Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/120302

An Observation on The Double Laplace-ARA Transform and Integral- Partial Differential Equations

Abdelilah Kamal H. Sedeeg[1,2,*]

[1] Department of Mathematics, Faculty of Education, Holy Quran and Islamic Sciences University, Sudan
[2] Merowe Institute of Technology, Merowe University of Technology- Abdulatif Alhamad, Merowe, Sudan

Received: 7 May 2025, Revised: 21 Jun. 2025, Accepted: 23 Aug. 2025
Published online: 1 Sep. 2025

Abstract: The major objective of this paper is to propose a novel double Laplace – ARA transform (DL-ARAT). This transform is employed to establish the convolution theorem, existence conditions, and other relevant theorems results, including derivative properties. Subsequently, the DL-ARAT is applied to solve a range of illustrative integro-differential equations. This approach was shown to be a powerful and efficient means to tack integro - differential equations.

Keywords: ARA transform; Laplace transform; Double Laplace - ARA transform; Volterra -integral equation; Volterra - integro differential equation and integro - Partial differential equations

1 Introduction

Integral transform methods are regarded among the basic and most widely used approach in solving partial differential equations. For instance, partial differential equations (PDEs) are utilized to model many phenomena in disciplines related to Mathematics such as Physics, Engineering and other scientific fields. These models are written using a variety of partial differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Integral transforms can be equally utilized to solve both integral and integral-differential equations. Thus, this paper primarily seeks to solve the aforementioned types through double Laplace-ARA transform. The double integral transform provides us with an efficient means whereby integral-differential equations can be easily transformed into algebraic equation so as to obtain the exact solutions. This study is rooted in the advances made by Scholars who exerted great efforts to develop the above method and apply them to a wide range of Mathematical problems. Example, include carried out the double Laplace transform [13, 14, 15, 16], double Sumudu transform [16, 17, 18, 19, 20], the double Laplace–Sumudu transform [21, 22], the double Elzaki transform [23], the double Kamal transform [24], the double formable transform [25], the double Laplace-ARA [26], the double ARA transform [27], among other.

Among the previous method is double Laplace-ARA Transform which has been increasingly used recently to solve partial differential equations (PDEs), integral equation (IEs) and integral-partial differential equations (IPDEs).

The main objective of our current study is to demonstrate that the DL-ARAT transforms has advantages in handling oscillatory functions, its ability to simplify and handle initial and boundary conditions, and its effectiveness in reducing the complexity of solving partial integro-differential equations compared to other well-known double transforms, such as the double Laplace transform and the double Sumudu transform. As shown in [26], the duality of the ARA transform relative to the Laplace transform gives it an advantage that allows it to overcome the singularity at $t = 0$. Given these advantages, we decided to construct this new combination of the Laplace and ARA transforms, so that we can reap the benefits of these two powerful transforms. We have named this new approach the double Laplace-ARA transforms.

This article is organized as following: In Section 2, Fundamental Facts of new integral ARA and Laplace transforms. In Section 3, we introduce a new double integral transform, the DL-ARAT, that combines the Laplace transform and the ARA transform and present

* Corresponding author e-mail: aelilah63@hotmail.com

some properties of this transform. In Section 4, we apply the DL-ARAT to IPDEs. In Section 5, some examples are presented and solved with the DL-ARAT, In Section 6, the results and discussion. Lastly, In Section 7, the conclusion.

2 Fundamental Facts of Laplace and ARA Transforms of order one:

In this section, we introduce the basic properties of single Laplace and ARA transforms.

2.1 Fundamental Facts of single Laplace transform: [28]

Definition 1. *Let $f(x)$ be a function of x specified for $x > 0$. Then Laplace transform of $f(x)$, denoted by $L[f(x)]$, is defined by*

$$L[f(x)](v) = F(v) = \int_0^\infty e^{-vx} f(x)\, dx, \quad v > 0, \quad (1)$$

and the inverse of Laplace transform is given by

$$L^{-1}[F(v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{vx}F(v)\, dv = f(x), \quad x > 0. \quad (2)$$

Theorem 1(Existence conditions). *If $f(x)$ is piecewise continuous function on the interval $[0,\infty)$ and of exponential order ϑ. Then $L[f(x)]$ exists for $Re(v) > \vartheta$ and satisfies*

$$|f(x)| \le \mu e^{\vartheta x},$$

where μ is positive constant. Then, the Laplace transform converges absolutely for $Re(v) > \vartheta$.

Proof. Using the definition of Laplace transform, we get

$$|F(v)| = \left|\int_0^\infty e^{-vx}[f(x)]\,dx\right| \le \int_0^\infty e^{-vx}|f(x)|\,dx$$
$$\le \mu \int_0^\infty e^{-(v-\vartheta)x}\,dx = \frac{\mu}{v-\vartheta}, \quad \mathrm{Re}(v) > \vartheta.$$

Therefore, the Laplace transform converges absolutely for $\mathrm{Re}(v) > \vartheta$.

In the following table, we present some properties of single Laplace transform.

2.2 Fundamental Facts of single ARA Transform of order one: [28]

Definition 2. *The single ARA transform of order one of a continuous function $f(t)$ on a given interval $(0,\infty)$ is given by*

$$G[f(t)](s) = Q(s) = s\int_0^\infty e^{-st} f(t)\, dt, \quad s > 0, \quad (3)$$

Table 1: Laplace transform of frequently encountered functions

$f(x)$	$L[f(x)](v)$
1	$\frac{1}{v}$
x^m	$\frac{\Gamma(m+1)}{v^{m+1}}, \quad m \ge 0$
e^{ax}	$\frac{1}{v-a}, \quad a \in \mathbb{R}$
$\sin(ax)$	$\frac{a}{v^2+a^2}, \quad a \in \mathbb{R}$
$\sinh(ax)$	$\frac{a}{v^2-a^2}, \quad a \in \mathbb{R}$
$f'(x)$	$vF(v) - f(0)$
$f^{(n)}(x)$	$v^nF(v) - \sum_{k=1}^{n} v^{n-k} f^{(k-1)}(0), \quad n = 0,1,2,3,\dots$

and the inverse ARA transform is expressed as

$$G^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} Q(s)\, ds = f(t). \quad (4)$$

Theorem 2(Existence conditions). *If the function $f(t)$ is continuous in each domain $0 \le t \le \alpha$ and satisfying*

$$|f(t)| \le Me^{\alpha t},$$

where M is a positive constant, then, the ARA transform of order one converges absolutely for $Re(s) > \alpha$.

In the following table, we present some properties of single ARA transform of order one.

Table 2: ARA transform of order one of frequently encountered functions

$f(t)$	$G[f(t)](s)$
1	1
t^b	$\frac{\Gamma(b+1)}{s^b}, \quad b \ge 0$
e^{bt}	$\frac{s}{s-b}, \quad b \in \mathbb{R}$
$\sin(bt)$	$\frac{bs}{s^2+b^2}, \quad b \in \mathbb{R}$
$\sinh(bt)$	$\frac{a}{v^2-a^2}, \quad a \in \mathbb{R}$
$f'(t)$	$sQ(s) - sf(0)$
$f^{(n)}(t)$	$s^nQ(s) - \sum_{k=1}^{n} s^{n-k+1} f^{(k-1)}(0), \quad n = 0,1,2,3,\dots$

All the above results can be obtained from the definition of Laplace and ARA transforms and simple calculations.

3 Double Laplace-ARA transform of order one (DL-ARAT):

In this section, a new integral transform, DL-ARAT, that combines the Laplace transform and the ARA transform of order one is introduced. We present basic properties concerning the existence conditions, linearity and the inverse of this transform. Moreover, some essential properties and results are used to compute the DL-ARAT for some basic functions. We introduce the convolution theorem and the derivatives properties of the new transform. Recall that the ARA transform of order one of a piecewise continuous function $f(t)$ on $[0,\infty)$ is given as:

Let $u(x,t)$ be continuous function of two positive variables x and t. Then the DL-ARAT of $u(x,t)$ is defined as:

$$L_xG_t[u(x,t)] = Q(v,s) = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[u(x,t)]\,dx\,dt, \quad v,s>0. \tag{5}$$

Clearly, the DL-ARAT is a linear integral transformation as shown below:

$$\begin{aligned} L_xG_t[Au(x,t)+Bw(x,t)] &= \\ s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}&[Au(x,t)+Bw(x,t)]\,dx\,dt \\ &= As\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[u(x,t)]\,dx\,dt \\ &\quad + Bs\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[w(x,t)]\,dx\,dt \\ &= AL_xG_t[u(x,t)] + BL_xG_t[w(x,t)]. \end{aligned}$$

where A and B are constants.

And, the inverse of the DL-ARAT is given as:

$$L_x^{-1}[G_t^{-1}[Q(v,s)]] = \left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty} e^{vx}\,dv \left(\frac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s}Q(v,s)\,ds = u(x,t). \tag{6}$$

3.1 DL-ARAT of some basic functions:

–Let $u(x,t)=1$, $x>0$, $t>0$. Then:

$$L_xG_t[1] = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}\,dx\,dt = \int_0^{\infty} e^{-vx}\,dx\cdot s\int_0^{\infty} e^{-st}\,dt = 1, \quad \mathrm{Re}(s)>0.$$

–Let $u(x,t)=x^{\alpha}t^{\beta}$, $x>0$, $t>0$ and α,β are constants. Then:

$$L_xG_t[x^{\alpha}t^{\beta}] = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[x^{\alpha}t^{\beta}]\,dx\,dt = \int_0^{\infty} e^{-vx}[x^{\alpha}]\,dx\cdot s\int_0^{\infty} e^{-st}[t^{\beta}]\,dt = L_x[x^{\alpha}]\cdot G_t[t^{\beta}].$$

Thus,

$$L_xG_t[x^{\alpha}t^{\beta}] = \frac{\alpha!\beta!}{v^{\alpha+1}s^{\beta}}, \quad \mathrm{Re}(\alpha)>-1, \mathrm{Re}(s)>0.$$

–Let $u(x,t)=e^{\alpha x+\beta t}$, $x>0$, $t>0$ and α,β are constants. Then:

$$L_xG_t[e^{\alpha x+\beta t}] = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[e^{\alpha x+\beta t}]\,dx\,dt = \int_0^{\infty} e^{-vx}[e^{\alpha x}]\,dx\cdot s\int_0^{\infty} e^{-st}[e^{\beta t}]\,dt = L_x[e^{\alpha x}]\cdot G_t[e^{\beta t}].$$

Thus,

$$L_xG_t[e^{\alpha x+\beta t}] = \frac{s}{(v-\alpha)(s-\beta)}, \quad \{\mathrm{Re}(\alpha)+\mathrm{Re}(s)\}>0.$$

Similarly,

$$L_xG_t[e^{i(\alpha x+\beta t)}] = \frac{s}{(v-i\alpha)(s-i\beta)}.$$

Using the property of complex analysis, we have:

$$L_xG_t[e^{i(\alpha x+\beta t)}] = \frac{s(sv-\alpha\beta)+is(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

Using Euler's formulas:

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix}+e^{-ix}}{2},$$

and the formulas:

$$\sinh x = \frac{e^{x}-e^{-x}}{2}, \quad \cosh x = \frac{e^{x}+e^{-x}}{2}.$$

Therefore, we conclude the following:

$$\begin{aligned} L_xG_t[\sin(\alpha x+\beta t)] &= \frac{s(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}, \\ L_xG_t[\cos(\alpha x+\beta t)] &= \frac{s(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}, \\ L_xG_t[\sinh(\alpha x+\beta t)] &= \frac{s(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}, \\ L_xG_t[\cosh(\alpha x+\beta t)] &= \frac{s(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}. \end{aligned}$$

Where $\mathrm{Im}(\alpha) < \mathrm{Re}(s)$.

–$L_xG_t[J_0(\lambda\sqrt{xt})] = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[J_0(\lambda\sqrt{xt})]\,dx\,dt = \int_0^{\infty}[J_0(\lambda\sqrt{xt})]e^{-vx}\,dx\cdot s\int_0^{\infty} g(t)e^{-st}\,dt = \frac{4s}{4vs+\lambda^2}$.

–Let $u(x,t)=f(x)g(t)$, $x>0$, $t>0$. Then:

$$L_xG_t[f(x)g(t)] = s\int_0^{\infty}\int_0^{\infty} e^{-(vx+st)}[f(x)g(t)]\,dx\,dt = \int_0^{\infty} e^{-vx}[f(x)]\,dx\cdot s\int_0^{\infty} e^{-st}[g(t)]\,dt = L_x[f(x)]\cdot G_t[g(t)].$$

3.2 Existence conditions for DL-ARAT:

Let $u(x,t)$ be function of exponential order α and β as $x \to \infty$ and $t \to \infty$. If there exists a positive N such that $\forall x > X$ and $t > T$, we have:

$$|u(x,t)| \leq Ne^{\alpha x + \beta t}.$$

We can write $u(x,t) = O(e^{\alpha x + \beta t})$ as $x \to \infty$ and $t \to \infty$, $v > \alpha$ and $s > \beta$.

Theorem 3. *Let $u(x,t)$ be a continuous function on the region $[0,X) \times [0,T)$ of exponential order α and β. Then $L_xG_t[u(x,t)]$ exists for v and s provided $Re(v) > \alpha$ and $Re(s) > \beta$.*

Proof. Using the definition of DL-ARAT, we get:

$$\begin{aligned}
|Q(v,s)| &= \left| s\int_0^\infty \int_0^\infty e^{-(vx+st)}[u(x,t)]\,dx\,dt \right| \\
&\leq s\int_0^\infty \int_0^\infty e^{-(vx+st)}|u(x,t)|\,dx\,dt \\
&\leq N\int_0^\infty e^{-(v-\alpha)x}\,dx \cdot s\int_0^\infty e^{-(s-\beta)t}\,dt \\
&= \frac{Ns}{(v-\alpha)(s-\beta)}, \quad \text{Re}(v) > \alpha \text{ and } \text{Re}(s) > \beta.
\end{aligned}$$

3.3 Some important theorems of DL-ARAT:

Theorem 4 (Shifting Property). *Let $u(x,t)$ be a continuous function and $L_xG_t[u(x,t)] = Q(v,s)$. Then:*

$$L_xG_t[e^{\alpha x+\beta t}u(x,t)] = \frac{s}{s-\beta}Q(v-\alpha, s-\beta). \tag{7}$$

Proof.

$$\begin{aligned}
L_xG_t[e^{\alpha x+\beta t}u(x,t)] &= s\int_0^\infty \int_0^\infty e^{-(v-\alpha)x-(s-\beta)t}[u(x,t)]\,dx\,dt \\
&= \frac{s}{s-\beta}(s-\beta)\int_0^\infty \int_0^\infty e^{-(v-\alpha)x}e^{-(s-\beta)t}u(x,t)\,dx\,dt \\
&= \frac{s}{s-\beta}Q(v-\alpha, s-\beta).
\end{aligned}$$

Theorem 5 (Periodic Function). *Let $L_xG_t[u(x,t)]$ exists, where $u(x,t)$ periodic function of periods α and β such that:*

$$u(x+\alpha, t+\beta) = u(x,t), \quad \forall x, y.$$

Then:

$$L_xG_t[u(x,t)] = \frac{s\int_0^\alpha \int_0^\beta e^{-(vx+st)}u(x,t)\,dx\,dt}{1-e^{-(v\alpha+s\beta)}}. \tag{8}$$

Proof. Using the definition of DL-ARAT, we get:

$$L_xG_t[u(x,t)] = s\int_0^\infty \int_0^\infty e^{-(vx+st)}u(x,t)\,dx\,dt. \tag{9}$$

Using the property of improper integral, Eq. (9) can be written as:

$$L_xG_t[u(x,t)] = s\int_0^\alpha \int_0^\beta e^{-(vx+st)}u(x,t)\,dx\,dt + s\int_\alpha^\infty \int_\beta^\infty e^{-(vx+st)}u(x,t)\,dx\,dt. \tag{10}$$

Putting $x = \alpha + \rho$ and $t = \beta + \tau$ on the second integral in Eq. (10). We obtain

$$Q(v,s) = s\int_0^\alpha \int_0^\beta e^{-(vx+st)} u(x,t)\,dx\,dt \\ + s\int_0^\infty \int_0^\infty e^{-(v(\alpha+\rho)+s(\beta+\tau))} u(\alpha+\rho, \beta+\tau)\,d\rho\,d\tau. \tag{11}$$

Using the periodicity of the function $u(x,t)$, Eq. (11) can be written by:

$$Q(v,s) = s\int_0^\alpha \int_0^\beta e^{-(vx+st)} u(x,t)\,dx\,dt \\ + e^{-(v\alpha+s\beta)} s\int_0^\infty \int_0^\infty e^{-(v\rho+s\tau)} u(\rho,\tau)\,d\rho\,d\tau. \tag{12}$$

Using the definition of DL-ARAT, we get:

$$Q(v,s) = s\int_0^\alpha \int_0^\beta e^{-(vx+st)} u(x,t)\,dx\,dt + e^{-(v\alpha+s\beta)} Q(v,s). \tag{13}$$

Thus, Eq. (13) can be simplified into:

$$Q(v,s) = \frac{1}{1-e^{-(v\alpha+s\beta)}} \left(s\int_0^\alpha \int_0^\beta e^{-(vx+st)} u(x,t)\,dx\,dt \right).$$

Theorem 6(Heaviside Function). *Let $L_xG_t[u(x,t)]$ exists and $L_xG_t[u(x,t)] = Q(v,s)$, then:*

$$L_xG_t[u(x-\delta, t-\varepsilon)H(x-\delta, t-\varepsilon)] = e^{-v\delta - s\varepsilon} Q(v,s). \tag{14}$$

where $H(x-\delta, t-\varepsilon)$ is the Heaviside unit step function defined as:

$$H(x-\delta, t-\varepsilon) = \begin{cases} 1, & x > \delta, t > \varepsilon \\ 0, & \textit{Otherwise.} \end{cases}$$

Proof. Using the definition of DL-ARAT, we get:

$$L_xG_t[u(x-\delta, t-\varepsilon)H(x-\delta, t-\varepsilon)] \\ = s\int_0^\infty \int_0^\infty e^{-(vx+st)} (u(x-\delta, t-\varepsilon)H(x-\delta, t-\varepsilon))\,dx\,dt \\ = s\int_\delta^\infty \int_\varepsilon^\infty e^{-(vx+st)} u(x-\delta, t-\varepsilon)\,dx\,dt. \tag{15}$$

Putting $x - \delta = \rho$ and $t - \varepsilon = \tau$ in Eq. (15). We obtain

$$L_xG_t[u(x-\delta, t-\varepsilon)H(x-\delta, t-\varepsilon)] = s\int_0^\infty \int_0^\infty e^{-v(\delta+\rho)-s(\varepsilon+\tau)} u(\rho,\tau)\,d\rho\,d\tau. \tag{16}$$

Thus, Eq. (16) can be simplified into:

$$L_xG_t[u(x-\delta, t-\varepsilon)H(x-\delta, t-\varepsilon)] \\ = e^{-v\delta - s\varepsilon} \left(s\int_0^\infty \int_0^\infty e^{-v\rho - s\tau} u(\rho,\tau)\,d\rho\,d\tau \right) \\ = e^{-v\delta - s\varepsilon} Q(v,s).$$

Theorem 7(Convolution Theorem). *Let $L_xG_t[u(x,t)]$ and $L_xG_t[w(x,t)]$ are exists and $L_xG_t[u(x,t)] = Q(v,s)$, $L_xG_t[w(x,t)] = W(v,s)$, then:*

$$L_xG_t[u**w(x,t)] = \frac{1}{s} Q(v,s)W(v,s). \tag{17}$$

where

$$u**w(x,t) = \int_0^x \int_0^t u(x-\rho, t-\tau)w(\rho,\tau)\,d\rho\,d\tau,$$

and the symbol $**$ denotes the double convolution with respect to x and t.

*Proof.*Using the definition of DL-ARAT, we get:

$$
\begin{aligned}
&L_xG_t[u**w(x,t)] \\
&= s\int_0^\infty\int_0^\infty e^{-(vx+st)}(u**w(x,t))\,dx\,dt \\
&= s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau\right)dx\,dt. \quad (18)
\end{aligned}
$$

Using the Heaviside unit step function, Eq. (18) can be written as

$$
\begin{aligned}
&L_xG_t[u**w(x,t)] \\
&= s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^\infty\int_0^\infty u(x-\rho,t-\tau)H(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau\right)dx\,dt. \quad (19)
\end{aligned}
$$

Thus, Eq. (19) can be written as

$$
\begin{aligned}
&L_xG_t[u**w(x,t)] \\
&= \int_0^\infty\int_0^\infty w(\rho,\tau)\,d\rho\,d\tau\left(s\int_0^\infty\int_0^\infty e^{-v(x+\rho)-s(t+\tau)}u(x-\rho,t-\tau)H(x-\rho,t-\tau)\,dx\,dt\right) \\
&= \int_0^\infty\int_0^\infty w(\rho,\tau)\,d\rho\,d\tau\left(e^{-v\rho-s\tau}Q(v,s)\right) \\
&= Q(v,s)\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}w(\rho,\tau)\,d\rho\,d\tau \\
&= \frac{1}{s}Q(v,s)W(v,s).
\end{aligned}
$$

Theorem 8(Derivatives Properties). *Let $u(x,t)$ be a continuous function and $L_xG_t[u(x,t)] = Q(v,s)$. Then, we get the following derivatives properties:*

1. $L_xG_t\left[\frac{\partial u(x,t)}{\partial t}\right] = sQ(v,s) - sL[u(x,0)]$,
2. $L_xG_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vQ(v,s) - G[u(0,t)]$,
3. $L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2Q(v,s) - s^2L[u(x,0)] - sL\left[\frac{\partial u(x,0)}{\partial t}\right]$,
4. $L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2Q(v,s) - vG[u(0,t)] - G\left[\frac{\partial u(0,t)}{\partial x}\right]$,
5. $L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - sG[u(0,t)] - vsL[u(x,0)] + sG[u(0,0)]$.

Proof. 1.

$$
\begin{aligned}
L_xG_t\left[\frac{\partial u(x,t)}{\partial t}\right] &= s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial t}\right]dx\,dt \\
&= \int_0^\infty e^{-vx}\,dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt.
\end{aligned}
$$

Using integration by parts, we obtain

$$
s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt = s\left(-u(x,0) + s\int_0^\infty e^{-st}u(x,t)\,dt\right).
$$

Therefore,

$$
L_xG_t\left[\frac{\partial u(x,t)}{\partial t}\right] = sQ(v,s) - sL[u(x,0)]. \quad (20)
$$

2.

$$L_xG_t\left[\frac{\partial u(x,t)}{\partial x}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial x}\right]dx\,dt$$
$$= s\int_0^\infty e^{-st}\,dt\cdot\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx.$$

Using integration by parts, we obtain

$$\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx = -u(0,t)+v\int_0^\infty e^{-vx}u(x,t)\,dx.$$

Therefore,

$$L_xG_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vQ(v,s)-G[u(0,t)]. \tag{21}$$

3.

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]dx\,dt$$
$$= \int_0^\infty e^{-vx}\,dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt.$$

Using integration by parts, we obtain

$$s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt = s\left(-\frac{\partial u(x,0)}{\partial t}+s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt\right).$$

Using Eq. (20), we have

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2Q(v,s)-s^2L[u(x,0)]-sL\left[\frac{\partial u(x,0)}{\partial t}\right]. \tag{22}$$

4.

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]dx\,dt$$
$$= s\int_0^\infty e^{-st}\,dt\cdot\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx.$$

Using integration by parts, we obtain

$$\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx = -\frac{\partial u(0,t)}{\partial x}+v\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx.$$

Using Eq. (21), we have

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2Q(v,s)-vG[u(0,t)]-G\left[\frac{\partial u(0,t)}{\partial x}\right]. \tag{23}$$

5.

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dx\,dt$$
$$= s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dt\cdot\int_0^\infty e^{-vx}\,dx.$$

Table 3: DL-ARAT to some basic functions.

$u(x,t)$	$L_xG_t[u(x,t)] = Q(v,s)$
1	1
$x^{\alpha}t^{\beta}$	$\frac{\alpha!\beta!}{v^{\alpha+1}s^{\beta}}$
$e^{\alpha x+\beta t}$	$\frac{s}{(v-\alpha)(s-\beta)}$
$\sin(\alpha x+\beta t)$	$\frac{s(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$
$\cos(\alpha x+\beta t)$	$\frac{s(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}$
$\sinh(\alpha x+\beta t)$	$\frac{s(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}$
$\cosh(\alpha x+\beta t)$	$\frac{s(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}$
$J_0(\lambda\sqrt{xt})$	$\frac{4sv}{4vs+\lambda^2}$
$e^{\alpha x+\beta t}u(x,t)$	$\frac{s}{s-\beta}Q(v-\alpha,s-\beta)$
$u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)$	$e^{-v\delta-s\varepsilon}Q(v,s)$
$u**w(x,t)$	$\frac{1}{s}Q(v,s)W(v,s)$
$u_t(x,t)$	$sQ(v,s)-sL[u(x,0)]$
$u_x(x,t)$	$vQ(v,s)-G[u(0,t)]$
$u_{tt}(x,t)$	$s^2Q(v,s)-s^2L[u(x,0)]-sL[u_t(x,0)]$
$u_{xx}(x,t)$	$v^2Q(v,s)-vG[u(0,t)]-G[u_x(0,t)]$
$u_{xt}(x,t)$	$vsQ(v,s)-sG[u(0,t)]-vsL[u(x,0)]+sG[u(0,0)]$

Using integration by parts, we obtain

$$s\int_0^{\infty}\int_0^{\infty}e^{-(st+vx)}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dx\,dt$$
$$=-\int_0^{\infty}e^{-st}\left(\frac{\partial u(0,t)}{\partial t}\right)dt+vs\int_0^{\infty}\int_0^{\infty}e^{-(st+vx)}\left(\frac{\partial u(x,t)}{\partial t}\right)dx\,dt.$$

And, using Eq. (21) and single ARA transform of $u_t(0,t)$, we have

$$L_xG_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right]=vsQ(v,s)-sG[u(0,t)]-vsL[u(x,0)]+sG[u(0,0)]. \tag{24}$$

The previous results of DL-ARAT to some basic functions, some theorems and basic derivatives are summed up in the Table below:

4 Presentation of Double Laplace-ARA Transform Method in Solving IDEs

To illustrate the basic idea of this method for solving integral partial differential equations, we consider instances of Volterra integral equations, Volterra integro-partial differential equations, and integro-partial differential equations.

4.1 Volterra Integral Equation

Consider the following Volterra integral equation:

$$u(x,t)=f(x,t)+\gamma\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau \tag{25}$$

where $u(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions, and γ is constant.

The main idea of this method is to apply the DL-ARAT to Eq. (25) as follows:

$$L_xG_t[u(x,t)]=L_xG_t\left[f(x,t)+\gamma\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau\right] \tag{26}$$

Using the differentiation property of the DL-ARAT of Eq. (26) and Theorem 7, we have:

$$Q(v,s) = F(v,s) + \gamma\left(\frac{1}{s}Q(v,s)W(v,s)\right) \tag{27}$$

Eq. (27) can be simplified as:

$$Q(v,s) = \frac{F(s,v)}{1-\frac{\gamma}{s}W(v,s)} \tag{28}$$

Operating with the inverse of DL-ARAT on both sides of Eq. (28) gives:

$$u(x,t) = L_x^{-1}G_t^{-1}\left[\frac{F(s,v)}{1-\frac{\gamma}{s}W(v,s)}\right] \tag{29}$$

where $u(x,t)$ represents the term arising from the known functions $f(x,t)$ and $w(x,t)$.

4.2 Volterra Integro-Partial Differential Equations

Consider the following Volterra integro-partial differential equation:

$$\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = f(x,t) + \gamma\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau \tag{30}$$

with the following initial condition:

$$u(x,0) = h(x), \tag{31}$$

and the following boundary condition:

$$u(0,t) = g(t). \tag{32}$$

where $u(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions, and γ is constant.

Applying Laplace transform to the initial condition in Eq. (31), we get:

$$L[u(x,0)] = L[h(x)] = H(v)$$

Applying ARA transform to the boundary condition in Eq. (32), we get:

$$G[u(0,t)] = G[g(t)] = G(s)$$

Applying DL-ARAT to Eq. (30), we get:

$$L_xG_t\left[\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right] = L_xG_t\left[f(x,t) + \gamma\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho\,d\tau\right] \tag{33}$$

Using the differentiation property of the DL-ARAT of Eq. (33), initial and boundary conditions, and Theorem 7, we have:

$$vQ(v,s) - G + sQ(v,s) - sH = F(v,s) + \frac{\gamma}{s}Q(v,s)W(v,s) \tag{34}$$

Eq. (34) can be simplified as:

$$Q(v,s) = \frac{F(s,v)+G+sH}{v+s-\frac{\gamma}{s}W(v,s)} \tag{35}$$

Operating with the inverse of DL-ARAT on both sides of Eq. (35) gives:

$$u(x,t) = L_x^{-1}G_t^{-1}\left[\frac{F(s,v)+G+sH}{v+s-\frac{\gamma}{s}W(v,s)}\right] \tag{36}$$

where $u(x,t)$ represents the term arising from the known functions $f(x,t)$, $w(x,t)$, $h(x)$ and $g(t)$.

4.3 Integro-Partial Differential Equations

Consider the following integro-differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) + \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau)\, d\rho\, d\tau = f(x,t) \tag{37}$$

with the following initial conditions:

$$u(x,0) = h_1(x), \quad u_t(x,0) = h_2(x), \tag{38}$$

and the following boundary conditions:

$$u(0,t) = g_1(t), \quad u_x(0,t) = g_2(t). \tag{39}$$

where $u(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions.

Applying Laplace transform to the initial conditions in Eq. (38), we get:

$$L[u(x,0)] = L[h_1(x)] = H_1(v), \quad L[u_t(x,0)] = L[h_2(x)] = H_2(v)$$

Applying ARA transform to the boundary conditions in Eq. (39), we get:

$$G[u(0,t)] = G[g_1(t)] = G_1(s), \quad G[u_x(0,t)] = G[g_2(t)] = G_2(s)$$

Applying DL-ARAT to Eq. (37), we get:

$$L_x G_t \left[\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) + \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau)\, d\rho\, d\tau \right] = L_x G_t [f(x,t)] \tag{40}$$

Using the differentiation property of the DL-ARAT of Eq. (40), initial and boundary conditions, and Theorem 7, we have:

$$[s^2 Q(v,s) - s^2 H_1 - sH_2] - [v^2 Q(v,s) - vG_1 - G_2] + Q(v,s) + \frac{1}{s} Q(v,s) W(v,s) = F(v,s) \tag{41}$$

Eq. (41) can be simplified as:

$$Q(v,s) = \frac{s(F(s,v) + s^2 H_1 + sH_2 - vG_1 - G_2)}{s^3 - sv^2 + s + W(v,s)} \tag{42}$$

Operating with the inverse of DL-ARAT on both sides of Eq. (42) gives:

$$u(x,t) = L_x^{-1} G_t^{-1} \left[\frac{s(F(s,v) + s^2 H_1 + sH_2 - vG_1 - G_2)}{s^3 - sv^2 + s + W(v,s)} \right] \tag{43}$$

where $u(x,t)$ represents the term arising from the known functions $f(x,t)$, $h_1(x)$, $g_1(t)$, $h_2(x)$, $g_2(t)$, and $w(x,t)$.

4.4 General Methodology

The general methodology for applying the DL-ARAT method to solve the integral equations defined in this section:

1. Apply DL-ARAT to the main equation
2. Solve the algebraic equation resulting from the effect of DL-ARAT on the main equation
3. Obtain the final solution using the inverse DL-ARAT method

And the general methodology for applying the DL-ARAT method to solve the partial integral equations:

1. Apply the Laplace transform to the initial conditions
2. Apply the ARA transformation to the boundary conditions
3. Apply DL-ARAT to the main equation
4. Solve the algebraic equation resulting from the effect of DL-ARAT on the main equation
5. Obtain the final solution using the inverse DL-ARAT method

5 Applications

In this section, we introduce five interesting examples of IPDEs and solve them by the current method.

5.1 Application 5.1

Consider the following Volterra integral equation:

$$u(x,t) = \lambda - \gamma \int_0^x \int_0^t u(\rho,\tau)\, d\rho\, d\tau \tag{44}$$

Applying the DL-ARAT to Eq. (44), we have:

$$Q(v,s) = \frac{\lambda}{v} - L_x G_t[1 **u(x,t)] = \frac{\lambda}{v} - \gamma\left[\frac{1}{s}Q(v,s)\right] \tag{45}$$

Eq. (45) can be simplified as:

$$Q(v,s) = \frac{\lambda s}{vs+\gamma} = \frac{4\lambda s}{4vs+(2\sqrt{\gamma})^2} \tag{46}$$

Taking the inverse of DL-ARAT to Eq. (46), then the solution of Eq. (44) is:

$$u(x,t) = L_x^{-1} G_t^{-1}\left[\frac{4\lambda s}{4vs+(2\sqrt{\gamma})^2}\right] = \lambda J_0(2\sqrt{\gamma x t})$$

5.2 Application 5.2

Consider the following Volterra integral equation:

$$4t = \gamma \int_0^x \int_0^t u(x-\rho, t-\tau) u(\rho,\tau)\, d\rho\, d\tau \tag{47}$$

Applying the DL-ARAT to Eq. (47), we have:

$$\frac{4}{sv} = \frac{\gamma}{s} Q(v,s)\cdot Q(v,s) = \frac{\gamma}{s}(Q(v,s))^2 \tag{48}$$

Eq. (48) can be simplified as:

$$Q(v,s) = \frac{2}{\sqrt{\gamma v}} \tag{49}$$

Taking the inverse of DL-ARAT to Eq. (49), we get the solution of Eq. (47):

$$u(x,t) = L_x^{-1} G_t^{-1}\left[\frac{2}{\sqrt{\gamma}\cdot\sqrt{v}}\right] = \frac{2}{\sqrt{\pi\gamma x}}$$

5.3 Application 5.3

Consider the following Volterra integral equation:

$$xe^x - xe^{x-t} = \int_0^x \int_0^t e^{\rho-\tau} u(x-\rho, t-\tau)\, d\rho\, d\tau \tag{50}$$

Applying the DL-ARAT to Eq. (50), we have:

$$\frac{1}{(v-1)^2} - \frac{s}{(v-1)^2(s+1)} = \frac{Q(v,s)}{(v-1)(s+1)} \tag{51}$$

Eq. (51) can be simplified as:

$$Q(v,s) = \left(\frac{1}{(v-1)^2} - \frac{s}{(v-1)^2(s+1)}\right)\cdot(v-1)(s+1) = \frac{s+1-s}{(v-1)^2(s+1)}\cdot(v-1)(s+1) = \frac{1}{v-1} \tag{52}$$

Taking the inverse of DL-ARAT to Eq. (52), we get the solution of Eq. (50):

$$u(x,t) = L_x^{-1} G_t^{-1}\left[\frac{1}{v-1}\right] = e^x$$

5.4 Application 5.4

Consider the following Volterra integro-partial differential equation:

$$\frac{\partial u(x,t)}{\partial x}+\frac{\partial u(x,t)}{\partial t}=-1+e^{x}+e^{t}+e^{x+t}+\int_{0}^{x}\int_{0}^{t}u(x-\rho,t-\tau)\,d\rho\,d\tau \tag{53}$$

with the initial condition $u(x,0)=e^{x}$ and boundary condition $u(0,t)=e^{t}$.

By substituting the values of functions $H=\frac{1}{v-1}$, $G=\frac{s}{s-1}$, $F=-\frac{1}{v}+\frac{1}{v-1}+\frac{s}{v(s-1)}+\frac{s}{(v-1)(s-1)}$ and $W=\frac{1}{v}$ in the general form in Eq. (35), we obtain:

$$Q(v,s)=\frac{-\frac{1}{v}+\frac{1}{v-1}+\frac{s}{v(s-1)}+\frac{s}{(v-1)(s-1)}+\frac{s}{s-1}+\frac{s}{v-1}}{v+s-\frac{1}{sv}} \tag{54}$$

Eq. (54) can be simplified as:

$$Q(v,s)=\left(\frac{sv^{2}+s^{2}v-1}{v(v-1)(s-1)}\right)\left(\frac{sv}{sv^{2}+s^{2}v-1}\right)=\frac{sv}{v(v-1)(s-1)} \tag{55}$$

Applying the inverse of DL-ARAT to Eq. (55), then the solution of Eq. (53) is:

$$u(x,t)=L_{x}^{-1}G_{t}^{-1}\left[\frac{s}{(v-1)(s-1)}\right]=e^{x+t}$$

5.5 Application 5.5

Consider the following integro-partial differential equation:

$$\frac{\partial^{2}u(x,t)}{\partial t^{2}}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}+u(x,t)+\int_{0}^{x}\int_{0}^{t}e^{x-\rho+t-\tau}u(x-\rho,t-\tau)\,d\rho\,d\tau=e^{x+t}+xte^{x+t} \tag{56}$$

with the initial conditions $u(x,0)=e^{x}$, $u_{t}(x,0)=e^{x}$ and boundary conditions $u(0,t)=e^{t}$, $u_{x}(0,t)=e^{t}$.

Applying the Laplace transform to the initial conditions and the ARA transform to the boundary conditions, we have:

$$H_{1}=H_{2}=\frac{1}{v-1},\quad G_{1}=G_{2}=\frac{s}{s-1}$$

By substituting the values of functions $H_{1}=H_{2}=\frac{1}{v-1}$, $G_{1}=G_{2}=\frac{s}{s-1}$, $F=\frac{s}{(v-1)(s-1)}+\frac{s}{(s-1)^{4}}$ and $W=\frac{s^{2}}{(v^{2}-1)(s^{2}-1)}$ in the general form in Eq. (42), we obtain:

$$Q(v,s)=\frac{s}{(s-1)(v-1)} \tag{57}$$

Applying the inverse of DL-ARAT to Eq. (57), then the solution of Eq. (56) is:

$$u(x,t)=L_{x}^{-1}G_{t}^{-1}\left[\frac{s}{(s-1)(v-1)}\right]=e^{x+t}$$

6 Results and Discussion

This study presents the Double Laplace-ARA Transform (DL-ARAT), an innovative hybrid technique that integrally combines the features of both the Laplace and ARA transforms. DL-ARAT has demonstrated its ability to simplify the process of solving integral and partial differential equations, offering both theoretical improvements and practical advantages over traditional approaches.

The fundamental mathematical properties of DL-ARAT were rigorously established, including:

- Linearity
- Convolution properties
- Existence conditions
- Shifting properties
- Periodicity handling
- Response to the Heaviside unit step function
- Derivative behavior

These properties affirm the theoretical soundness and high flexibility of the transform. To validate its practical applicability, DL-ARAT was applied to a range of problems, including:

- Volterra integral equations
- Integro-partial differential equations
- More complex integral systems

In all cases, DL-ARAT enabled the derivation of exact closed-form solutions with notable efficiency. For instance:

- Volterra integral equations were converted into algebraic equations, significantly reducing computational complexity
- Integro-partial differential equations were solved through a simplified process

The illustrative examples provided highlight the effectiveness and reliability of this approach, reinforcing DL-ARAT as a powerful tool in mathematical physics. Compared to conventional methods, it offers:

- Enhanced accuracy
- Computational ease

Future work is recommended to explore the extension of DL-ARAT to:

- Nonlinear integral equations
- Partial differential equations

potentially expanding its applicability across diverse scientific and engineering fields.

7 Conclusion

This paper presents a new double integral transform, called the double Laplace-ARA transform, which is constructed by combining the Laplace transform with the ARA transform of order one. The basic properties of this transform, involving:

- Linearity
- Displacement
- Convolution
- Derivatives

are extracted and proven precisely.

We applied the DL-ARAT transforms to a variety of illustrative examples including:

- Volterra integral equations
- Integral partial differential equations
- Volterra partial integrodifferential equations

The transformation successfully reduces these equations to algebraic forms, allowing for accurate solutions. Compared to existing double transforms, such as:

- Double Laplace transform
- Double Sumudu transform

DL-ARAT demonstrates clear advantages in:

- Handling boundary and initial conditions simultaneously
- Simplifying the overall solution process

These advantages are particularly evident in the fourth section, where we explicitly solve complex equations and verify the consistency of the results with the specified properties. Thus, DL-ARAT can be considered a powerful analytical tool for solving a wider class of:

- Integral equations
- Integro-differential equations

in physics and engineering mathematics.

Future research may extend this method to include:

- Fractional order systems
- Numerical approximations

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