa and Abdelwahab [9] derived estimators using the Weighted Symmetric (WS) method derived from the ARPD(2) model. Additionally, they investigated various properties of the OLS estimator for the ARPD(2) model parameters, particularly in the absence of homogeneity. The study examined the linearity, bias, variance, and asymptotic consistency of the estimator and mathematically derived its asymptotic distribution.

Gonçalves and Perron [6] studied the bias and efficiency of alternative estimators for ARPD models characterized by AR(1) process disturbances ARPD(1) and nonstationary regressors. They introduced a novel Combined GMM estimator that integrates the strengths of the Arellano-Bond GMM [2] and Arellano and Bover system GMM [3] estimators. Through simulations and empirical applications, they studied that the Combined GMM estimator outperforms existing estimators in terms of efficiency. The underlying model for their analysis was a dynamic panel data model with ARPD(1) errors and nonstationary regressors.

The research aims to reuse the WLS method using different weights to reduce the MSE of the estimator. To achieve this, an estimator for the ARPD(1) model is derived using the Weighted Least Squares (WLS) method. The properties of this estimator are discussed. A Monte Carlo simulation study is performed to compare the proposed estimator with other methods across various sample sizes. Additionally, the ARPD(1) model is applied to real data for further evaluation. The structure of the article is as follows: Section 2 presents the model and its assumptions. Section 3 derives the WLS estimator for the ARPD(1) model and explores its properties. Section 4 conducts simulation studies to compare the OLS estimator with the proposed WLS estimators (WLSA and WLSB). Section 5 applies these estimators to real data to assess their practical effectiveness, reliability, and ability to accurately model various applications. Finally, Section 6 provides a conclusion that summarizes the theoretical insights and the results from the simulation study.

2 The Model and Assumptions

The first-order autoregressive panel data model takes the following form:

$$y_{it} = \alpha_i + \phi y_{i(t-1)} + \varepsilon_{it}, \quad i = 1,2,3,\ldots,N \quad \text{and} \quad t = 2,\ldots,T \tag{4}$$

where $y_{i(t-1)}$ is an explanatory variable, y_{i0} is fixed, and ϕ is a coefficient such that $|\phi| < 1$ for every $i = 1,2,3,\ldots,N$. α_i is an unobserved individual-specific time-invariant effect, which allows for heterogeneity in the means of the y_{it} series across individuals. We assume the cross-section dimension to be N and the time dimension T to be of the same order of magnitude, that is, $N = N(T) = O(T)$.

Summing Eq. (4) on both sides and dividing by T results in taking averages over the time dimension. Subtracting the results yields a simpler implementation by applying the within transformation, which accounts for the disappearance of individual effects by transforming the data into deviations with respect to individual means:

$$y_{it} - \bar{y}_i = \alpha_i + \phi \left(y_{i(t-1)} - \bar{y}_{i(-1)}\right) + (\varepsilon_{it} - \bar{\varepsilon}_i), \tag{5}$$

where

$$\bar{y}_i = \frac{1}{T}\sum_{t=2}^{T} y_{it}, \quad \bar{y}_{i(-1)} = \frac{1}{T}\sum_{t=2}^{T} y_{i(t-1)}, \quad \bar{\varepsilon}_i = \frac{1}{T}\sum_{t=2}^{T} \varepsilon_{it}.$$

Equation (5) can be simplified as follows:

$$k_{it} = \phi k_{i(t-1)} + v_{it}, \quad i = 1,2,3,\ldots,N \quad \text{and} \quad t = 2,\ldots,T. \tag{6}$$

where:

$$k_{it} = y_{it} - \bar{y}_i, \quad k_{i(t-1)} = y_{i(t-1)} - \bar{y}_{i(-1)}, \quad v_{it} = \varepsilon_{it} - \bar{\varepsilon}_i$$

is the unobservable error term with certain properties specified below. The following assumptions are applied:

1. The unknown parameter ϕ is constrained to $|\phi| < 1$ for stationarity.
2. k_{i0} is fixed, and when $N \to \infty$, the effect of ε_{i1} will be negligible and tends to zero.
3. v_{it} is independent and identically distributed (i.i.d.) with a Gaussian distribution with mean 0 and variance $\sigma^2_{i\varepsilon}$, and the fourth moment of v_{it} exists. Therefore, the function of variance determines the functional form of the conditional heteroskedasticity.
4. $E(k_{i(t-1)}, v_{it}) = 0$, meaning that the independent variables are predetermined in the sense that they are orthogonal to the contemporaneous error term for every $t = 2,3,\ldots,T$, $i = 1,2,\ldots,N$.
5. $E(v_{ij}, v_{it}) = 0$ for all $i \neq j$ and $t \neq s$.
6. $E(v_{it} v_{js} \mid k_{i(t-1)}, k_{j(s-1)}) = 0$ for all $i \neq j$ and $t \neq s$.

3 The Proposed Estimator and Its Properties

In this section, the estimator of the ARPD(1) panel data model and its properties will be derived using the WLS method.

Lemma 1. *Based on the model of Eq. (6) with the same assumptions as above, by applying weighted least squares, we get the within estimator of the Fixed Effects model:*

$$\hat{\phi}_{WLS} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} k_{it} k_{i(t-1)}}{\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} k^2_{i(t-1)}}$$

Proof. Let Q be the weighted sum of squares of the random factors of model (**??**) of the estimated residuals:

$$Q = \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} v^2_{it} = \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} \left(k_{it} - \phi k_{i(t-1)}\right)^2 \tag{7}$$

After differentiating equation (7) with respect to ϕ and setting the derivative to zero, we get:

$$\hat{\phi}_{WLS} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} k_{it} k_{i(t-1)}}{\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it} k^2_{i(t-1)}} \tag{8}$$

The weighted w_{it} suggested by Issa [8] will be reused in ARPD models as follows:

$$A.\ w_{it} = |k_{i(t-1)}|^{-2\gamma_i}$$
$$B.\ w_{it} = |k_{i(t-1)}|^{\gamma_i - 1}$$

where γ_i is the coefficient of heteroscedasticity according to Brewer [14]. Substitute the value of weight number (A) in equation (8) to get:

$$\hat{\phi}_{WLS.A} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}|k_{i(t-1)}|^{-2\gamma_i}k_{it}k_{i(t-1)}}{\sum_{i=1}^{N}\sum_{t=2}^{T}|k_{i(t-1)}|^{-2\gamma_i}k_{i(t-1)}^2} \tag{9}$$

When $\gamma = 0$, we have $\hat{\phi}_{WLS.A} = \hat{\phi}_{OLS}$, and we revert to the form presented by Levin and Lin [2] in Eq. (2). Substitute the value of weight number (B) in Eq. (8). to get:

$$\hat{\phi}_{WLS.B} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}|k_{i(t-1)}|^{\gamma_i-1}k_{it}k_{i(t-1)}}{\sum_{i=1}^{N}\sum_{t=2}^{T}|k_{i(t-1)}|^{\gamma_i-1}k_{i(t-1)}^2} \tag{10}$$

When $\gamma = 1$, we have $\hat{\phi}_{WLS.B} = \hat{\phi}_{OLS}$, and we assume γ_i is a fixed effect for all times and individuals.

Lemma 2. *Based on the model of the parameter of the WLS estimator in Eq. (8)., we can study the linearity, unbiasedness, and variance.*

$$var(\hat{\phi}_{WLS}) = \sum_{i=1}^{N}\sum_{t=2}^{T}\left(w_{it}k_{i(t-1)}\left(\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it}k_{i(t-1)}^2\right)^{-1}\right)^2\sigma_{\varepsilon_i}^2$$

Proof. Equation (8). can be rewritten as follows. It is easy to verify that $\hat{\phi}_{WLS}$ can be rewritten in linear form:

$$\hat{\phi}_{WLS} = \sum_{i=1}^{N}\sum_{t=2}^{T}z_{it}k_{it} \tag{11}$$

where:

$$z_{it} = w_{it}k_{i(t-1)}\left(\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it}k_{i(t-1)}^2\right)^{-1} \tag{12}$$

By substituting k_{it} from Eq. (6) into Eq. (11), we get:

$$\hat{\phi}_{WLS} = \phi + \sum_{i=1}^{N}\sum_{t=2}^{T}z_{it}v_{it} \tag{13}$$

By taking the expectation of Eq. (13) and using assumption (3), we get:

$$E(\hat{\phi}_{WLS}) = \phi$$

Since:

$$\text{var}(\hat{\phi}_{WLS}) = E\left[\hat{\phi}_{WLS} - \phi\right]^2 \tag{14}$$

Equation (13) can be rewritten as:

$$E\left[\hat{\phi}_{WLS} - \phi\right]^2 = E\left[\sum_{i=1}^{N}\sum_{t=2}^{T}z_{it}v_{it}\right]^2$$

$$E\left[\hat{\phi}_{WLS} - \phi\right]^2 = E\left(\sum_{i=1}^{N}\sum_{t=2}^{T}v_{it}^2z_{it}^2\right) + 2\sum_{i<j}^{N}\sum_{t<s}^{T}\sum_{s=2}^{T}z_{it}z_{js}v_{it}v_{js}$$

By using assumptions (4) and (5), we get:

$$E\left[\hat{\phi}_{WLS} - \phi\right]^2 = E\left(\sum_{i=1}^{N}\sum_{t=2}^{T}v_{it}^2z_{it}^2\right)$$

By using assumption (3), we get:

$$E\left[\hat{\phi}_{WLS} - \phi\right]^2 = E\left(\sum_{i=1}^{N}\sum_{t=2}^{T}v_{it}^2z_{it}^2\right) = \sum_{i=1}^{N}\sum_{t=2}^{T}z_{it}^2\sigma_{\varepsilon_i}^2 \tag{15}$$

By substituting the value of z_{it} in Eq. (15), we obtain our proof.

Note: By substituting the values of weighted A and B in Eq. (15)., we can obtain the form of the variance for the different weights.

Table 1: ARPD (1) Model Estimation When $\gamma = 0.3$ and positive φ

(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	9.8653	0.5925	0.5771	15.5340	0.4085	0.4021	29.3439	0.2334	0.2313
	AIC	-1717.69	34.7721	108.9326	-1695.75	29.5787	98.1899	-1740.99	29.2586	98.2333
	BIC	-1714.04	38.4287	112.5893	-1692.09	33.2353	101.8466	-1737.33	32.9153	101.8899
10-50	MSE	10.3450	0.5892	0.5726	16.3177	0.4041	0.3977	31.0651	0.2268	0.2248
	AIC	-3476.58	82.7537	279.5601	-3543.26	88.5938	266.1192	-3433.75	83.6207	256.2021
	BIC	-3470.84	88.4898	285.2962	-3537.53	94.3298	271.8553	-3428.02	89.3567	261.9382
10-100	MSE	10.5820	0.5872	0.5696	16.6988	0.4022	0.3959	31.8697	0.2239	0.2219
	AIC	-7179.37	107.6889	413.7131	-6770.07	155.9942	522.1895	-6694.52	210.1597	689.7848
	BIC	-7171.55	115.5044	421.5286	-6762.25	163.8097	530.0050	-6686.70	217.9753	697.6003
25-25	MSE	9.8639	0.5925	0.5772	15.5503	0.4088	0.4024	29.3392	0.2334	0.2313
	AIC	-1750.34	47.7524	137.7717	-1840.37	69.9339	181.0234	-1699.01	73.7937	192.5395
	BIC	-1746.69	51.4090	141.4283	-1836.71	73.5906	184.6800	-1695.35	77.4503	196.1962
25-50	MSE	10.3435	0.5893	0.5727	16.3218	0.4042	0.3978	31.0643	0.2268	0.2248
	AIC	-3462.85	63.1579	217.7020	-3384.90	73.4376	225.9134	-3366.23	96.3213	310.5586
	BIC	-3457.11	68.8940	223.4380	-3379.16	79.1737	231.6495	-3360.49	102.0574	316.2946
25-100	MSE	10.5798	0.5870	0.5693	16.6974	0.4022	0.3958	31.8646	0.2239	0.2219
	AIC	-6733.13	158.9565	512.0868	-6795.42	33.2238	277.8930	-6686.33	148.8547	462.2712
	BIC	-6725.32	166.7720	519.9023	-6787.60	41.0393	285.7086	-6678.51	156.6702	470.0867
50-25	MSE	9.8641	0.5924	0.5771	15.5476	0.4087	0.4023	29.3384	0.2335	0.2314
	AIC	-1749.71	16.6312	75.7734	-1781.58	38.9308	122.6424	-1665.51	92.1251	233.8779
	BIC	-1746.06	20.2878	79.4300	-1777.92	42.5875	126.2991	-1661.86	95.7817	237.5345
50-50	MSE	10.3470	0.5890	0.5724	16.3216	0.4042	0.3979	31.0505	0.2268	0.2248
	AIC	-3447.08	71.3939	233.5438	-3400.65	48.5380	195.4869	-3477.63	92.0927	276.1130
	BIC	-3441.34	77.1300	239.2799	-3394.91	54.2740	201.2230	-3471.89	97.8288	281.8491
50-100	MSE	10.5796	0.5870	0.5693	16.6962	0.4022	0.3959	31.8678	0.2239	0.2219
	AIC	-6642.47	182.3233	605.2486	-6747.58	232.1638	752.5092	-6820.25	106.0016	384.2487
	BIC	-6634.66	190.1388	613.0642	-6739.76	239.9793	760.3247	-6812.43	113.8171	392.0642

4 Simulation Study

This section aims to investigate the properties of the proposed estimation methods through the simulation study output with OLS and WLS. The model is generated as follows:

1. ARPD(1) model without constant is generated. The errors are generated $\sim$ IIDN$(0,1)$, and the autoregressive parameter ϕ is chosen to be -0.3, -0.5, -0.1, 0.1, 0.3, and 0.5.
2. Different sample time and individuals for each combination of (N,T) have been used:
   - $(10,25)$, $(10,50)$, $(10,100)$
   - $(25,25)$, $(25,50)$, $(25,100)$
   - $(50,25)$, $(50,50)$, and $(50,100)$
3. Different values of γ have been used as: $\gamma_i = 0.3, 0.5, 0.7$.
4. MSE, AIC, and BIC as criteria of comparison.
5. All Monte Carlo experiments involved 10000 replications.

The simulation results for when $\gamma_i = (0.3, 0.5, 0.7)$ and $\phi = (0.1, 0.3, 0.5, -0.1, -0.3, -0.5)$ are used. The tables (1 - 6) present a detailed comparison of three methods (OLS, WLS.A, and WLS.B) for estimating an ARPD(1) model.

Table 2: ARPD (1) Model Estimation When $\gamma = 0.5$ and positive φ

(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	9.8653	0.5029	0.6231	15.5381	0.3695	0.4212	29.3373	0.2199	0.2376
	AIC	-1721.68	97.4040	61.3539	-1774.06	80.1410	48.1270	-1755.97	108.1744	64.0495
	BIC	-1718.03	101.0606	65.0106	-1770.40	83.7976	51.7836	-1752.31	111.8310	67.7061
10-50	MSE	10.3400	0.4875	0.6212	16.3192	0.3643	0.4166	31.0641	0.2139	0.2307
	AIC	-3380.81	146.1729	81.4126	-3432.89	225.3226	137.5207	-3424.64	171.0454	100.2360
	BIC	-3375.07	151.9090	87.1487	-3427.15	231.0587	143.2568	-3418.91	176.7815	105.9721
10-100	MSE	10.5815	0.4751	0.6206	16.7011	0.3621	0.4147	31.8702	0.2115	0.2278
	AIC	-6636.59	451.5107	273.3162	-7203.46	546.6053	332.7054	-6841.49	371.1251	221.3277
	BIC	-6628.77	459.3262	281.1317	-7195.64	554.4208	340.5209	-6833.68	378.9406	229.1432
25-25	MSE	9.8709	0.5029	0.6232	15.5482	0.3693	0.4212	29.3513	0.2198	0.2376
	AIC	-1713.13	134.1808	88.8945	-1710.25	66.3277	36.5311	-1757.28	108.0809	68.2768
	BIC	-1709.47	137.8374	92.5511	-1706.60	69.9843	40.1878	-1753.63	111.7375	71.9334
25-50	MSE	10.3479	0.4879	0.6213	16.3169	0.3644	0.4166	31.0542	0.2140	0.2308
	AIC	-3426.26	234.6900	139.2377	-3559.95	224.4677	136.5296	-3552.49	90.6584	37.4562
	BIC	-3420.52	240.4261	144.9737	-3554.21	230.2038	142.2657	-3546.75	96.3944	43.1922
25-100	MSE	10.5782	0.4748	0.6205	16.6926	0.3622	0.4146	31.8725	0.2115	0.2278
	AIC	-6932.87	311.8806	182.5601	-7128.39	554.6122	346.3290	-6910.50	566.1498	338.8934
	BIC	-6925.05	319.6961	190.3756	-7120.58	562.4277	354.1445	-6902.68	573.9653	346.7089
50-25	MSE	9.8647	0.5021	0.6228	15.5433	0.3695	0.4213	29.3409	0.2199	0.2376
	AIC	-1805.32	145.0608	91.0359	-1691.93	175.0328	116.0339	-1699.03	111.0869	71.6060
	BIC	-1801.66	148.7174	94.6925	-1688.27	178.6894	119.6906	-1695.37	114.7435	75.2626
50-50	MSE	10.3457	0.4879	0.6213	16.3167	0.3646	0.4166	31.0573	0.2140	0.2307
	AIC	-3375.82	243.4030	155.2937	-3419.58	203.2097	118.9686	-3384.50	219.8100	135.1243
	BIC	-3370.08	249.1391	161.0298	-3413.84	208.9458	124.7047	-3378.77	225.5460	140.8604
50-100	MSE	10.5808	0.4751	0.6206	16.6925	0.3621	0.4146	31.8693	0.2115	0.2278
	AIC	-6805.77	431.3942	257.0127	-6745.77	310.5323	172.7281	-6740.12	318.2737	185.2941
	BIC	-6797.96	439.2097	264.8283	-6737.96	318.3478	180.5436	-6732.30	326.0892	193.1096

Here is a more in-depth analysis of the results, focusing on the impact of varying sample sizes (N,T), ϕ, and the performance of each method within different N,T settings:

4.1 In Case of $\gamma_i = (0.3, 0.5, 0.7)$ and ϕ are positive sign

In this case, the commentary is divided into two sections. The first section discusses the results of the Comparative Analysis of Methods, while the second section addresses the results of the Intra Method Comparison. This division is applicable only if the values of ϕ are positive sign.

4.1.1 Comparative Analysis of Methods

The performance of the Ordinary Least Squares (OLS) method and the newly proposed Weighted Least Squares methods (WLS.A and WLS.B) is evaluated across different metrics: MSE, AIC, and BIC under various parameter settings. The analysis is divided into three distinct scenarios with different values of $\phi = (0.1, 0.3, 0.5)$, and is summarized in tables (1, 2, and 3).

First: When $\phi = 0.1$:

– **MSE:** WLS.A and WLS.B consistently outperform OLS across all sample sizes and time periods. The largest improvements are observed for larger samples and longer time periods, with WLS.B showing the most significant reduction in MSE.

Table 3: ARPD (1) Model Estimation When $\gamma = 0.7$ and positive φ

(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
(N,T)	Criteria	$\varphi = 0.1$			$\varphi = 0.3$			$\varphi = 0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	9.8584	0.4730	0.6374	15.5508	0.3560	0.4274	29.3361	0.2149	0.2396
	AIC	-1804.19	163.7381	63.9493	-1655.68	81.5940	18.5334	-1711.21	170.2621	66.1746
	BIC	-1800.54	167.3947	67.6059	-1652.02	85.2506	22.1901	-1707.55	173.9187	69.8312
10-50	MSE	10.3559	0.4551	0.6369	16.3172	0.3506	0.4226	31.0531	0.2093	0.2327
	AIC	-3445.24	261.3330	87.7023	-3447.96	246.6041	76.0606	-3592.60	324.3485	111.2099
	BIC	-3439.50	267.0690	93.4384	-3442.23	252.3401	81.7967	-3586.87	330.0846	116.9460
10-100	MSE	10.5816	0.4371	0.6364	16.6995	0.3478	0.4207	31.8621	0.2070	0.2297
	AIC	-7283.96	515.3762	167.1367	-6951.62	645.9204	222.4474	-7254.18	654.0460	207.2372
	BIC	-7276.15	523.1918	174.9522	-6943.81	653.7359	230.2629	-7246.37	661.8615	215.0527
25-25	MSE	9.8689	0.4741	0.6377	15.5437	0.3560	0.4273	29.3297	0.2150	0.2396
	AIC	-1695.25	123.3938	45.5896	-1718.95	117.9463	36.6502	-1696.63	71.7041	9.3715
	BIC	-1691.59	127.0504	49.2463	-1715.30	121.6029	40.3068	-1692.97	75.3608	13.0282
25-50	MSE	10.3445	0.4547	0.6367	16.3242	0.3506	0.4227	31.0527	0.2093	0.2327
	AIC	-3524.35	253.6045	86.9561	-3539.50	225.5890	67.5385	-3326.86	257.8484	85.8169
	BIC	-3518.62	259.3405	92.6921	-3533.77	231.3250	73.2746	-3321.12	263.5844	91.5530
25-100	MSE	10.5818	0.4368	0.6364	16.6966	0.3480	0.4207	31.8654	0.2070	0.2297
	AIC	-6751.01	396.4496	107.3660	-6989.37	452.1356	121.3836	-7041.64	667.1644	209.0347
	BIC	-6743.20	404.2651	115.1815	-6981.55	459.9511	129.1991	-7033.82	674.9799	216.8502
50-25	MSE	9.8629	0.4740	0.6377	15.5347	0.3559	0.4273	29.3419	0.2148	0.2395
	AIC	-1695.66	110.1157	32.7953	-1823.04	86.7898	21.0633	-1747.36	158.1484	63.1430
	BIC	-1692.00	113.7723	36.4520	-1819.38	90.4465	24.7199	-1743.71	161.8051	66.7997
50-50	MSE	10.3469	0.4549	0.6368	16.3196	0.3506	0.4227	31.0512	0.2093	0.2327
	AIC	-3570.75	245.5100	78.3397	-3466.21	299.2092	102.4199	-3340.87	158.1716	31.7058
	BIC	-3565.01	251.2461	84.0758	-3460.48	304.9453	108.1559	-3335.13	163.9077	37.4419
50-100	MSE	10.5800	0.4369	0.6363	16.6962	0.3479	0.4207	31.8731	0.2070	0.2297
	AIC	-6781.97	443.8374	125.2546	-6810.07	525.0661	181.0855	-6784.70	539.8646	166.0688
	BIC	-6774.15	451.6529	133.0701	-6802.25	532.8816	188.9011	-6776.88	547.6801	173.8844

- **AIC:** Both WLS.A and WLS.B exhibit lower AIC values compared to OLS, indicating better model fit, particularly for moderate to large sample sizes and longer time periods.
- **BIC:** Similar to AIC, WLS.A and WLS.B generally achieve lower BIC values, reflecting improved model performance over OLS.

Second: When $\phi = 0.3$:

- **MSE:** WLS.A and WLS.B maintain superior performance over OLS. The reduction in MSE is more pronounced for larger sample sizes and longer time periods, with WLS.B again showing notable efficiency.
- **AIC:** The AIC values for WLS.A and WLS.B are consistently lower than those for OLS, suggesting that these methods offer a better balance between model complexity and fit.
- **BIC:** Lower BIC values are observed for WLS.A and WLS.B compared to OLS, supporting their effectiveness in handling larger datasets and extended time periods.

Third: When $\phi = 0.5$:

- **MSE:** The trend continues with WLS.A and WLS.B outperforming OLS. The improvements in MSE are most substantial with increasing sample size and time period.
- **AIC:** Both WLS.A and WLS.B show lower AIC values, particularly in larger sample sizes and longer time periods, indicating superior model fit.
- **BIC:** Lower BIC values for WLS.A and WLS.B reinforce their advantage over OLS, especially with larger datasets.

Table 4: ARPD (1) Model Estimation When $\gamma = 0.3$ and positive φ

(N,T)	Criteria		$\varphi = -0.1$			$\varphi = -0.3$			$\varphi = -0.5$	
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
(N,T)	Criteria		$\varphi = -0.1$			$\varphi = -0.3$			$\varphi = -0.5$	
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	6.9196	0.7397	0.7086	5.0600	0.8122	0.7629	3.6440	0.8265	0.7703
	AIC	-1736.70	34.3075	105.5233	-1778.24	41.3682	117.4269	-1833.57	45.4446	135.0723
	BIC	-1733.04	37.9641	109.1799	-1774.59	45.0249	121.0836	-1829.91	49.1012	138.7289
10-50	MSE	7.2687	0.7278	0.6909	5.3349	0.7806	0.7201	3.8176	0.7811	0.7144
	AIC	-3490.50	68.7350	220.1372	-3435.72	104.7995	294.1674	-3448.10	64.6698	220.8952
	BIC	-3484.76	74.4711	225.8733	-3429.98	110.5355	299.9034	-3442.36	70.4059	226.6312
10-100	MSE	7.4518	0.7189	0.6766	5.4731	0.7581	0.6881	3.9063	0.7505	0.6759
	AIC	-7155.89	250.0717	767.5050	-6776.09	162.1410	529.8706	-6924.70	271.6001	795.1747
	BIC	-7148.08	257.8872	775.3205	-6768.27	169.9565	537.6861	-6916.89	279.4157	802.9902
25-25	MSE	6.9181	0.7393	0.7083	5.0625	0.8124	0.7631	3.6427	0.8265	0.7703
	AIC	-1800.11	50.6094	143.0832	-1724.53	8.8483	67.0772	-1719.69	30.8764	101.3055
	BIC	-1796.45	54.2660	146.7398	-1720.87	12.5049	70.7339	-1716.03	34.5330	104.9621
25-50	MSE	7.2684	0.7279	0.6910	5.3363	0.7807	0.7201	3.8196	0.7810	0.7143
	AIC	-3427.93	100.3277	300.2953	-3611.16	129.8351	396.5201	-3497.61	74.1893	258.3955
	BIC	-3422.20	106.0638	306.0314	-3605.42	135.5711	402.2562	-3491.87	79.9254	264.1315
25-100	MSE	7.4495	0.7195	0.6773	5.4785	0.7587	0.6886	3.9073	0.7506	0.6760
	AIC	-6794.21	106.7458	414.5926	-6906.00	174.4955	550.1713	-6722.09	133.7106	475.5320
	BIC	-6786.40	114.5614	422.4082	-6898.18	182.3110	557.9868	-6714.28	141.5261	483.3476
50-25	MSE	6.9186	0.7394	0.7084	5.0632	0.8114	0.7621	3.6439	0.8267	0.7704
	AIC	-1718.24	38.8694	113.8351	-1690.88	18.8384	83.5892	-1672.48	16.4649	78.5443
	BIC	-1714.58	42.5260	117.4918	-1687.23	22.4950	87.2458	-1668.83	20.1216	82.2010
50-50	MSE	7.2738	0.7280	0.6911	5.3358	0.7809	0.7204	3.8158	0.7804	0.7137
	AIC	-3454.57	26.0021	161.2374	-3436.63	130.4298	366.4524	-3551.69	42.9456	182.4433
	BIC	-3448.83	31.7382	166.9735	-3430.89	136.1659	372.1885	-3545.95	48.6817	188.1794
50-100	MSE	7.4500	0.7195	0.6774	5.4771	0.7583	0.6883	3.9073	0.7500	0.6753
	AIC	-6811.28	160.0356	533.9021	-6830.51	128.6325	441.0150	-6920.41	132.5387	466.7110
	BIC	-6803.46	167.8511	541.7176	-6822.69	136.4480	448.8305	-6912.59	140.3543	474.5265

4.1.2 Intra method Comparisons

This part evaluates the performance of each estimation method OLS, WLS.A, and WLS.B by analyzing their results across various conditions. It includes two main comparisons: first, how each method performs with different sample sizes while keeping either N or T fixed, and second, the impact of changing N and T on each method?s performance. Each method will be presented and analyzed separately to highlight its performance under different scenarios.

- **OLS Method:**
  - **Effect of Sample Size (N):** With fixed T, OLS generally shows increased MSE, AIC, and BIC values as N grows. The performance deteriorates with larger N, indicating less robustness in larger datasets.
  - **Effect of Time (T):** With fixed N, OLS performance improves with longer time periods, showing reduced MSE, AIC, and BIC values. This trend reflects OLS?s capacity to handle extended time series more effectively.
- **WLS.A Method:**
  - **Effect of Sample Size (N):** WLS.A consistently performs better than OLS, with MSE, AIC, and BIC improving as sample size N increases, demonstrating robustness in handling larger datasets.
  - **Effect of Time (T):** For fixed N, WLS.A shows a strong performance improvement with longer time periods, with substantial reductions in MSE, AIC, and BIC, indicating enhanced efficiency in extended datasets.
- **WLS.B Method:**
  - **Effect of Sample Size (N):** WLS.B outperforms both OLS and WLS.A in most cases, with notable improvements in MSE, AIC, and BIC as sample size N increases. This method shows the best performance among the three, especially with larger sample sizes.

Table 5: ARPD (1) Model Estimation When $\gamma = 0.5$ and Negative φ

(N,T)	Criteria	$\varphi = -0.1$			$\varphi = -0.3$			$\varphi = -0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	6.9176	0.5697	0.8029	5.0618	0.5644	0.9171	3.6426	0.5665	0.9544
	AIC	-1697.56	133.8154	87.7228	-1777.32	97.1482	58.7241	-1756.17	101.7173	59.8468
	BIC	-1693.91	137.4721	91.3794	-1773.66	100.8048	62.3807	-1752.51	105.3739	63.5034
10-50	MSE	7.2703	0.5197	0.8004	5.3353	0.4770	0.9073	3.8162	0.4794	0.9303
	AIC	-3476.58	169.5512	97.4207	-3417.15	169.2008	99.7294	-3483.24	268.4006	163.6742
	BIC	-3470.84	175.2873	103.1568	-3411.42	174.9368	105.4655	-3477.51	274.1367	169.4103
10-100	MSE	7.4471	0.4710	0.7986	5.4793	0.4029	0.9002	3.9054	0.4215	0.9165
	AIC	-6790.23	482.0713	289.6638	-6722.90	494.7315	283.7602	-6896.49	457.2959	274.5616
	BIC	-6782.42	489.8868	297.4793	-6715.09	502.5470	291.5757	-6888.68	465.1114	282.3772
25-25	MSE	6.9173	0.5694	0.8027	5.0619	0.5644	0.9177	3.6460	0.5663	0.9546
	AIC	-1745.70	116.5619	74.4522	-1746.70	86.1982	48.5233	-1791.50	97.5767	60.3172
	BIC	-1742.04	120.2185	78.1088	-1743.04	89.8549	52.1799	-1787.84	101.2333	63.9738
25-50	MSE	7.2720	0.5192	0.8008	5.3354	0.4768	0.9075	3.8184	0.4801	0.9315
	AIC	-3417.25	169.7302	94.3977	-3465.06	272.2907	170.9059	-3501.30	203.5320	123.9465
	BIC	-3411.52	175.4663	100.1338	-3459.32	278.0268	176.6420	-3495.56	209.2681	129.6825
25-100	MSE	7.4489	0.4721	0.7990	5.4750	0.4046	0.9002	3.9084	0.4215	0.9166
	AIC	-6717.23	415.8513	248.2406	-6813.14	337.8615	197.5822	-6919.48	507.3212	306.3516
	BIC	-6709.41	423.6668	256.0561	-6805.33	345.6770	205.3977	-6911.66	515.1367	314.1671
50-25	MSE	6.9170	0.5689	0.8025	5.0641	0.5643	0.9174	3.6444	0.5659	0.9540
	AIC	-1695.91	54.9805	26.8249	-1717.77	130.4930	80.8944	-1716.50	135.0491	84.7892
	BIC	-1692.26	58.6371	30.4815	-1714.11	134.1496	84.5510	-1712.84	138.7057	88.4458
50-50	MSE	7.2708	0.5193	0.8007	5.3363	0.4762	0.9073	3.8183	0.4800	0.9312
	AIC	-3506.99	169.6904	99.4678	-3488.15	196.2946	119.5992	-3365.50	201.3273	124.0887
	BIC	-3501.26	175.4265	105.2039	-3482.41	202.0306	125.3352	-3359.76	207.0634	129.8248
50-100	MSE	7.4502	0.4723	0.7993	5.4761	0.4042	0.9003	3.9082	0.4218	0.9168
	AIC	-6737.16	610.1064	351.2496	-6816.73	490.5285	307.8790	-7104.62	346.1799	195.3501
	BIC	-6729.35	617.9219	359.0651	-6808.92	498.3440	315.6945	-7096.81	353.9954	203.1656

- **Effect of Time (T):** Like WLS.A, WLS.B shows significant improvements with longer time periods. The method?s efficiency increases notably, with the lowest MSE, AIC, and BIC values observed.

Overall, both WLS.A and WLS.B demonstrate clear advantages over OLS across various metrics, particularly in handling larger datasets and longer time periods. The comparative analysis indicates that WLS.B is the most effective method, offering superior performance across all scenarios..

4.2 In Case of $\gamma_i = (0.3, 0.5, 0.7)$ and φ are Negative Sign

In this case, the commentary is divided into two sections. The first section discusses the results of the Comparative Analysis of Methods, while the second section addresses the results of the Intra Method Comparison. This division is applicable only if the values of φ are negative.

4.2.1 Comparison across Methods (MSE, AIC, BIC)

The proposed methods WLS.A and WLS.B were compared with the traditional OLS method using the criteria MSE, AIC, and BIC for different values of the autoregressive parameter φ (-0.1, -0.3, and -0.5). The results, presented in tables (4,

Table 6: ARPD (1) Model Estimation When $\gamma = 0.7$ and Negative φ

(N,T)	Criteria	$\varphi = -0.1$			$\varphi = -0.3$			$\varphi = -0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
(N,T)	Criteria	$\varphi = -0.1$			$\varphi = -0.3$			$\varphi = -0.5$		
		OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B	OLS	WLS.A	WLS.B
10-25	MSE	6.9170	0.5221	0.8342	5.0557	0.5063	0.9719	3.6524	0.5133	1.0252
	AIC	-1727.79	102.843	29.894	-1683.47	72.9586	10.926	-1712.38	163.3414	62.3924
	BIC	-1724.13	106.4997	33.5507	-1679.81	76.6153	14.5826	-1708.72	166.998	66.049
10-50	MSE	7.2685	0.461	0.8352	5.3398	0.4102	0.9712	3.8194	0.4254	1.0124
	AIC	-3339.62	485.305	177.9824	-3531.84	242.2699	74.3528	-3559.05	252.8204	84.4003
	BIC	-3333.88	491.041	183.7184	-3526.11	248.006	80.0889	-3553.31	258.5565	90.1364
10-100	MSE	7.4513	0.4028	0.8361	5.4806	0.3294	0.969	3.9088	0.3681	1.0041
	AIC	-6677.27	689.2402	203.9398	-6728.41	549.5299	172.2117	-6749.24	425.1904	122.5856
	BIC	-6669.46	697.0557	211.7553	-6720.6	557.3454	180.0272	-6741.43	433.0059	130.4011
25-25	MSE	6.9169	0.5216	0.8336	5.0624	0.5073	0.9727	3.6463	0.513	1.024
	AIC	-1785.43	126.9742	44.2544	-1684.54	215.829	86.1799	-1840.37	172.9335	66.5789
	BIC	-1781.78	130.6308	47.911	-1680.88	219.4857	89.8365	-1836.71	176.5901	70.2355
25-50	MSE	7.2718	0.461	0.8353	5.3362	0.4079	0.9702	3.8172	0.4246	1.0113
	AIC	-3376.56	209.3109	59.288	-3480.8	270.1758	88.4327	-3438.71	199.9613	59.1594
	BIC	-3370.83	215.0469	65.0241	-3475.06	275.9119	94.1687	-3432.97	205.6974	64.8955
25-100	MSE	7.4512	0.4026	0.8361	5.4765	0.329	0.9688	3.9061	0.3683	1.0038
	AIC	-6959.83	513.2732	149.1171	-7039.47	503.3703	134.1362	-7030.01	457.6635	143.1951
	BIC	-6952.01	521.0887	156.9326	-7031.66	511.1858	141.9517	-7022.2	465.479	151.0106
50-25	MSE	6.9207	0.5227	0.8343	5.0635	0.5067	0.9724	3.6463	0.514	1.0249
	AIC	-1713.79	194.4504	81.3052	-1735.28	166.8572	68.4712	-1699.75	202.31	82.9239
	BIC	-1710.13	198.1071	84.9618	-1731.62	170.5138	72.1279	-1696.1	205.9666	86.5805
50-50	MSE	7.2717	0.4614	0.8355	5.3374	0.4088	0.9708	3.8183	0.4251	1.0114
	AIC	-3388.67	300.224	107.2046	-3575.84	180.1739	46.1446	-3349.65	202.5652	53.1203
	BIC	-3382.94	305.96	112.9407	-3570.1	185.91	51.8807	-3343.91	208.3012	58.8563
50-100	MSE	7.451	0.4032	0.8361	5.4779	0.3287	0.9688	3.9058	0.3681	1.0035
	AIC	-7227.33	409.0175	102.8343	-6887.71	619.0341	190.1353	-6758.16	365.4327	96.9974
	BIC	-7219.52	416.833	110.6498	-6879.89	626.8496	197.9508	-6750.34	373.2482	104.8129

5 and 6), reveal consistent trends. In terms of MSE, both WLS.A and WLS.B consistently outperform OLS, especially as φ decreases, indicating a more accurate estimation. For AIC and BIC, WLS.A and WLS.B often show lower values compared to OLS, suggesting better model selection criteria, especially when T increases relative to N. This trend is evident across all tables, demonstrating the efficiency of WLS methods over OLS, particularly in scenarios with higher autoregressive parameters and larger T.

4.2.2 Comparison within Each Method (MSE, AIC, BIC)

- **OLS Method:** When analyzing the OLS method alone, it is observed that as T increases while N is fixed, the MSE generally decreases, indicating improved estimation accuracy. However, the AIC and BIC values do not always decrease correspondingly, suggesting that while the model fit improves with more data points, the complexity and penalty terms may impact the overall model selection criteria.
- **WLS.A Method:** For the WLS.A method, a similar trend is noticed with a decreasing MSE as T increases. Notably, the AIC and BIC values for WLS.A tend to improve significantly as T increases, particularly when φ is higher, indicating that WLS.A adapts well to larger datasets and higher autoregressive parameters, making it a robust choice for model estimation.
- **WLS.B Method:** WLS.B shows the most significant reduction in MSE as T increases, even more so than WLS.A. This suggests that WLS.B may be more sensitive to increases in T, providing the most accurate estimates in large-sample scenarios. The AIC and BIC values for WLS.B also tend to decrease as T increases, particularly for higher values of φ, reinforcing its effectiveness in both model fit and complexity management in larger datasets.

Summary

- **Ordinary Least Squares (OLS):** Performs better with positive initial values of φ, as indicated by lower MSE, AIC, and BIC values in tables (1, 2 and 3) compared to tables (4, 5 and 6).
- **WLS.A and WLS.B:** Both methods consistently outperform OLS regardless of whether the initial values of φ are positive or negative. However, they show slightly better performance with positive φ, as indicated by lower MSE, AIC, and BIC values in tables (1, 2 and 3).

Overall, while WLS methods (both A and B) demonstrate superior performance compared to OLS in both scenarios, they achieve the best results with positive initial values of φ.

5 Real Data Application

The feasibility of the proposed estimators are illustrated using productivity across 74 countries from 1992 to 2016 using Zakaryan [15]. Figure 1 shows this data that consist of 25 year observations of 74 countries. The preliminary analysis of the data indicates that the time series in all sectors lack stationarity in terms of variance and mean. Therefore, logarithmic transformation was applied, followed by taking the first differences of the data. This approach ensures that the data becomes stationary. By using Levin, Lin & Chu and Augmented Dickey-Fuller tests for stationarity assumption of the time series, the tests were used. The results were as follows:

Table 7: Output of the Unit Root Test

Method	Statistic	Prob.	Cross-sections	Obs
Null: Unit root (assumes common unit root process)				
Levin, Lin & Chu t*	-19.2015	0.0000	74	1628
Null: Unit root (assumes individual unit root process)				
Im, Pesaran and Shin W-stat	-22.2587	0.0000	74	1628
ADF - Fisher Chi-square	765.833	0.0000	74	1628
PP - Fisher Chi-square	1204.49	0.0000	74	1702

Probabilities for Fisher tests are computed using an asymptotic Chi-square distribution. All other tests assume asymptotic normality.

Based on the results of the unit root tests, the data becomes stationary after taking the first differences and applying the logarithmic transformation. We can use the ACF and PACF, as shown in the following figure, to determine whether the data follows ARPD(1) models. The results are illustrated in the figure 1 below. To compare between (OLS), (WLS.A),

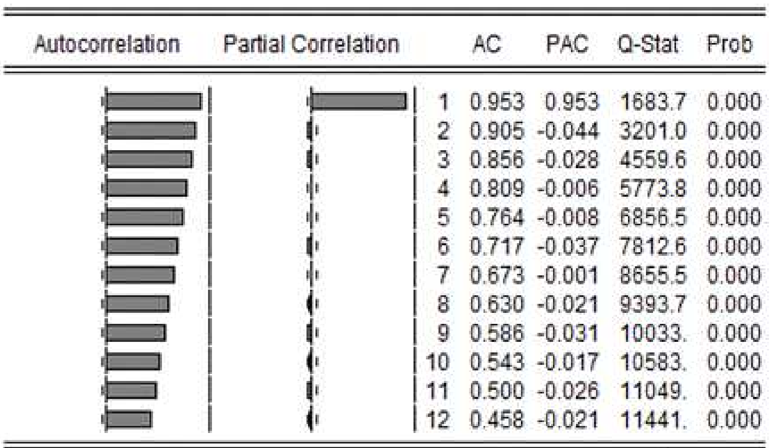

Fig. 1: Sample ACF and PACF for Annual Productivity

and (WLS.B) methods of estimation for the parameter of ARPD (1) model in the different values of γ_i. To measure the accuracy, (MSE), (AIC), and (BIC) are computed. The results of tables (7, 8) indicated that the new methods (WLS.A and WLS.B) give good performance for the values of (MSE), (AIC), and (BIC) for different values of γ with respect to other methods.

Table 8: MSE, AIC, and BIC for Different Estimators

ρ	Estimator	MSE	AIC	BIC
0.3	WLS.A	0.002302869	-0.5222403	-0.5217419
	WLS.B	0.002303158	-0.5744420	-0.5738938
	OLS	0.252301952	-5.7453508	-5.7398687
0.5	WLS.A	0.002304197	-0.5221472	-0.5216488
	WLS.B	0.002302611	-0.5744842	-0.5739360
	OLS	0.252301952	-5.7453508	-5.7398687
0.7	WLS.A	0.002307912	-0.5218871	-0.5213888
	WLS.B	0.002302206	-0.5745154	-0.5739672
	OLS	0.252301952	-5.7453508	-5.7398687

Conclusion

This article delves into estimating parameters for ARPD(1) models using a technique called Weighted Least Squares (WLS). It proposes two different weighting schemes and calculates the variances of the estimated parameters under various scenarios, while also exploring some properties of the WLS estimator. Building upon Issa's work, the study extends AR(1) models to handle constant terms and missing data points. A Monte Carlo simulation compares the performance of three estimators: Ordinary Least Squares (OLS), WLS with weighting (WLS.A), and WLS with weighting (WLS.B). The simulation assesses results across different sample sizes (N) and time periods (T), considering both positive and negative initial values for the model parameter. The results consistently show that both WLS methods (A and B) achieve lower Mean Squared Error (MSE) compared to OLS, indicating a more accurate fit for the model. This advantage extends to the AIC and BIC, where WLS methods generally produce lower values. WLS methods are therefore preferable when selecting the best model. Furthermore, the analysis of real data confirms that both WLS methods outperform OLS regardless of whether the initial values are positive or negative, with slightly better performance observed for positive initial values, as reflected in lower MSE, AIC, and BIC values. Overall, while WLS methods demonstrate superior performance compared to OLS in both scenarios, they achieve the best results with positive initial values.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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