Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/190410

Analysis of Iterative Method for Impulsive Neutral Functional Integro-Differential Equation Using Degree Theory

Kottakkaran Sooppy Nisar[1,2,*], *Kanagaraj Muthuselvan* [3], *Muhannad Ibrahim Al-Shartab*[1], *Ahmed M. Shehata*[4], *and Mohamed R. Eid*[5]

[1]Department of Mathematics, College of Science and Humanities in Al Kharj, Prince Sattam bin Abdulaziz University, Al Kharj, 11942, Saudi Arabia
[2] Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Jordan
[3] Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Chennai - 601103, India
[4] Department of Mathematics, College of Sciences, University of Bisha, Bisha 61922, Saudi Arabia
[5]Center for Scientific Research and Entrepreneurship, Northern Border University, Arar 73213, Saudi Arabia

Received: 15 Jan. 2025, Revised: 1 Apr. 2025, Accepted: 29 Apr. 2025
Published online: 1 Jul. 2025

Abstract: The fundamental objective of this paper is to develop the impulsive neutral functional integro-differential equation by employing an iterative process to establish comprehensive analytical results. Using the topological degree method, we derive sufficient and necessary conditions for the existence of solutions to the given dynamical system. To further validate these results, we utilize Gronwall's inequality, which plays a crucial role in proving the uniqueness of the solutions under specific conditions. Additionally, numerical computations are provided to illustrate the theoretical findings and to assess their accuracy and applicability. These computations highlight the effectiveness of the proposed method.

Keywords: Functional differential equation; Degree theory; Iterative process; Numerical computations.

1 Introduction

One of the primary foundations of mathematical modeling is the differential equation, which offers an effective framework for characterizing the manner in which values change across time or space. In a wide range of scientific fields, including physics, engineering, biology, and economics, these equations describe dynamic processes by expressing relationships between unknown functions and their derivatives [1–3]. In this manuscript, we consider the following form of the impulsive neutral functional integro-differential equation:

$$\begin{aligned}
\frac{d}{d\varkappa}[\mathfrak{Q}(\varkappa)-\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)] &= \mathscr{H}\mathfrak{Q}(\varkappa)+\xi\left(\varkappa,\mathfrak{Q}_\varkappa,\int_0^\varkappa \varpi(\varkappa,\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z}\right),\\
&\quad \varkappa\in\mathscr{J}^*:=[0,T]\backslash\{\varkappa_1,\ldots,\varkappa_\mathscr{M}\}\\
\Delta\mathfrak{Q}(\varkappa) &= \mathfrak{Q}(\varkappa_\mathfrak{J}^+)-\mathfrak{Q}(\varkappa_\mathfrak{J}^-)=I_\mathfrak{J}(\mathfrak{Q}(\varkappa_\mathfrak{J}^-)),\\
&\quad \mathfrak{J}=1,2,3,\ldots,\mathscr{M}. \qquad (1)\\
\mathfrak{Q}(\varkappa) &= \Upsilon(\varkappa), \qquad \varkappa\in\mathfrak{B}:=(-\delta,0].
\end{aligned}$$

where $\mathscr{H}:\mathscr{D}(\mathscr{H})\subset\Sigma\to\Sigma$ is a closed linear operator and -$\mathscr{H}$ generates a strongly continuous semi-group $\mathfrak{W}(\varkappa)(\varkappa\geq 0)$ on Σ. The nonlinear function $\mathfrak{A}:\mathscr{J}^*\times\mathfrak{B}\to\Sigma$, $\xi:\mathscr{J}^*\times\mathfrak{B}\times\Sigma\to\Sigma$, $\varpi:\mathscr{J}^*\times\mathscr{J}^*\times\mathfrak{B}\to\Sigma$ are continuous, $\mathscr{J}^*:=[0,T]$, $0=\varkappa_0<\varkappa_1<\varkappa_2<\varkappa_3<\ldots.<\varkappa_\mathscr{M}<\varkappa_{\mathscr{M}+1}=T$, $I_\mathfrak{J}\in C(\Sigma,\Sigma)$ and $\mathfrak{B}=\{-\delta<\varkappa\leq 0:\delta>0\}$. Let $\mathfrak{Q}_\varkappa$ be defined by $\mathfrak{Q}_\varkappa(\mathfrak{z})=\mathfrak{Q}(\varkappa+\mathfrak{z}),\mathfrak{z}\leq 0$ and Υ be a initial condition. Dynamic processes in the natural and engineering sciences require an extensive comprehension of differential equations in order to be modelled. The differential equations are used in many disciplines to describe and understand the behavior of complex systems and phenomena. In this paper, we discuss a impulsive neutral functional integro-differential equation with an initial condition in a Banach space Σ.

Differential equations that exhibit instantaneous changes or impulses, at specific points in time are known as impulsive differential equations. These impulses can occur discontinuously, causing sudden jumps in the state variables of the system [4, 5]. Compared to other

* Corresponding author e-mail: n.sooppy@psau.edu.sa

conventional methods like fixed point procedures [6, 7], the topological degree method (TDM) makes it simpler to assess whether an equation has a solution or not. For traditional ways to solve differential equations and other similar problems, stronger criteria are needed. However, TDM does not mandate these kinds of standards. When utilizing the TDM, it is more beneficial for finding solutions under weaker rather than stronger conditions. Equations that incorporate components of differential and integral equations and have delays in both the differential and integral terms are known as neutral functional integro-differential equations. For more specific studies on neutral functional differential equations, readers are referred to these articles [8, 9]. We were motivated to accomplish this comprehensive study since the aforementioned prior research works did not address the present concept.

The following are primary significances of this proposed work:

- The proposed study utilizes a combination of mathematical tools such as closed linear operator, strongly continuous operator, and an iterative process are used derive the solution representation.
- The existence of a solution is achived through the application of specialized conditions, including $\mathfrak{S}$-Lipschitz condition, $\mathfrak{S}$- condensing mapping, $\mathfrak{S}$-contraction mapping. To attain uniqueness of the solution by using Gronwall's inequality.
- The novel aspect of this manuscript is the exploration of an iterative process for impulsive neutral functional integro-differential equation employed through the TDM and it is helpful to attain the solution under weaker conditions compared to other traditional methods such as fixed point method.
- Unlike prior studies, this research exhibit semi-group theory into the analysis of impulsive systems, significantly enhancing the effectiveness of the proposed methodology. The inclusion of semi-group theory addresses gaps in previous works, such as those referenced in [10, 11], by providing a deeper understanding of the evolution of solutions over time. This novel approach allows for a more comprehensive exploration of the dynamical behavior of the system.

The following are the five sections of this paper: Section 2, which provides the dynamical system's vital results, and lemmas. Section 3 focuses on establishing the necessary and sufficient conditions for given dynamical system. These conditions are derived using the TDM and Gronwall's inequality. Section 4 strengthens the applicability and validity of the proposed problem by presenting two numerical examples. In Section 5, we examine the conclusion of the manuscript.

2 Prefaces

In this section, we present basic key facts, and lemmas that have been chosen to ensure that the objective. Let Σ be a Banach space with $\|.\| = \sup_{t\in\mathscr{J}^*} |\mathfrak{Q}(\varkappa)|$.

Lemma 1. *[12] Let $\mathscr{H} : \mathscr{D}(\mathscr{H}) \subset \Sigma \to \Sigma$ be a closed linear operator and let -$\mathscr{H}$ generates a C_0-semigroup $\mathfrak{W}(\varkappa)$ $(\varkappa \geq 0)$ on Σ. Then $\exists \mathfrak{K} > 0$ and $\Lambda \in \mathbb{R} \ni$:*

$$\|\mathfrak{W}(\varkappa)\| \leq \mathfrak{K}e^{\Lambda\varkappa}.$$

Lemma 2. *[13, 14] The Kuratowski measure of noncompactness is the map $\mathfrak{S} : \mathbb{B} \to \mathbb{R}_+$ defined by*

$$\mathfrak{S}(B) = inf\{\varepsilon > 0 : B \subset \bigcup_{j=1}^{m} M_j \; and \; diam(M_j) \leq \varepsilon\},$$

Definition 1. *A function $\mathfrak{Q} \in C(\mathscr{J}^*, \Sigma)$ is said to be a mild solution of the problem (1) if it satisfies*

$$\mathfrak{Q}(\varkappa) = \begin{cases} \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0,\Upsilon(0))] + \mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa) \\ \quad - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \\ \quad + \int_0^\varkappa \mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\int_0^\mathfrak{z} \varpi(\mathfrak{z},m,\mathfrak{Q}_m)dm)d\mathfrak{z} \\ \quad + \mathfrak{W}(\varkappa-\varkappa_1)I_1(\mathfrak{Q}(\varkappa_1^-)), \varkappa \in [0,\varkappa_1]; \\ \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0,\Upsilon(0))] + \mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa) \\ \quad - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \\ \quad + \int_0^\varkappa \mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\int_0^\mathfrak{z} \varpi(\mathfrak{z},m,\mathfrak{Q}_m)dm)d\mathfrak{z} \\ \quad + \sum_{\mathfrak{Z}=1}^{2} \mathfrak{W}(\varkappa-\varkappa_\mathfrak{Z})I_\mathfrak{Z}(\mathfrak{Q}(\varkappa_\mathfrak{Z}^-)), \varkappa \in (\varkappa_1,\varkappa_2]; \\ . \\ . \\ . \\ \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0,\Upsilon(0))] + \mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa) \\ \quad - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \\ \quad + \int_0^\varkappa \mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\int_0^\mathfrak{z} \varpi(\mathfrak{z},m,\mathfrak{Q}_m)dm)d\mathfrak{z} \\ \quad + \sum_{\mathfrak{Z}=1}^{\mathscr{M}} \mathfrak{W}(\varkappa-\varkappa_\mathfrak{Z})I_\mathfrak{Z}(\mathfrak{Q}(\varkappa_\mathfrak{Z}^-)), \varkappa \in (\varkappa_\mathscr{M},T]; \end{cases} \quad (2)$$

3 Main Results

The main objective of this section is to examine the existence and uniqueness results of equation (1). These ideas are based on the following presumptions.

$\mathscr{N}(1)$.There are constants $\mathscr{C}_\xi, \mathscr{B}_\mathfrak{Z} \in [0,1) \ni$:
(i)$|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa^*) - \mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa^{**})| \leq \mathscr{C}_\xi|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}|$, $\forall \mathfrak{Q}^*, \mathfrak{Q}^{**} \in C(\mathscr{J}^*,\Sigma)$.
(ii)$|I_\mathfrak{Z}(\mathfrak{Q}^*(\varkappa_\mathfrak{Z}^-)) - I_\mathfrak{Z}(\mathfrak{Q}^*(\varkappa_\mathfrak{Z}^-))\mathfrak{J}(\mathfrak{Q}^{**}(\varkappa_\mathfrak{Z}^-))| \leq \mathscr{B}_\mathfrak{Z}|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}|$, $\forall \mathfrak{Q}^*, \mathfrak{Q}^{**} \in C(\mathscr{J}^*,\Sigma)$.
$\mathscr{N}(2)$.There are constants $\mathscr{B}_1 > 0, \mathscr{B}_2 > 0, \mathscr{L}_\xi > 0 \ni$:
(i)$|\xi(\varkappa,\mathfrak{Q}_\varkappa^*,\Lambda_\varkappa^*) - \xi(\varkappa,\mathfrak{Q}_\varkappa^{**},\Lambda_\varkappa^{**})| \leq \mathscr{B}_1|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}| + \mathscr{B}_2|\Lambda_\varkappa^* - \Lambda_\varkappa^{**}|$.
(ii)$|\Lambda_\varkappa^* - \Lambda_\varkappa^{**}| \leq \mathscr{L}_\xi|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}|$.

Theorem 3.1. The existence of a solution for problem (1) is equivalent to the existence of a fixed point for operator γ^*.

Proof. Let $\alpha^* : C(\mathscr{J}^*, \Sigma) \to C(\mathscr{J}^*, \Sigma)$ is determined in the following manner:

$$(\alpha^*\mathfrak{Q})(\varkappa) = \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0, \Upsilon(0))] + \mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa), \forall \varkappa \in \mathscr{J}^*.$$

The operator $\beta^* : C(\mathscr{J}^*, \Sigma) \to C(\mathscr{J}^*, \Sigma)$ is specified in the following manner:

$$\begin{aligned}(\beta^*\mathfrak{Q})(\varkappa) = & \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z})\xi(s, \mathfrak{Q}_\mathfrak{z}, \int_0^\mathfrak{z} \varpi(\mathfrak{z}, m, \mathfrak{Q}_m)dm)d\mathfrak{z} \\ & - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \\ & + \sum_{\mathfrak{J}=1}^{\mathscr{M}} \mathfrak{W}(\varkappa - \varkappa_\mathfrak{J})I_\mathfrak{J}(\mathfrak{Q}(\varkappa_\mathfrak{J}^-)), \ \forall \varkappa \in \mathscr{J}^*.\end{aligned}$$

$$\begin{aligned} Then,\ (\gamma^*\mathfrak{Q})(\varkappa) &= (\alpha^*\mathfrak{Q})(\varkappa) + (\beta^*\mathfrak{Q})(\varkappa). \\ &= \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0, \Upsilon(0))] + \mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa) \\ & \quad - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z})d\mathfrak{z}. \\ & \quad + \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z}) \\ & \quad \times \xi(s, \mathfrak{Q}_\mathfrak{z}, \int_0^\mathfrak{z} \varpi(\mathfrak{z}, m, \mathfrak{Q}_m)dm)d\mathfrak{z} \\ & \quad + \sum_{\mathfrak{J}=1}^{\mathscr{M}} \mathfrak{W}(\varkappa - \varkappa_\mathfrak{J})I_\mathfrak{J}(\mathfrak{Q}(\varkappa_\mathfrak{J}^-)), \varkappa \in \mathscr{J}^*. \\ (\gamma^*\mathfrak{Q})(\varkappa) &= \mathfrak{Q}(\varkappa).\end{aligned}$$

Theorem 3.2. The operator α^* is $\mathfrak{S}$-Lipschitz with $\mathscr{C}_\xi$.

Proof. Let $\alpha^* : C(\mathscr{J}^*, \Sigma) \to C(\mathscr{J}^*, \Sigma)$ is specified as follows:

$$(\alpha^*\mathfrak{Q})(\varkappa) = \mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0, \Upsilon(0))] + \mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa).$$

Then for any $\mathfrak{Q}^*(\varkappa), \mathfrak{Q}^{**}(\varkappa) \in \Sigma$, we have

$$\begin{aligned}\|(\alpha^*\mathfrak{Q}^*)(\varkappa) - (\alpha^*\mathfrak{Q}^{**})(\varkappa)\| = & \sup_{\varkappa \in \mathscr{J}^*} |\mathfrak{W}(\varkappa)[\Upsilon(0) \\ & - \mathfrak{A}(0, \Upsilon(0))] + \mathfrak{A}(\varkappa, \mathfrak{Q}^*_\varkappa) - \mathfrak{W}(\varkappa)[\Upsilon(0) \\ & - \mathfrak{A}(0, \Upsilon(0))] + \mathfrak{A}(\varkappa, \mathfrak{Q}^{**}_\varkappa)\big| \le \mathscr{C}_\xi |\mathfrak{Q}^*_\varkappa - \mathfrak{Q}^{**}_\varkappa|, \\ \therefore \|(\alpha^*\mathfrak{Q}^*)(\varkappa) - (\alpha^*\mathfrak{Q}^{**})(\varkappa)\| \le & \ \mathscr{C}_\xi |\mathfrak{Q}^*_\varkappa - \mathfrak{Q}^{**}_\varkappa|.\end{aligned}$$

By utilizing the proposition (1.20) in [14] we obtained α^* is $\mathfrak{S}$-Lipschitz with $\mathscr{C}_\xi$.

Theorem 3.3. The operator $\beta^* : C(\mathscr{J}^*, \Sigma) \to C(\mathscr{J}^*, \Sigma)$ is continuous. Furthermore, β^* fulfills the following prerequisite:

$$\|(\beta^*\mathfrak{Q})(\varkappa)\| \le \tau_\xi, \ \forall\, t \in \mathscr{J}^* \ and \ \Xi \in C(\mathscr{J}^*, \Sigma). \quad (3)$$

For brevity let us take

$$\begin{aligned}\Lambda_\mathfrak{z} &= \int_0^\mathfrak{z} \varpi(\mathfrak{z}, m, \mathfrak{Q}_m)dm. \\ \Lambda_\mathfrak{z}^v &= \int_0^\mathfrak{z} \varpi(\mathfrak{z}, m, \mathfrak{Q}_m^v)dm. \\ \tau_\xi &= \sup_{\varkappa \in \mathscr{J}^*} \Bigg| \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z})\xi(s, \mathfrak{Q}_\mathfrak{z}, \int_0^\mathfrak{z} \varpi(\mathfrak{z}, m, \mathfrak{Q}_m)dm)d\mathfrak{z} \\ & \quad - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \Bigg|.\end{aligned}$$

Proof. The operator β^* can be formulated as follows:

$$\begin{aligned}(\beta^*\mathfrak{Q})(\varkappa) = & \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z})\xi(s, \mathfrak{Q}_\mathfrak{z}, \Lambda_\mathfrak{z})d\mathfrak{z} \\ & - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \\ & + \sum_{\mathfrak{J}=1}^{\mathscr{M}} \mathfrak{W}(\varkappa - \varkappa_\mathfrak{J})I_\mathfrak{J}(\mathfrak{Q}(\varkappa_\mathfrak{J}^-)), \\ \|(\beta^*\mathfrak{Q}^v)(\varkappa) - (\beta^*\mathfrak{Q})(\varkappa)\| = & \sup_{\varkappa \in \mathscr{J}^*} \Bigg\{ \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z}) \\ & \times \xi(s, \mathfrak{Q}_\mathfrak{z}^v, \Lambda_\mathfrak{z}^v)d\mathfrak{z} - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z}^v)d\mathfrak{z} \\ & - \Bigg\{ \int_0^\varkappa \mathfrak{W}(\varkappa - \mathfrak{z})\xi(s, \mathfrak{Q}_\mathfrak{z}, \Lambda_\mathfrak{z})d\mathfrak{z} \\ & - \int_0^\varkappa \mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\mathfrak{A}(\mathfrak{z}, \mathfrak{Q}_\mathfrak{z})d\mathfrak{z} \Bigg\} \\ & + \sum_{\mathfrak{J}=1}^{\mathscr{M}} \mathfrak{W}(\varkappa - \varkappa_\mathfrak{J}) \left[I_\mathfrak{J}(\mathfrak{Q}^v(\varkappa_\mathfrak{J}^-)) - I_\mathfrak{J}(\mathfrak{Q}(\varkappa_\mathfrak{J}^-)) \right] \Bigg\} \\ \le & \int_0^\varkappa \|\mathfrak{W}(\varkappa - \mathfrak{z})\| \times (\mathscr{B}_1 + \mathscr{B}_2\mathscr{L}_\xi)\|\mathfrak{Q}_\mathfrak{z}^v - \mathfrak{Q}_\mathfrak{z}\|ds \\ & - \int_0^\varkappa \|\mathscr{H}\mathfrak{W}(\varkappa - \mathfrak{z})\| \times \mathscr{C}_\xi \|\mathfrak{Q}_\mathfrak{z}^v - \mathfrak{Q}_\mathfrak{z}\|d\mathfrak{z} \\ & + \sum_{\mathfrak{J}=1}^{\mathscr{M}} \|\mathfrak{W}(\varkappa - \varkappa_\mathfrak{J})\| \times \mathfrak{B}_\mathfrak{J} \|\mathfrak{Q}_\mathfrak{z}^v - \mathfrak{Q}_\mathfrak{z}\|.\end{aligned}$$

Since $\mathfrak{Q}_\mathfrak{z}^v \to \mathfrak{Q}_\mathfrak{z}$ as $v \to \infty$, then $\|(\beta^*\mathfrak{Q}^v)(\varkappa) - (\beta^*\mathfrak{Q})(\varkappa)\| \to 0, \forall t \in \mathscr{J}^*$.

Theorem 3.4. The operator β^* is compact. Consequently, β^* is $\mathfrak{S}-$Lipschitz with zero constant.

Proof. Let us consider the two arbitrary elements $\tau_1, \tau_2 \in \mathscr{J}^*$ and relation between τ_1, τ_2 is $\tau_1 < \tau_2$ then we explore that the operator $\beta^* : C(\mathscr{J}^*, \Sigma) \to C(\mathscr{J}^*, \Sigma)$ is equicontinuous on $\mathscr{J}^*$,

$$\|(\beta^*\mathfrak{Q})(\tau_2)-(\beta^*\mathfrak{Q})(\tau_1)\| = \sup_{\varkappa\in\mathscr{J}^*}\left\{\int_0^{\tau_2}\mathfrak{W}(\tau_2-s)\times\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})d\mathfrak{z}-\int_0^{\tau_2}\mathscr{H}\mathfrak{W}(\tau_2-s)\times\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z}+\sum_{\mathfrak{Z}=1}^{\mathscr{M}}\mathfrak{W}(\tau_2-\varkappa_\mathfrak{Z})I_\mathfrak{Z}(\mathfrak{Q}(\varkappa_\mathfrak{Z}^-)) - \left\{\int_0^{\tau_1}\mathfrak{W}(\tau_1-s)\times\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})d\mathfrak{z} - \int_0^{\tau_1}\mathscr{H}\mathfrak{W}(\tau_1-s)\times\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z} + \sum_{\mathfrak{Z}=1}^{\mathscr{M}}\mathfrak{W}(\tau_1-\varkappa_\mathfrak{Z})\times I_\mathfrak{Z}(\mathfrak{Q}(\varkappa_\mathfrak{Z}^-))\right\}\right\}$$
$$= \left\{\int_0^{\tau_2}\|\mathfrak{W}(\tau_2-s)\|ds-\int_0^{\tau_1}\|\mathfrak{W}(\tau_1-s)\|ds\right\}\times\|\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})\|-\left\{\int_0^{\tau_2}\|\mathscr{H}\mathfrak{W}(\tau_2-s)\|ds-\int_0^{\tau_1}\|\mathscr{H}\mathfrak{W}(\tau_1-s)\|ds\right\}\times\|\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})\| + \sum_{\mathfrak{Z}=1}^{\mathscr{M}}\|\mathfrak{W}(\tau_2-\varkappa_\mathfrak{Z})-\mathfrak{W}(\tau_1-\varkappa_\mathfrak{Z})\|\times\|I_\mathfrak{Z}(\mathfrak{Q}(\varkappa_\mathfrak{Z}^-))\|.$$

As $\tau_2\to\tau_1$, we have $\|(\beta^*\mathfrak{Q})(\tau_2)-(\beta^*\mathfrak{Q})(\tau_1)\|\to 0$ then by implementing proposition (1.19) in [14], we get the operator is $\mathfrak{S}-$Lipschitz with zero constant.

Theorem 3.5. The system (1) has a solution on $C(\mathscr{J}^*,\Sigma)$ and the set of solution is bounded.

Proof. From Theorem 1.2 in [14], we want to exhibit that S is bounded for every $\varkappa\in(0,\varkappa_1]$

$$\begin{aligned}\|\mathfrak{Q}(\varkappa)\| &= \|\lambda\gamma^*\mathfrak{Q}(\varkappa)\|\\ &\le \{\lambda\,\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]+\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa) - \int_0^{\varkappa}\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z} + \int_0^{\varkappa}\mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})d\mathfrak{z} + \mathfrak{W}(\varkappa-\varkappa_1)I_1(\mathfrak{Q}(\varkappa_1^-))\|\},\\ \|\mathfrak{Q}(\varkappa)\| &\le \lambda\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]\|+\|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)\| - \int_0^{\varkappa}\|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z}\| + \int_0^{\varkappa}\|\mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})d\mathfrak{z}\| + \|\mathfrak{W}(\varkappa-\varkappa_1)I_1(\mathfrak{Q}(\varkappa_1^-))\|.\end{aligned}$$

By using equation (3), we get,

$$\|\mathfrak{Q}(\varkappa)\| \le \lambda\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]\| + \|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)\|+\tau_1. \tag{4}$$

Similarly, if $\varkappa\in(\varkappa_1,\varkappa_2]$,

$$\|\mathfrak{Q}(\varkappa)\| \le \lambda\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]\|+\|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)\| - \int_0^{\varkappa}\|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z})d\mathfrak{z}\| + \int_0^{\varkappa}\|\mathfrak{W}(\varkappa-\mathfrak{z})\xi(s,\mathfrak{Q}_\mathfrak{z},\Lambda_\mathfrak{z})d\mathfrak{z}\| + \|\mathfrak{W}(\varkappa-\varkappa_1)I_1(\mathfrak{Q}(\varkappa_1^-))\| + \|\mathfrak{W}(\varkappa-\varkappa_2)I_2(\mathfrak{Q}(\varkappa_2^-))\|.$$

Again utilizing equation (3), we obtain the subsequent inequality:

$$\|\mathfrak{Q}(\varkappa)\| \le \lambda\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]\| + \|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)\|+\tau_2. \tag{5}$$

In general, if $t\in(\varkappa_\mathscr{M},T]$,

$$\|\mathfrak{Q}(\varkappa)\| \le \lambda\|\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]\| + \|\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa)\|+\tau_\xi. \tag{6}$$

From the inequality (6), the solution is bounded and has a solution on $C(\mathscr{J}^*,\Sigma)$.

Theorem 3.6. The dynamical system (1) has only one solution if the hypotheses $\mathscr{N}(1)$ and $\mathscr{N}(2)$ are true.
Proof. Let $\mathfrak{Q}^*(\varkappa)$ and $\mathfrak{Q}^{**}(\varkappa)$ are defined as follows:

$$\mathfrak{Q}^*(\varkappa)=\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]+\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa^*) - \int_0^{\varkappa}\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\times\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z}^*)d\mathfrak{z}+\int_0^{\varkappa}\mathfrak{W}(\varkappa-\mathfrak{z})\times\xi(\mathfrak{z},\mathfrak{Q}_\mathfrak{z}^*,\Lambda_\mathfrak{z}^*)d\mathfrak{z}+\mathfrak{W}(\varkappa-\varkappa_1)\times I_1(\mathfrak{Q}^*(\varkappa_1^-)),\forall t\in(0,\varkappa_1].$$
$$\mathfrak{Q}^{**}(\varkappa)=\mathfrak{W}(\varkappa)[\Upsilon(0)-\mathfrak{A}(0,\Upsilon(0))]+\mathfrak{A}(\varkappa,\mathfrak{Q}_\varkappa^{**}) - \int_0^{\varkappa}\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\times\mathfrak{A}(\mathfrak{z},\mathfrak{Q}_\mathfrak{z}^{**})d\mathfrak{z}+\int_0^{\varkappa}\mathfrak{W}(\varkappa-\mathfrak{z})\times\xi(\mathfrak{z},\mathfrak{Q}_\mathfrak{z}^{**},\Lambda_\mathfrak{z}^{**})d\mathfrak{z}+\mathfrak{W}(\varkappa-\varkappa_1)\times I_1(\mathfrak{Q}^{**}(\varkappa_1^-)),\forall t\in(0,\varkappa_1].$$
$$Consider\ \|\mathfrak{Q}^*(\varkappa)-\mathfrak{Q}^{**}(\varkappa)\|\le\mathscr{C}_\xi\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\| + \mathscr{C}_\xi\int_0^{\varkappa}\|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\|\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_\mathfrak{z}^{**}\|d\mathfrak{z} + \int_0^{\varkappa}\|\mathfrak{W}(\varkappa-\mathfrak{z})\|\times(\mathscr{B}_1+\mathscr{B}_2\mathscr{L}_\xi)\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_\mathfrak{z}^{**}\|d\mathfrak{z}+\|\mathfrak{W}(\varkappa-\varkappa_1)\|\times\mathfrak{B}_1\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\|.$$

Similarly, if $\varkappa\in(\varkappa_\mathscr{M},T]$ then

$$\|\mathfrak{Q}^*(\varkappa)-\mathfrak{Q}^{**}(\varkappa)\|\le\mathscr{C}_\xi\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\| + \mathscr{C}_\xi\int_0^{\varkappa}\|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\|\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_s^{**}\|d\mathfrak{z}+\int_0^{\varkappa}\|\mathfrak{W}(\varkappa-\mathfrak{z})\|\times(\mathscr{B}_1+\mathscr{B}_2\mathscr{L}_\xi)\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_\mathfrak{z}^{**}\|d\mathfrak{z}+\sum_{\mathfrak{Z}=1}^{\mathscr{M}}\|\mathfrak{W}(\varkappa-\varkappa_\mathfrak{Z})\|\times\mathfrak{B}_\mathfrak{Z}\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\|.$$

Applying Gronwall's inequality yields,

$$\|\mathfrak{Q}^*(\varkappa)-\mathfrak{Q}^{**}(\varkappa)\|\le\mathscr{V}\times exp\left\{\mathscr{C}_\xi\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\| + \mathscr{C}_\xi\int_0^{\varkappa}\|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})\|\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_s^{**}\|d\mathfrak{z} + \int_0^{\varkappa}\|\mathfrak{W}(\varkappa-\mathfrak{z})\|\times(\mathscr{B}_1+\mathscr{B}_2\mathscr{L}_\xi)\times\|\mathfrak{Q}_\mathfrak{z}^*-\mathfrak{Q}_\mathfrak{z}^{**}\|d\mathfrak{z} + \sum_{\mathfrak{Z}=1}^{\mathscr{M}}\|\mathfrak{W}(\varkappa-\varkappa_\mathfrak{Z})\|\times\mathfrak{B}_\mathfrak{Z}\|\mathfrak{Q}_\varkappa^*-\mathfrak{Q}_\varkappa^{**}\|\right\}.$$

Let $\mathscr{V} \to 0 \implies \mathfrak{Q}^*(\varkappa) = \mathfrak{Q}^{**}(\varkappa)$ for every $\varkappa \in (\varkappa_{\mathscr{M}}, T]$.

4 Numerical Examples

Example 4.1. Consider the following form:

$$\frac{d}{d\varkappa}\left(\mathfrak{Q}(\varkappa) - \frac{1}{7}sin\left(\mathfrak{Q}\left(2\pi+\varkappa\right)\right)\right) = \mathscr{H}\mathfrak{Q}(\varkappa) + \frac{1}{7}\int_0^5 cosec\left(\mathfrak{Q}\left(\frac{\pi}{2}+\mathfrak{z}\right)\right)d\mathfrak{z},$$
$$\varkappa \in [0,5]\backslash\{1,2,3,4\},$$
$$\Delta\mathfrak{Q}(\varkappa) = \frac{1}{81}sin\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-), \mathfrak{J} = 1,2,3,4. \quad (7)$$
$$\mathfrak{Q}(0) = \Upsilon(0).$$

Table 1: Illustration of assumptions and interpretations for Example 4.1.

SI.No	Symbol	Interpretations	Assumptions
1.	$\mathscr{H}$	Closed linear operator	$D(\mathscr{H}) = \{\varepsilon \in \Sigma : \varepsilon(0) = \varepsilon(5) = 0\}$
2.	$\xi(\varkappa, \mathfrak{Q}_\varkappa, \Lambda_\varkappa)$	Integro-Differential function	$\frac{1}{7}\int_0^5 cosec\left(\Xi\left(\frac{\pi}{2}+\mathfrak{z}\right)\right)d\mathfrak{z}$
3.	$\Delta\mathfrak{Q}(\varkappa)$	Impulsive function	$\frac{1}{81}sin\Xi(\varkappa_{\mathfrak{J}}^-)$
4.	$\mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa)$	Neutral term	$\frac{1}{7}sin\left(\Xi\left(2\pi+\varkappa\right)\right)$

Table 1 represents the symbols and presumption of equation (7). Let $\lambda = \frac{3}{7}$, and we assume that the function $\xi : \mathscr{J}^* \times \mathfrak{B} \times \Sigma \to \Sigma$ be defined as follows:

$$\|\xi(\varkappa, \mathfrak{Q}_\varkappa, \Lambda_\varkappa)\| \leq \frac{1}{7}\left\|\int_0^5 cosec\left(\mathfrak{Q}\left(\frac{\pi}{2}+\mathfrak{z}\right)\right)d\mathfrak{z}\right\|.$$

The impulsive function $I_{\mathfrak{J}} : \Sigma \to \Sigma$ is defined by $\|I_{\mathfrak{J}}(\mathfrak{Q}(\varkappa))\| = \frac{1}{81}\|sin\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-)\|$. The norm $\|\mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa)\| \leq \frac{1}{7}\|sin\left(\mathfrak{Q}\left(2\pi+\varkappa\right)\right)\|$. By using the Theorem 3 we have,

$$\begin{aligned}\|\mathfrak{Q}(\varkappa)\| \leq & \frac{3}{7}\|\mathfrak{W}(\varkappa)[\Upsilon(0) - \mathfrak{A}(0, \Upsilon(0))]\| \\ & + \frac{1}{7}\|sin\left(\mathfrak{Q}\left(2\pi+\varkappa\right)\right)\| \\ & - \frac{1}{7}\int_0^5 \|\mathscr{H}\mathfrak{W}(\varkappa-\mathfrak{z})sin\left(\Xi\left(2\pi+\varkappa\right)\right)\|d\mathfrak{z} \\ & + \frac{1}{7}\int_0^5 \left\|\mathfrak{W}(\varkappa-\mathfrak{z})cosec\left(\mathfrak{Q}\left(\frac{\pi}{2}+\mathfrak{z}\right)\right)\right\|d\mathfrak{z} \\ & + \frac{1}{81}\sum_{\mathfrak{J}=1}^{5}\|\mathfrak{W}(\varkappa-\varkappa_{\mathfrak{J}})sin(\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-))\|.\end{aligned}$$

From above inequality, our presumptions are satisfied with the problem (4.1).

Example 4.2. Consider the following form of a system:

$$\frac{d}{d\varkappa}\left(\mathfrak{Q}(\varkappa) - \frac{4}{7}sec\left(\mathfrak{Q}\left(2\pi+\varkappa\right)\right)\right) = \mathscr{H}\mathfrak{Q}(\varkappa) + \frac{\pi}{7}\int_0^3 cos\left(\mathfrak{Q}\left(\frac{\pi}{9}+\mathfrak{z}\right)\right)d\mathfrak{z},\ t \in [0,3]\backslash\{1,2\},$$
$$\Delta\mathfrak{Q}(\varkappa) = \frac{8}{9}tan\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-), \mathfrak{J} = 1,2. \quad (8)$$
$$\mathfrak{Q}(0) = \Upsilon(0).$$

Table 2: Illustration of assumptions and interpretations for Example 4.2.

SI.No	Symbol	Interpretations	Assumptions
1.	$\mathscr{H}$	Closed linear operator	$D(\mathscr{H}) = \{\varepsilon \in \Sigma : \varepsilon(0) = \varepsilon(3) = 0\}$
2.	$\xi(\varkappa, \mathfrak{Q}_\varkappa, \Lambda_\varkappa)$	Integro-Differential function	$\frac{\pi}{7}\int_0^3 cos\left(\mathfrak{Q}\left(\frac{\pi}{9}+\mathfrak{z}\right)\right)d\mathfrak{z}$
3.	$\Delta\mathfrak{Q}(\varkappa)$	Impulsive function	$\frac{8}{9}tan\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-)$
4.	$\mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa)$	Neutral term	$\frac{4}{7}sec\left(\mathfrak{Q}\left(2\pi+\varkappa\right)\right)$

Table 2 represents the symbols and presumption of equation (8). Let $\mathscr{C}_\xi = \frac{2}{5}, \mathfrak{B}_{\mathfrak{J}} = \frac{9}{8}, \mathscr{B}_1 = \frac{6}{7}, \mathscr{B}_2 = \frac{2}{3}$ and $\mathscr{L}_\xi = \frac{5}{6}$. The function $\xi : \mathscr{J}^* \times \mathfrak{B} \times \Sigma \to \Sigma$ is defined as follows:

$$\begin{aligned}\|\mathfrak{Q}(\varkappa, \mathfrak{Q}_\varkappa^*, \Lambda_\varkappa^*) & - \mathfrak{Q}(\varkappa, \mathfrak{Q}_\varkappa^{**}, \Lambda_\varkappa^{**})\| \leq \frac{\pi}{7}\left\|\int_0^3 cos\left(\mathfrak{Q}^*\left(\frac{\pi}{9}+\mathfrak{z}\right)\right)d\mathfrak{z}\right. \\ & \left. - \int_0^3 cos\left(\mathfrak{Q}^{**}\left(\frac{\pi}{9}+\mathfrak{z}\right)\right)d\mathfrak{z}\right\|, \\ & \leq 0.4487\|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}\|.\end{aligned}$$

Let $\|\mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa)\| \leq \frac{4}{7}\|sec\left(\mathfrak{Q}\left(2\pi+t\right)\right)\|$ and manipulating the hypothesis $\mathscr{N}(1)(i)$, we have

$$\begin{aligned}\|\mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa^*) - \mathfrak{A}(\varkappa, \mathfrak{Q}_\varkappa^{**})\| \leq & \frac{4}{7}\|sec\left(\mathfrak{Q}^*\left(2\pi+\varkappa\right)\right) \\ & - sec\left(\mathfrak{Q}^{**}\left(2\pi+\varkappa\right)\right)\|, \\ \leq & 0.5714\|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}\|.\end{aligned}$$

The impulsive function $I_{\mathfrak{J}} : \Sigma \to \Sigma$ is defined by $\|I_{\mathfrak{J}}(\mathfrak{Q}(\varkappa))\| = \frac{8}{9}\|tan\mathfrak{Q}(\varkappa_{\mathfrak{J}}^-)\|$ and satisfies the hypothesis $\mathscr{N}(1)(ii)$, we have

$$\begin{aligned}\|I_{\mathfrak{J}}(\mathfrak{Q}^*(\varkappa)) - \|I_{\mathfrak{J}}(\mathfrak{Q}^{**}(\varkappa))\|\| \leq & \frac{8}{9}\|tan\mathfrak{Q}^*(\varkappa_{\mathfrak{J}}^-) \\ & - tan\mathfrak{Q}^{**}(\varkappa_{\mathfrak{J}}^-)\|, \\ \leq & 0.8888\|\mathfrak{Q}_\varkappa^* - \mathfrak{Q}_\varkappa^{**}\|.\end{aligned}$$

Utilizing Theorem 3, we can determine unique solution to problem (8).

5 Conclusion

In this study, we have investigated the iterative process for solving given dynamical system using TDM. This approach has proven to be a powerful tool, enabling us to establish both the existence and uniqueness of solutions under weaker conditions compared to traditional methods. Our analysis has provided valuable insights into the behavior of solutions to these equations, as well as contributing to the development of efficient numerical methods for their solution. Our work contributes to advancing the understanding and numerical treatment of given system, addressing challenging problems in applied mathematics and engineering.

Conflict of interest

Authors has no conflict of interest.

Acknowledgments

The authors extend their appreciation to Northern Border University, Saudi Arabia, for supporting this work through project number (NBU-CRP-2025-3021). This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445). The authors are thankful to the Deanship of Graduate Studies and Scientific Research at University of Bisha for supporting this work through the Fast-Track Research Support Program.

References

[1] K. Muthuselvan, B. Sundaravadivoo, S. Alsaeed, K. S. Nisar. New interpretation of topological degree method of Hilfer fractional neutral functional integro-differential equation with nonlocal condition[J], *AIMS Mathematics*, **8**(7) (2023) 17154-17170.

[2] M. Shqair, M. Al-Smadi, S. Momani, E. El-Zahar, Adaptation of Conformable Residual Power Series Scheme in Solving Nonlinear Fractional Quantum Mechanics Problems. *Applied Sciences*, **10**(3) (2020), 890.

[3] L. Zada, M. Al-Hamami, R. Nawaz, S. Jehanzeb, A. Morsy, A. Abdel-Aty and K. S. Nisar, A New Approach for Solving Fredholm Integro-Differential Equations, *Information Sciences Letters*, **10**(3) (2021), 407-415.

[4] Z. Li, X. Shu, F. Xu , The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem, *AIMS Mathematics*, **5**(6) (2020), 6189-210.

[5] K. Muthuselvan, B. Sundaravadivoo, Analyze existence, uniqueness and controllability of impulsive fractional functional differential equations, *Advanced Studies: Euro-Tbilisi Mathematical Journal*, **10**, (2022), 171-190.

[6] K.S. Nisar, K. Munusamy, C. Ravichandran, Results on existence of solutions in nonlocal partial functional integrodifferential equations with finite delay in nondense domain, *Alexandria Engineering Journal*, **73** (2023), 377-384.

[7] L. Wang, X.B. Shu, Y. Cheng, R. Cui, Existence of periodic solutions of second-order nonlinear random impulsive differential equations via topological degree theory, *Results in Applied Mathematics*, **12** (2021), 100215.

[8] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Computers & Mathematics with Applications*, **59** (3) (2010), 1095-1100.

[9] M. Benchohra, S. K. Ntouyas, Nonlocal Cauchy problems for neutral functional differential and integro-differential inclusions in Banach spaces, *Journal of Mathematical Analysis and Applications*, **258** (2001), 573–590.

[10] E. Abada, H. Lakhal, M. Maouni, Topological degree method for fractional Laplacian system, *Bulletin of Mathematical Analysis and Applications*, **13**(2) (2021) 10-19.

[11] C. Allalou, K. Hilal, Weak solution to $p(x)$-Kirchoff type problems under no-flux boundary condition by Topological Degree, *Boletim da Sociedade Paranaense de Matemática*, **41**, (2023), 1-12.

[12] K.D. Kucche, Existence and controllability results for mixed function integro-differential equation with infinite delay, *Demonstratio Mathematica*, **47**(4) (2014).

[13] K. Muthuselvan, B. Sundaravadivoo, K.S. Nisar and S. Alsaeed, Discussion on iterative process of nonlocal controllability exploration for Hilfer neutral impulsive fractional integro-differential equation, *AIMS Mathematics*, **8** (7), (2023), 16846–16863.

[14] Y. Zhou, J. Wang, L. Zhang, Basic Theory of Fractional Differential Equation, *World Scientific*, (2016).