Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/140307

Reliability Analysis and Parameter Estimation for Censored Data in Extended Gompertz Distribution

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Received: 5 Oct. 2024, Revised: 12 Feb. 2025, Accepted: 2 Apr. 2025
Published online: 1 May 2025

Abstract: This article explores the estimation of parameters and lifetime indices of the extended Gompertz distribution under progressively Type-II censored schemes. The study employs maximum likelihood, Bayes, and two parametric bootstrap methods for parameter estimation, alongside the computation of reliability and hazard rate functions. Additionally, approximate confidence intervals and an asymptotic variance-covariance matrix are derived. The Markov chain Monte Carlo technique, specifically the Gibbs sampler within the Metropolis-Hastings algorithm, is utilized to generate samples from posterior density functions. Bayesian estimates are computed using both symmetric and asymmetric loss functions. Through Monte Carlo simulations, the efficacy of these methods is evaluated using metrics such as mean squared errors, average interval lengths, and coverage probabilities. Finally, a real dataset is analyzed to demonstrate the practical application of the developed inferential procedures.

Keywords: Extended Gompertz distribution; Progressive Type-II censoring; Bayesian inference; Bootstrap methods; MCMC.

1 Introduction

Censored data plays a crucial role in various industries, especially in reliability and engineering contexts. Censoring occurs when the full observation of a variable is not available or is incomplete, either due to limitations in measurement capabilities, time constraints, or ethical considerations. There are two primary types of censored data: right-censored and left-censored. Right-censored data indicates that the observation is known to be above a certain threshold but not precisely quantifiable beyond that point. Conversely, left-censored data suggests that the observation is below a known threshold. In industries such as manufacturing, where testing destructive to the product might be costly or impractical, right-censored data allows for the estimation of product lifetimes or failure rates without the full destruction of every unit. In reliability engineering, censored data allows for the modeling of survival times or failure rates of components that may not have failed by the end of a study period. This is crucial for predicting maintenance schedules and ensuring product reliability over its lifecycle.

Moreover, in clinical trials and epidemiology, censoring plays a critical role in longitudinal studies where subjects may drop out or where the study may end before all events (e.g., deaths or disease progression) have occurred. Statistical methods like Kaplan-Meier estimators and Cox proportional hazards models are employed to analyze censored data, providing insights into time-to-event outcomes without complete data observation. The reliability and engineering sectors rely heavily on accurate interpretation of censored data to make informed decisions about product design, maintenance strategies, and safety protocols. Advanced statistical techniques continue to evolve to handle various forms of censoring, ensuring that industries can derive reliable predictions and optimize their processes effectively. Thus, understanding and appropriately analyzing censored data are indispensable for maintaining high standards of quality and safety across diverse industrial applications.

Due to losing of the object of interest in industrial life testing and reliability studies, it's preferable to obtain statistical inferences based on censored samples. The obtained sample is called a censored sample (an incomplete

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sample). Censoring schemes are used to reduce both the cost and time of the experiment. Besides accelerating the performance of the design. In lifetime and reliability analysis, we are concerned with obtaining results that allow us to make inferences about the processes or populations involved. The two common censoring schemes that are used in reliability and life testing are Type-I censoring schemes and Type-II censoring schemes. In Type-I censoring schemes, the experiment is terminated at a pre-fixed time T, whereas in Type-II censoring schemes, the experiment is terminated at pre-fixed number of failures. But both Type-I and Type-II censoring schemes do not support the removal of units at points except the terminal point of the experiment. Therefore,we will use a more general censoring called progressive Type-II censoring (PT2C). Which allows to eliminate survival units from the experiment at different stages due to a pre-fixed time or pre-fixed number of failures.

PT2C can be described as follows: Consider that we have n independent components that are placed in a life test experiment with continuous identically distributed failure times $Y_1, Y_2, ..., Y_n$ and $m \leq n$ failures are observed with a prefixed censoring scheme $R = R_1, R_2, ..., R_m$. When the 1$^{\text{st}}$ failure time $Y_{1:m:n}$ occurs, the surviving components R_1 are randomly removed from the remaining $(n-1)$ surviving components. At the 2$^{\text{nd}}$ failure time $Y_{2:m:n}$, R_2 survival components are randomly removed from the remaining $(n-R_1-2)$ surviving components. Similarly, untill the m$^{\text{th}}$ failure $(Y_{m:m:n})$ has occured, the remaining r_m survival components are removed from the remaining $(n-m-\sum_{i=1}^{m-1} R_i)$ surviving components.

As a result, $Y_{1:m:n} \leq Y_{2:m:n} \leq \cdots \leq Y_{m:m:n}$ with the censoring scheme $R = R_1, R_2, ..., R_m$ are called PT2C samples. Several authors have examined inference under PT2C with various lifespan distributions see, for instance Chacko and Mohan [1], Qin and Gui [2], Kumar et al. [3], EL-Sagheer et al. [4], Alotaibi et al. [5], Alotaibi et al. [6], Almetwally et al. [7] and Maiti and Kayal [8]. The outstanding review article and a comprehensive discussion of the topic of progressive censoring are provided in Balakrishnan [9]. An algorithm was created by Aggarwala and Balakrishnan [10] to simulate general PT2C samples from the uniform distribution or any other continuous distribution. The likelihood function can be written as:

$$L(y_1, y_2, ..., y_m) = C \prod_{i=1}^{m} f(y_i)[1-F(y_i)]^{R_i}, \tag{1}$$

with $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-\sum_{i=1}^{m-1}(R_i+1))$ being regular constants. The Gompertz distribution is widely used in various fields, primarily in reliability engineering and actuarial sciences. It describes the distribution of lifetimes for items that experience an increasing hazard rate over time, making it suitable for modeling aging processes and mortality rates. In reliability engineering, the Gompertz distribution helps predict failure rates for components that are more likely to fail as they age. It also finds application in demography to analyze mortality patterns in populations. Actuaries use the Gompertz distribution to model human mortality, aiding in the development of life insurance policies and pension plans. Its flexibility and ability to capture the increasing risk of failure over time make the Gompertz distribution a valuable tool in understanding and managing risks associated with aging and mortality. In recent years, researchers have been more interested in the idea of generating new extended distributions from classical distributions, one of which we will use in our study is the extended Gompertz distribution (EGD), which was proposed by Eliwa et al. [11].

The EGD expands upon the traditional Gompertz model by introducing additional shape parameter to enhance its flexibility in modeling survival data. This distribution is particularly useful in fields such as biomedical sciences and economics, where complex aging processes and survival analysis are studied. It accommodates scenarios where the hazard rate might not increase exponentially with age but may instead vary in more nuanced ways. In biomedical research, the EGD is employed to analyze disease progression and survival times in clinical trials, providing insights into treatment effectiveness and patient prognosis. Economists use it to study longevity risk in pension planning and insurance, helping to estimate future liabilities based on mortality trends. On the opposite side, for low levels of infant mortality, the Gompertz distribution force of mortality is extended across the entire life span of populations, and no observed mortality slowing is noted, see Vaupel [12]. EGD's versatility in capturing diverse hazard rate patterns makes the EGD a powerful tool for understanding and predicting survival outcomes across various disciplines. A random variable Y has an EGD if its probability density function (PDF) and the corresponding cumulative distribution function (CDF) are given by

$$F(y) = 1 - e^{-\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}}, y > 0, \lambda, \theta, \beta > 0, \tag{2}$$

and

$$f(y) = \lambda\theta(e^{\beta y}-1)^{\theta-1} e^{\beta y} e^{-\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}}, y > 0, \lambda, \theta, \beta > 0, \tag{3}$$

while its reliability and hazard functions are given by

$$S(y) = e^{-\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}}, y > 0, \lambda, \theta, \beta > 0, \tag{4}$$

$$h(y) = \lambda\theta(e^{\beta y}-1)^{\theta-1}e^{\beta y}, y > 0, \lambda, \theta, \beta > 0, \tag{5}$$

One can easily show that, the hazard rate function of EGD is increasing function for $\theta \geq 1$ whereas bathtub-shaped when $\theta < 1$. To illustrate that, we plot the hazard rate function of EGD in Figure 2. Figure 2 shows that the hazard rate function can be take different shapes. The EGD with parameters λ, β and θ denoted by $EGD(\lambda,\beta,\theta)$ tends to the exponential model when the parameter β tends to zero and the parameter $\theta = 1$. Figure 1 shows the pdf for various values of the parameters. The pdf can be taken unimodal or decreasing-shaped.

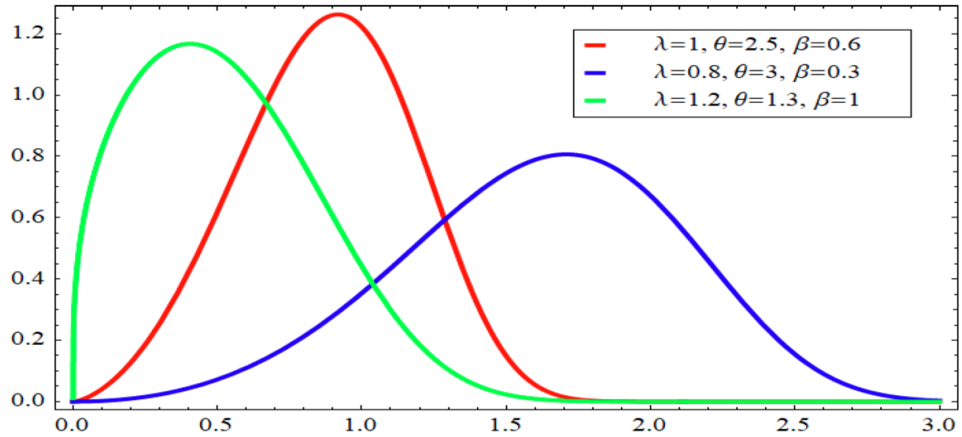

Fig. 1: The PDF for EGD with different parameters values

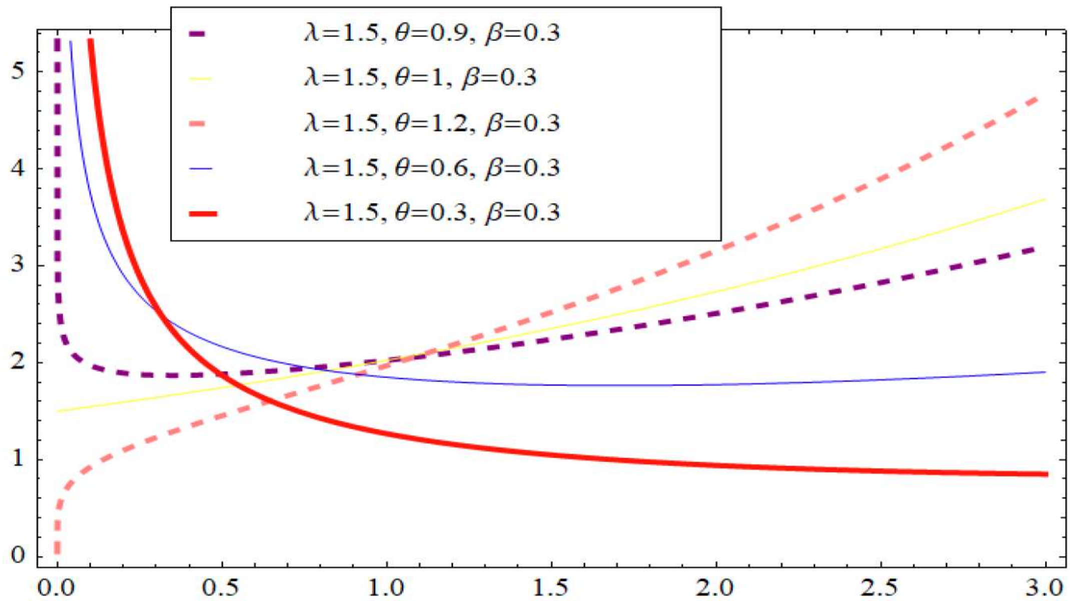

Fig. 2: The hazard rate function of EGD with various parameters values

Frequently probability models are applied, such as the exponential, Gompertz, Weibull, Gumbel, etc., to analyze the lifetime data. Therefore, in order to describe a variety of phenomena in many of fields, researchers have been developing various univariate and bivariate extensions of these distributions. See for example, Hamid et al. [13], Ali et al. [14], El-Gohary et al. [15], Khan et al. [16], El-Bassiouny et al. [17], Haitham et al. [18], El-Bassiouny and El-Morshedy [19], El-Morshedy et al. [20], Mohamed et al. [21], Eliwa et al. [22,23], Jehhan et al. [24], El-Morshedy and Eliwa [25], Alizadeh et al. [26], Eliwa and El-Morshedy [27], Salah et al. [28], among others. A bathtub-shaped hazard rate function, such as that seen in machine life cycles, cannot be utilized to model lifetime data when the hazard function of the probability distribution is constant, increasing, or decreasing. Therefore, we suggested in this study an extension of the Gompertz distribution with three parameters, which we refer to as the EGD.

The main aim of this paper is to investigate parameter estimation for the EGD. Maximum likelihood estimates of these parameters are computed via the Newton-Raphson iteration method to solve non-linear equations. Hazard and reliability functions are among the parameters estimated. To establish approximate confidence intervals for these parameters as well as reliability and hazard and functions, we employ the parametric bootstrap technique. Furthermore, Bayesian estimation is explored using Markov chain Monte Carlo methods. The reliability and hazard functions, along with Bayesian parameter estimates, are derived using the Metropolis algorithm within the Gibbs sampler framework. Throughout, these estimation methodologies are demonstrated with the analysis of a real-world dataset.

The remainder of this paper is structured as follows. Section 2 focuses on the maximum likelihood estimation and asymptotic confidence intervals. Section 3 introduces two parametric bootstrap approaches. Bayesian estimation using the MCMC technique is detailed in Section 4. In Section 5, a simulation study is carried out to assess and compare the effectiveness of these estimation methodologies. Section 6 includes the presentation of a real-world dataset to exemplify the application of the proposed inference procedures. Lastly, Section 7 provides a concise conclusion to summarize the findings and contributions of this research.

2 Parameters Estimation

Maximum likelihood estimation (MLE) is a powerful statistical method used to estimate the parameters of a probability distribution. The core idea behind MLE is to find the set of parameter values that maximize the likelihood function, which measures how likely the observed data are under the given distributional assumptions. One of the key properties of MLE is consistency, meaning that as the sample size increases, the estimates converge to the true parameter values. Additionally, MLE tends to be asymptotically efficient, meaning that the estimated parameters have the smallest possible variance among all consistent estimators.

Advantages of maximum likelihood estimation include its simplicity in implementation, especially when compared to other methods like Bayesian estimation, and its robustness when the sample size is large. MLE does not require prior information about the parameters, making it particularly useful in situations where prior knowledge is limited or unavailable. Moreover, MLE provides a framework for hypothesis testing and model selection, enabling researchers to assess the goodness of fit of their models rigorously. Overall, MLE is widely used in fields ranging from biology to economics due to its solid theoretical foundation and practical applicability.

As a sort of model complexity, we have extended the EGD with three parameters in order to achieve high accuracy and improved data fitting. Additionally, we compute the estimate and the approximate confidence intervals for the survival function and hazard rate function, which have, to the best of our knowledge, not discussed a lot in the literature. Suppose that $Y_1 < Y_2 < \cdots < Y_m$ are PT2C sample drawn from EGD with the censoring scheme $R = (R_1, R_2, ..., R_m)$. From (1), (2) and (3), the likelihood function is given by

$$L(\lambda,\beta,\theta|\underline{y}) = C\lambda^m\theta^m\prod_{i=1}^{m}(e^{\beta y}-1)^{\theta-1}\prod_{i=1}^{m}e^{\beta y-\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}}\prod_{i=1}^{m}e^{-R_i\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}}. \tag{6}$$

Ignoring the constant term, The log-likelihood function $\ell = \log L(\lambda,\beta,\theta|y)$ is obtained from (6) as

$$\ell = m\log\lambda + m\log\theta + (\theta-1)\sum_{i=1}^{m}\log\left(e^{\beta y_i}-1\right) + \beta\sum_{i=1}^{m}y_i - \sum_{i=1}^{m}\left(\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}\right) - \sum_{i=1}^{m}R_i\left(\frac{\lambda}{\beta}(e^{\beta y}-1)^{\theta}\right). \tag{7}$$

2.1 Point estimation

Upon differentiating (7) w.r.t λ, β and θ and equating each to zero, the MLEs of λ, β and θ can be obtained from the following

$$\frac{\partial\ell}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m}\frac{R_i}{\beta}\left(e^{\beta y_i}-1\right)^{\theta}, \tag{8}$$

$$\frac{\partial \ell}{\partial \beta} = (\theta-1)\sum_{i=1}^{m}\frac{y_i e^{\beta y_i}}{e^{\beta y_i}-1} + \sum_{i=1}^{m} y_i - \sum_{i=1}^{m}\left[\frac{-\lambda}{\beta^2}\left(e^{\beta y_i}-1\right)^{\theta} + \frac{\lambda}{\beta}\theta\left(e^{\beta y_i}-1\right)^{\theta-1} y_i e^{\beta y_i}\right]$$
$$-\sum_{i=1}^{m} R_i\left[\frac{-\lambda}{\beta^2}\left(e^{\beta y_i}-1\right)^{\theta} + \frac{\lambda}{\beta}\theta\left(e^{\beta y_i}-1\right)^{\theta-1} y_i e^{\beta y_i}\right], \tag{9}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{m}{\theta} + \sum_{i=1}^{m}\log(e^{\beta y_i}-1) - \sum_{i=1}^{m}\left[\frac{\lambda}{\beta}\left(e^{\beta y_i}-1\right)^{\theta}\log(e^{\beta y_i}-1)\right] - \sum_{i=1}^{m}\left[R_i\frac{\lambda}{\beta}(e^{\beta y_i}-1)^{\theta}\log(e^{\beta y_i}-1)\right]. \tag{10}$$

From (8) we obtain the MLE of λ as

$$\hat{\lambda} = m\left[\sum_{i=1}^{m}\left(\frac{R_i}{\hat{\beta}}\right)\left(e^{\hat{\beta} y_i}-1\right)^{\hat{\theta}}\right]^{-1}, \tag{11}$$

there are no closed forms for (9) and (10) thus, the Newton–Raphson iteration method is used to obtain the estimates, see EL-sagheer [29]. The algorithm is described as follows:

1. Use the method of moments or any other methods to estimate the parameters λ, β and θ as starting point of iteration, denote the estimates as $(\lambda_0, \beta_0, \theta_0)$ and set $k = 0$.
2. Calculate $(\frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \theta})_{(\lambda_k,\, \beta_k,\, \theta_k)}$ and the observed Fisher Information matrix $I^{-1}(\lambda, \beta, \theta)$, given in the next paragraph.
3. Update (λ, β, θ) as

$$(\lambda_{k+1}, \beta_{k+1}, \theta_{k+1}) = (\lambda_k, \beta_k, \theta_k) + (\frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \theta})_{(\lambda_k,\, \beta_k,\, \theta_k)} \times\ I^{-1}(\lambda, \beta, \theta). \tag{12}$$

4. Set $k = k+1$ and then go back to Step 1.
5. Continue the iterative steps until $|(\lambda_{k+1}, \beta_{k+1}, \theta_{k+1}) - (\lambda_k, \beta_k, \theta_k)|$ is smaller than a threshold value. The final estimates of (λ, β, θ) are the MLE of the parameters, denoted as $(\hat{\lambda}, \hat{\beta}, \hat{\theta})$.

Moreover, using the invariance property of MLEs, the MLEs of $S(t)$ and $h(t)$ can be obtained after replacing λ, β and θ by $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\theta}$ as

$$\hat{S}(t) = e^{-\frac{\hat{\lambda}}{\hat{\beta}}(e^{\hat{\beta} y}-1)^{\hat{\theta}}},\ \hat{h}(t) = \hat{\lambda}\hat{\theta}(e^{\hat{\beta} y}-1)^{\hat{\theta}-1}e^{\hat{\beta} y}. \tag{13}$$

2.2 Asymptotic confidence intervals

The asymptotic normal distribution of the MLEs is the most popular way to make confidence bounds for the parameters, according to Vander Wiel and Meeker [30]. The approximate asymptotic variance–covariance matrix of the MLEs, $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\theta}$ can be obtained through the entries of the inverse of the Fisher information matrix as $I_{ij} = E[\frac{-\partial^2 l(\Phi)}{\partial \varphi_i \partial \varphi_j}]$ where $i, j = 1,2,3$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3) = (\lambda, \beta, \theta)$. Unfortunately, it can be difficult to find the exact closed forms for the expectations mentioned above. Thus, by removing the expectation operator E, we may derive the asymptotic variance-covariance matrix for the maximum likelihood estimators. For further details see Cohen [31]. The entries in the observed Fisher information matrix are second partial derivatives of the log-likelihood function, which are easily calculated. As a result, the observed information matrix is denoted by

$$\hat{I}(\lambda, \beta, \theta) = \begin{pmatrix} -\frac{\partial^2 l}{\partial \lambda^2} & -\frac{\partial^2 l}{\partial \lambda \partial \beta} & -\frac{\partial^2 l}{\partial \lambda \partial \theta} \\ -\frac{\partial^2 l}{\partial \beta \partial \lambda} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \theta} \\ -\frac{\partial^2 l}{\partial \theta \partial \lambda} & -\frac{\partial^2 l}{\partial \theta \partial \beta} & -\frac{\partial^2 l}{\partial \theta^2} \end{pmatrix}_{(\lambda=\hat{\lambda}, \beta=\hat{\beta}, \theta=\hat{\theta})}. \tag{14}$$

As a result, inverting the observed information matrix $\hat{I}(\lambda,\beta,\theta)$ yields the asymptotic variance-covariance matrix $[\hat{V}]$ for the MLEs.

$$[\hat{V}] = \hat{I}^{-1}(\lambda,\beta,\theta) = \begin{pmatrix} \widehat{var(\lambda)} & cov(\lambda,\beta) & cov(\lambda,\theta) \\ cov(\beta,\lambda) & \widehat{var(\beta)} & cov(\beta,\theta) \\ cov(\theta,\lambda) & cov(\theta,\beta) & \widehat{var(\theta)} \end{pmatrix}_{\downarrow(\hat{\lambda},\hat{\beta},\hat{\theta})}. \tag{15}$$

It is known that $(\hat{\lambda},\hat{\beta},\hat{\theta})$ is approximately distributed as multivariate normal with mean (λ,β,θ) and covariance matrix $I^{-1}(\lambda,\beta,\theta)$ under some conditions, see Lawless [32]. Consequently, the approximate confidence intervals (ACIs) of $(1-\gamma)100\%$ For λ, β and θ can obtained by

$$(\hat{\lambda} \pm Z_{\gamma/2}\sqrt{\widehat{var(\lambda)}}),\ (\hat{\beta} \pm Z_{\gamma/2}\sqrt{\widehat{var(\beta)}}),\ (\hat{\theta} \pm Z_{\gamma/2}\sqrt{\widehat{var(\theta)}}), \tag{16}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$. The variances of the reliability and hazard functions must also be determined in order to create the ACIs for each. We apply the delta method described in Greene [33] to obtain the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{h}(t)$. The variance of $\hat{S}(t)$ and $\hat{h}(t)$, can be approximated, respectively by

$$\hat{\sigma}^2_{\hat{S}(t)} = [\nabla\hat{S}(t)]^T[\hat{V}][\nabla\hat{S}(t)], \qquad \hat{\sigma}^2_{\hat{h}(t)} = [\nabla\hat{h}(t)]^T[\hat{V}][\nabla\hat{h}(t)], \tag{17}$$

where $\hat{S}(t)$and $\hat{h}(t)$ are the gradient of $\hat{S}(t)$ and $\hat{h}(t)$, respectively, with respect to λ,β and θ. Thus, the $(1-\gamma)100\%$ ACIs for $S(t)$ and $h(t)$ can obtained by:

$$(\hat{S}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{S}(t)}}), \quad (\hat{h}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{h}(t)}}), \tag{18}$$

The main issue with $(1-\gamma)100\%$ ACI is that it could return a negative value in the lower bound for a positive parameter. It is simple to confirm that the computed confidence intervals have a lower bound with a positive value if one of the computed confidence intervals has a negative lower bound, then this value is replaced by zero. The normal approximation can be used for the log-transformed MLE, suggested by Meeker and Escobar [34]. Therefore, A two-sided $(1-\gamma)100\%$ ACIs for $\Omega = (\lambda,\beta,\theta,S(t),h(t))$ are provided as

$$\left[\hat{\Omega}.\exp\{-\frac{Z_{\gamma/2}\sqrt{\widehat{var(\hat{\Omega})}}}{\hat{\Omega}}\},\hat{\Omega}.\exp\{-\frac{Z_{\gamma/2}\sqrt{\widehat{var(\hat{\Omega})}}}{\hat{\Omega}}\}\right]. \tag{19}$$

where $\hat{\Omega}$ =$(\hat{\lambda},\hat{\beta},\hat{\theta},\hat{S}(t),\hat{h}(t))$.

3 Bootstrap Confidence Intervals

Bootstrap estimation is a resampling technique used to estimate the sampling distribution of a statistic by drawing repeated samples with replacement from the original data. The fundamental idea is to simulate new datasets that resemble the original dataset to assess variability and uncertainty in statistical estimates. One of the key properties of bootstrap estimation is its simplicity and versatility it can be applied to a wide range of statistical problems without relying on complex theoretical assumptions. Bootstrap estimates are often robust and provide reliable confidence intervals and standard errors, especially when the underlying distribution of the data is unknown or non-normal. Moreover, bootstrap methods are computationally feasible with modern computing power, making them accessible for practical use in data analysis and inference. By generating numerous resamples, bootstrap estimation captures the variability inherent in the data, offering advantages over traditional methods like parametric assumptions or asymptotic approximations, particularly in smaller sample sizes or when data distributions are non-standard. To calculate the bootstrap confidence intervals of $\lambda,\beta,\theta,S(t)$ and $h(t)$, two parametric bootstrap algorithms are offered. The first is based on Efron's [35] concept of the percentile bootstrap (Boot-p) confidence interval method. The second is the bootstrap-t (Boot-t) confidence interval method, proposed by Hall [36]. An estimator of the variance of the MLE of $\lambda,\beta,\theta,S(t)$ and $h(t)$ is needed for Boot-t, which was developed based on a studentized "pivot.".

3.1 Parametric boot-p

1.Based on the original data y= $y_{1:m:n}, y_{2:m:n}, ..., y_{m:m:n}$ obtain $\hat{\lambda}, \hat{\beta}$ and $\hat{\theta}$ by maximizing Eqs. (8)–(11).
2.Based on the pre-specified progressive censoring scheme $(R_1, R_2, ..., R_m)$ generate a Type-II progressive censoring sample $y^* = y^*_{1:m:n}, y^*_{2:m:n}, ..., y^*_{m:m:n}$ from the EGD with parameters $\hat{\lambda}, \hat{\beta}$ and $\hat{\theta}$, using the algorithm described in Balakrishnan and Sandhu [37].
3.Obtain the MLEs based on the bootstrap sample and denote this bootstrap estimate by $\hat{\psi}^*$(in our case ψ could be λ, β, θ, $S(t)$ or $h(t)$.
4.Repeat Steps (2) and (3) Nboot times, and obtain. $\hat{\psi}^*_1, \hat{\psi}^*_2, ..., \hat{\psi}^*_{Nboot}$, where $\hat{\psi}^*_i = (\hat{\lambda}^*_i, \hat{\beta}^*_i, \hat{\theta}^*_i, \hat{S}^*_i(t), \hat{h}^*_i(t))$, $i = 1, 2, 3, ..., Nboot$.
5.Arrange $\hat{\psi}^*_i$, $i = 1, 2, 3, ..., Nboot$ in ascending orders and obtain $\hat{\psi}^*_1, \hat{\psi}^*_2, ..., \hat{\psi}^*_{Nboot}$.

Let $G_1(z) = P(\hat{\psi}^* \leq z)$ be the cumulative distribution function of $\hat{\psi}^*$. Define $\hat{\psi}_{boot-p} = \mathrm{G}_1^{-1}(z)$ for given z. The approximate bootstrap-p $(1-\gamma)100\%$ CI of $\hat{\psi}$, is obtained by

$$\left[\hat{\psi}_{boot-p}(\frac{\gamma}{2}),\ \hat{\psi}_{boot-p}(1-\frac{\gamma}{2})\right]. \tag{20}$$

3.2 Parametric boot-t

(1)–(3)The same as the parametric Boot-p.
(4)Based on the asymptotic variance–covariance matrix (15) and delta method (17), respectively, compute the variance–covariance matrix $I^{-1*}(\hat{\lambda}^*, \hat{\beta}^*, \hat{\theta}^*)$ and the approximate estimates of the variance $\hat{S}^*(t)$ and $\hat{h}^*(t)$.
(5)Compute the $T^{*\psi}$ statistic defined as

$$T^{*\psi} = \frac{(\hat{\psi}^* - \hat{\psi})}{\sqrt{\widehat{var(\hat{\psi}^*)}}}.$$

(6)Repeat Steps 2–5, NBoot times and obtain $T_1^{*\psi}, T_2^{*\psi}, ..., T_{NBoot}^{*\psi}$.
(7)Sort $T_1^{*\psi}, T_2^{*\psi}, ..., T_{NBoot}^{*\psi}$ in ascending orders and obtain the ordered sequences $T_{(1)}^{*\psi}, T_{(2)}^{*\psi}, ..., T_{(NBoot)}^{*\psi}$

Let $G_2(z) = P(T^* \leq z)$ be the cumulative distribution function of T^* for a given z, define

$$\hat{\psi}_{boot-t} = \hat{\psi} + G_2^{-1}(z)\sqrt{\widehat{var(\hat{\psi}^*)}}.$$

Then, the approximate bootstrap-t $(1-\gamma)100\%$ CI of $\hat{\psi} = (\hat{\lambda}, \hat{\beta}, \hat{\theta}, \hat{S}(t) or \hat{h}(t))$, is denoted by

$$\left[\hat{\psi}_{boot-t}(\frac{\gamma}{2}),\ \hat{\psi}_{boot-t}(1-\frac{\gamma}{2})\right]. \tag{21}$$

4 Bayes Estimation Using MCMC

Bayesian estimation hinges on the principle of updating beliefs about parameters in light of observed data. Central to this approach is the use of prior distributions, which encapsulate existing knowledge or assumptions about the parameters before any data is observed. These priors can be informative, reflecting strong prior beliefs, or uninformative, allowing the data to dominate. Through Bayes' theorem, the prior is combined with the likelihood function, representing the data's probability given the parameters, to produce the posterior distribution. This posterior distribution synthesizes prior beliefs with current data, providing a refined estimate of the parameters of interest. Bayesian estimation thus offers a coherent framework for incorporating both subjective beliefs and empirical evidence, allowing for robust inference and decision-making under uncertainty. This approach contrasts with frequentist methods by explicitly quantifying uncertainty and providing a probabilistic interpretation of parameters. By iteratively updating beliefs as more data becomes available, Bayesian estimation offers a flexible and powerful tool for a wide range of statistical applications.

The priors of the parameters must be chosen appropriately for the Bayesian deduction. According to Arnold and Press [38], there is definitely no way to conclude that one prior is superior to another from a strictly Bayesian viewpoint. One must presumably accept the lumps and bumps of one's subjective past and live with it. However, it is preferable to employ the informative prior(s), which may be chosen over all other options, if we have sufficient information on the parameter(s).

If not, using ambiguous or non-informative priors might be appropriate; refer to Upadhyay et al. [39]. As can be shown in Kundu and Howlader [40], the family of gamma distributions is recognized to be sufficiently adaptable to accommodate a wide range of prior assumptions held by the experimenter. Thus, we consider that the unknown parameters λ, β, and θ are stochastically independently distributed with a conjugate gamma prior distributions as follows

$$\pi_1(\lambda) = \frac{b_1^{a_1}}{\Gamma(a_1)}\lambda^{(a_1-1)}e^{-b_1\lambda} \quad , \lambda, a_1, b_1 > 0,$$
$$\pi_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{(a_2-1)}e^{-b_2\beta} \quad , \beta, a_2, b_2 > 0,$$
$$\pi_3(\theta) = \frac{b_3^{a_3}}{\Gamma(a_3)}\theta^{(a_3-1)}e^{-b_3\theta} \quad , \theta, a_3, b_3 > 0,$$

where, a_i and $b_i, i = 1,2,3$ are assumed to be known and non-negative. The joint prior function of the parameters λ, β, and θ is defined as

$$\pi(\lambda,\beta,\theta) = \frac{b_1^{a_1} b_2^{a_2} b_3^{a_3}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\lambda^{(a_1-1)}\beta^{(a_2-1)}\theta^{(a_3-1)}e^{-b_1\lambda-b_2\beta-b_3\theta} \tag{22}$$

The Bayes' theorem is created by multiplying the likelihood function(6) with the joint prior distribution (23). Thus, the posterior density function of λ, β and θ is defined using Bayes' theorem as follows:

$$\begin{aligned}\pi^*(\lambda,\beta,\theta|\underline{y}) &= \frac{L(\underline{y};\lambda,\beta,\theta)\times\pi(\lambda,\beta,\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\underline{y};\lambda,\beta,\theta)\times\pi(\lambda,\beta,\theta)d\lambda d\beta d\theta} \\ &= \lambda^{m+a_1-1}\theta^{m+a_3-1}\beta^{a_2-1}\prod_{i=1}^{m}\left[\begin{array}{l}(e^{\beta y_i}-1)^{\theta-1}e^{\beta y_i}e^{-\frac{\lambda}{\beta}(e^{\beta y_i}-1)^\theta} \\ \times e^{-R_i\frac{\lambda}{\beta}(e^{\beta y_i}-1)^\theta}e^{-b_1\lambda-b_2\beta-b_3\theta}\end{array}\right] \\ &= \pi_1^*\left(\lambda|\beta,\theta,\underline{y}\right)\pi_2^*\left(\beta|\lambda,\theta,\underline{y}\right)\pi_3^*\left(\theta|\lambda,\beta,\underline{y}\right),\end{aligned} \tag{23}$$

where,

$$\pi_1^*\left(\lambda|\beta,\theta,\underline{y}\right) \propto \lambda^{m+a_1-1}e^{-\lambda\left[b_1+\frac{1}{\beta}\Sigma\left(e^{\beta y_i}-1\right)^\theta(1+R_i)\right]}, \tag{24}$$

$$\pi_2^*\left(\beta|\lambda,\theta,\underline{y}\right) \propto \beta^{a_2-1}e^{-\beta[b_2-\Sigma y_i]+(\theta-1)\Sigma\log\left(e^{\beta y_i}-1\right)-\frac{\lambda}{\beta}\Sigma\left(e^{\beta y}-1\right)^\theta(1+R_i)}, \tag{25}$$

$$\pi_3^*\left(\theta|\lambda,\beta,\underline{y}\right) \propto \theta^{m+a_3-1}e^{-\theta\left[b_3-\Sigma\log\left(e^{\beta y}-1\right)\right]-\frac{\lambda}{\beta}\Sigma\left(e^{\beta y}-1\right)^\theta(1+R_i)}. \tag{26}$$

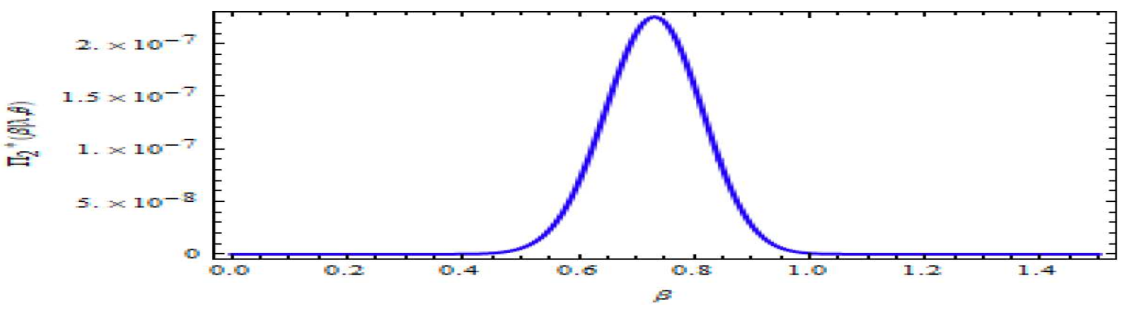

Fig. 3: The posterior density function $\pi_2^*\left(\beta|\lambda,\theta,\underline{y}\right)$ of β

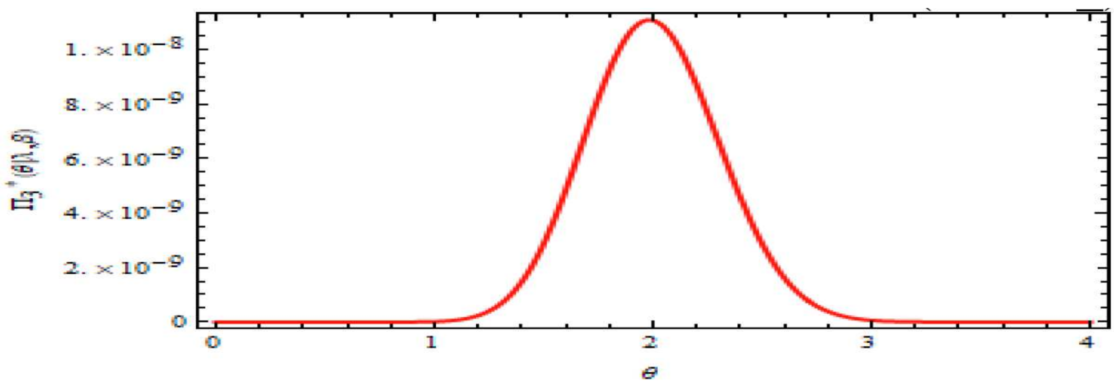

Fig. 4: The posterior density function $\pi_3^*\left(\theta|\lambda,\beta,\underline{y}\right)$ of θ

It's clear that the conditional posterior densities of λ given in (25) is gamma density with shape parameter$(m+a_1)$ and scale parameter $(b_1+\frac{1}{\beta}\sum(e^{\beta y}-1)^{\theta}(1+R_i))$. As a result, samples of λ can be easily generated using any gamma generating routine. Additionally, since β and θ in (26) and (27) do not give standard forms, but the plot of both of them indicates that they are similar to the normal distribution, see Figs. 3 and 4. Gibbs sampling is not an appropriate option. Using the Metropolis-Hasting (M-H) sampler is the most practical choice for the use of the MCMC methodology which is suggested by Metropolis [41]. Below is a hybrid algorithm that updates the parameter λ using Gibbs sampling steps and updates β and θ using M-H steps. We used the MLEs of $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\theta}$ to execute the Gibbs sampler algorithm. If a systematic pattern of convergence wasn't obtained, we took samples from each of the full conditionals, in turn, using the most recent values for all other conditioning variables. The following steps show the Gibbs sampling method's use of the Metropolis-Hastings algorithm, see Tierney [42]:

Step1:Start with $(\lambda^{(0)},\beta^{(0)},\theta^{(0)})=(\hat{\lambda},\hat{\beta},\hat{\theta})$.
step2:Set $j=1$.
step3:Generate $\lambda^{(j)}$ from Gamma$(m+a_1,\,b_1+\frac{1}{\beta}\sum(e^{\beta y}-1)^{\theta}(1+R_i))$.
step4:Using the following M-H algorithm, generate $\beta^{(j)}$ and $\theta^{(j)}$ from π_1^* $(\beta^{(j-1)}|\lambda^{(j)},\theta^{(j-1)},y)$ and π_2^* $(\theta^{(j-1)}|\beta^{(j)},\lambda^{(j)},y)$ with the normal proposal distributions $N(\beta^{(j-1)},var(\beta))$ and $N(\theta^{(j-1)},var(\theta))$.
(i)Generate a proposal β^* from $N(\beta^{(j-1)},var(\beta))$ and θ^* from $N(\theta^{(j-1)},var(\theta))$.
(ii)Evaluate the acceptance probabilities:

$$\eta_\beta=\min\left[1,\frac{\pi_2^*(\beta^*|\lambda^j,\theta^{j-1},\underline{y})}{\pi_2^*(\beta^{j-1}|\lambda^j,\theta^{j-1},\underline{y})}\right],\quad \eta_\theta=\min\left[1,\frac{\pi_3^*(\theta^*|\lambda^j,\beta^j,\underline{y})}{\pi_3^*(\theta^{j-1}|\lambda^j,\beta^j,\underline{y})}\right]. \tag{27}$$

(iii)Generate a u_1 and u_2 from a Uniform $(0,1)$.
(iv)If $u_1<\eta_\beta$, accept the proposal and set $\beta^{(j)}=\beta^*$, else set $\beta^{(j)}=\beta^{(j-1)}$.
(v)If $u_2<\eta_\theta$, accept the proposal and set $\theta^{(j)}=\theta^*$, else set $\theta^{(j)}=\theta^{(j-1)}$.
step5:Compute the reliability function, hazard function and coefficient of variation as

$$\begin{aligned}S^j(t)&=e^{-\frac{\lambda^j}{\beta^j}(e^{\beta^j t}-1)^{\theta^j}}\ ,\ t>0\\ h^j(t)&=\lambda^j\theta^j(e^{\beta^j t}-1)^{\theta^j-1}e^{\beta^j t}\ ,\ t>0\end{aligned} \tag{28}$$

step6:Set $j=j+1$.
step7:Repeat Steps (3)–(6) N times.

The first M simulated variants are deleted in order to ensure convergence and eliminate the affection of initial value selection. The chosen sample thus includes $\lambda^{(j)},\beta^{(j)},\theta^{(j)},S^{(j)}(t)$ and $h^{(j)}(t)$, $j=M+1,...,N$, for sufficiently large N, this sample provides an approximate posterior sample that may be used to develop the Bayes estimates of $\phi=\lambda,\beta,\theta,S(t)$ or $h(t)$ as:

$$\hat{\phi}_{MC}=\frac{1}{N-M}\sum_{j=M+1}^{N}\phi^j.$$

To determine the reliable intervals of λ, β, θ, $S(t)$ and $h(t)$, order $\lambda^{(j)}, \beta^{(j)}, \theta^{(j)}, S^{(j)}(t)$ and $h^{(j)}(t)$, $i = 1, ..., N$ as $\{\lambda^{(1)} < \cdots < \lambda^{(N)}\}$, $\{\beta^{(1)} < \cdots < \beta^{(N)}\}$, $\{\theta^{(1)} < \cdots < \theta^{(N)}\}$, $\{S^{(1)}(t) < \cdots < S^{(N)}(t)\}$ and $\{h^{(1)}(t) < \cdots < h^{(N)}(t)\}$. Therefore, the $100(1-\gamma)\%$ CRIs of $\phi = \lambda, \beta, \theta, S(t)$ or $h(t)$ become

$$\left[\phi_{(N\gamma/2)}\,,\,\phi_{(N(1-\gamma/2))}\right]. \tag{29}$$

5 Simulation Study

We provide a simulation study in this section in order to compare the performance of the estimates and confidence intervals created in the previous sections using various techniques. Here, we present the outcomes of the simulation in the case of $(\lambda, \beta, \theta) = (0.22, 1, 2)$. Consequently, the actual values of S(t) and h(t) at time $t = 0.4$ are evaluated to be 0.94818 and 0.32284, respectively. The mean square error (MSE), is used to evaluate the performance of estimators which calculated as:

$$MSE = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{\varphi}_k^i - \varphi_k\right)^2,$$

where, $k = 1, 2, ..., 5$, $\varphi_1 = \lambda, \varphi_2 = \beta, \varphi_3 = \theta, \varphi_4 = S(t)$,and $\varphi_5 = h(t)$ for the point estimates, additionally, for interval

estimations, average lengths (ALs) and coverage probability (CPs), which are calculated as the number of CIs that covered the true values divided by 1000. Based on $M = 12\,000$ MCMC samples, Bayes estimates and the highest posterior density CRIs are calculated, and the first values, $M_\circ = 2000$, are discarded as burn-in. Additionally, we consider the informative gamma priors for λ, β, and θ that is, when the hyperparameters are $a_i = 2$ and $b_i = 2$, $i = 1, 2, 3$. Furthermore, for every simulated sample, 95% CRIs were calculated. We take into consideration various sample sizes (n = 30, 50, and 80) in our study. The censoring schemes (SCs) are as follows:

SC1: $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.
SC2: $R_{(m+1)/2} = n - m, R_i = 0$ for $i \neq (m+1)/2$ if m odd,
$R_{m/2} = n - m, R_i = 0$ for $i \neq m/2$ if m even.
SC3: $R_m = n - m$, $R_i = 0$ for $i \neq m$.

Tables $1-6$ show the outcomes of the means, MSEs, ALs, and CPs of the estimates. Based on the results, we note the following:

– As expected, tables $1-6$ shown that: as sample size n increases, the MSEs and ALs decrease.

– Because Bayes estimates have the least MSEs and ALs for all parameters and reliability characteristics, they perform better than MLEs and bootstrap methods.

– In terms of MSEs and ALs, bootstrap methods outperform the ML method. In terms of MSEs and ALs, BT outperforms BP as well.

– The maximum posterior density credible intervals and coverage probability of the asymptotic confidence intervals are all close to the target level of 0.95.

Table 1: MSE of estimates for the parameters λ and β.

		λ				β			
(n,m)	CSs	ML	Boot-p	Boot-t	Bayes	ML	Boot-p	Boot-t	Bayes
$(30,15)$	SC1	0.0174	0.0157	0.0123	0.0099	0.2354	0.2147	0.1925	0.1655
	SC2	0.0196	0.0185	0.0146	0.0112	0.2792	0.2564	0.2258	0.1892
	SC3	0.0235	0.0214	0.0178	0.0135	0.3157	0.2841	0.2543	0.2144
$(30,20)$	SC1	0.0144	0.0128	0.0106	0.0082	0.1925	0.1905	0.1622	0.1347
	SC2	0.0165	0.0151	0.0123	0.0101	0.2254	0.2247	0.1936	0.1599
	SC3	0.0194	0.0186	0.0160	0.0119	0.2584	0.2598	0.2243	0.1797
$(50,25)$	SC1	0.0112	0.0102	0.0088	0.0075	0.1699	0.1689	0.1357	0.1148
	SC2	0.0132	0.0133	0.0109	0.0086	0.1865	0.1798	0.1479	0.1301
	SC3	0.0149	0.0147	0.0118	0.0093	0.1996	0.1992	0.1693	0.1497
$(50,35)$	SC1	0.0102	0.0093	0.0079	0.0066	0.1235	0.1199	0.0996	0.0887
	SC2	0.0117	0.0111	0.0087	0.0072	0.1456	0.1398	0.1191	0.0953
	SC3	0.0132	0.0129	0.0098	0.0087	0.1763	0.1752	0.1465	0.1199
$(80,40)$	SC1	0.0092	0.0085	0.0065	0.0051	0.1055	0.1101	0.0854	0.0768
	SC2	0.0102	0.0094	0.0071	0.0062	0.1245	0.1246	0.0935	0.0832
	SC3	0.0125	0.0123	0.0087	0.0074	0.1512	0.1523	0.1169	0.0961
$(80,50)$	SC1	0.0083	0.0081	0.0054	0.0044	0.0975	0.0971	0.0778	0.0625
	SC2	0.0091	0.0089	0.0065	0.0057	0.1112	0.1124	0.0834	0.0714
	SC3	0.0105	0.0104	0.0076	0.0068	0.1299	0.1289	0.0947	0.0835

Table 2: MSE of estimates for the parameter θ and $S(t=0.4)$.

		θ				$S(t=0.4)$			
(n,m)	CSs	ML	Boot-p	Boot-t	Bayes	ML	Boot-p	Boot-t	Bayes
$(30,15)$	SC1	0.4772	0.4563	0.3965	0.3377	0.0245	0.0227	0.0199	0.0179
	SC2	0.5247	0.5224	0.4367	0.3783	0.0276	0.0265	0.0223	0.0199
	SC3	0.5974	0.5836	0.4863	0.4112	0.0299	0.0286	0.0265	0.0227
$(30,20)$	SC1	0.3875	0.3799	0.3247	0.2647	0.0187	0.0185	0.0156	0.0132
	SC2	0.4125	0.4116	0.3665	0.2961	0.0199	0.0196	0.0174	0.0153
	SC3	0.4566	0.4561	0.3947	0.3299	0.0215	0.0211	0.0196	0.0174
$(50,25)$	SC1	0.2978	0.2972	0.2564	0.1998	0.0147	0.0142	0.0125	0.0109
	SC2	0.3214	0.3215	0.2765	0.2211	0.0168	0.0165	0.0146	0.0125
	SC3	0.3561	0.3566	0.3001	0.2504	0.0191	0.0189	0.0170	0.0151
$(50,35)$	SC1	0.2532	0.2530	0.1987	0.1647	0.0112	0.0110	0.0099	0.0087
	SC2	0.2863	0.2859	0.2346	0.1899	0.0136	0.0136	0.0111	0.0095
	SC3	0.3145	0.3151	0.2658	0.2183	0.0156	0.0151	0.0139	0.0112
$(80,40)$	SC1	0.2139	0.2137	0.1697	0.1359	0.0092	0.0091	0.0088	0.0079
	SC2	0.2458	0.2449	0.1963	0.1596	0.0106	0.0104	0.0093	0.0084
	SC3	0.2766	0.2699	0.2237	0.1895	0.0121	0.0119	0.0096	0.0089
$(80,50)$	SC1	0.1684	0.1677	0.1263	0.0997	0.0085	0.0085	0.0078	0.0071
	SC2	0.1855	0.1842	0.1501	0.1285	0.0092	0.0091	0.0083	0.0077
	SC3	0.2151	0.2143	0.1765	0.1491	0.0101	0.0099	0.0089	0.0083

6 Real Data Application

To illustrate the computation of methods proposed in this paper, we cover an example of real life data set. Considering the real data set of sample size 88 observed failure times of windshields reported in Murthy et al. [43], who have obtained it from Blischke and Murthy [44] and used in EL-Sagheer et al. [45], given in Table 7. We have computed the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distribution functions. It is 0.0829 and the associated p-value is 0.5809. Since the p- value is quite high, we cannot reject the null hypothesis that the data is coming from the EGD. In

Table 3: MSE of estimates for $h(t = 0.4)$.

(n,m)	CSs	ML	Boot-p	Boot-t	Bayes
(30,15)	SC1	0.0074	0.0072	0.0069	0.0061
	SC2	0.0079	0.0077	0.0073	0.0064
	SC3	0.0082	0.0081	0.0077	0.0069
(30,20)	SC1	0.0063	0.0061	0.0056	0.0051
	SC2	0.0066	0.0065	0.0059	0.0055
	SC3	0.0073	0.0073	0.0063	0.0058
(50,25)	SC1	0.0051	0.0052	0.0047	0.0042
	SC2	0.0056	0.0055	0.0049	0.0044
	SC3	0.0059	0.0059	0.0052	0.0047
(50,35)	SC1	0.0045	0.0044	0.0038	0.0032
	SC2	0.0048	0.0048	0.0042	0.0036
	SC3	0.0053	0.0052	0.0047	0.0039
(80,40)	SC1	0.0038	0.0038	0.0031	0.0026
	SC2	0.0042	0.0041	0.0035	0.0029
	SC3	0.0045	0.0044	0.0039	0.0033
(80,50)	SC1	0.0033	0.0032	0.0026	0.0021
	SC2	0.0036	0.0036	0.0029	0.0024
	SC3	0.0039	0.0038	0.0033	0.0028

Table 4: ALs (first row) and PCs (second row) of 95% ACIs for λ and β.

		λ				β			
(n,m)	CSs	MLE	Bootstrap		MCMC	MLE	Bootstrap		MCMC
		ACIs	boot-p	boot-t	CRIs	ACI	boot-p	boot-t	CRIs
(30,15)	SC1	1.0124	1.0564	1.0045	0.9953	2.3641	2.2578	2.1473	1.7865
		0.924	0.932	0.934	0.947	0.932	0.947	0.954	0.956
	SC2	1.1356	1.1284	1.0967	1.0364	2.4156	2.3572	2.1689	1.8278
		0.935	0.944	0.939	0.951	0.942	0.934	0.946	0.942
	SC3	1.2365	1.2121	1.1325	1.0993	2.5478	2.4791	2.2397	1.8897
		0.927	0.921	0.928	0.932	0.931	0.922	0.951	0.943
(30,20)	SC1	0.9524	0.9147	0.8832	0.8146	1.9647	1.8547	1.6978	1.4958
		0.938	0.941	0.942	0.953	0.941	0.937	0.949	0.954
	SC2	1.0956	1.0899	0.9564	0.8769	2.1254	1.9684	1.7628	1.5536
		0.937	0.951	0.947	0.945	0.928	0.954	0.941	0.953
	SC3	1.1325	1.1014	1.0963	0.9463	2.2367	2.1479	1.8345	1.6347
		0.932	0.947	0.957	0.961	0.935	0.937	0.919	0.928
(50,25)	SC1	0.9135	0.9055	0.7967	0.7201	1.5789	1.5536	1.3478	0.9998
		0.952	0.947	0.957	0.947	0.951	0.948	0.938	0.944
	SC2	0.9632	0.9499	0.8456	0.7732	1.7473	1.6954	1.5768	1.1973
		0.947	0.959	0.953	0.951	0.941	0.936	0.951	0.957
	SC3	1.0587	0.0124	0.9136	0.8554	1.8997	1.8755	1.6845	1.3014
		0.939	0.945	0.929	0.948	0.923	0.948	0.957	0.929
(50,35)	SC1	0.8635	0.8555	0.7217	0.6648	1.3258	1.3147	1.0478	0.8756
		0.947	0.942	0.938	0.960	0.919	0.927	0.953	0.961
	SC2	0.9258	0.9097	0.7684	0.6999	1.4982	1.5011	1.2347	0.9957
		0.953	0.943	0.954	0.957	0.947	0.937	0.945	0.955
	SC3	0.9638	0.9499	0.8153	0.7456	1.6235	1.5997	1.3451	1.1249
		0.942	0.947	0.961	0.955	0.929	0.951	0.938	0.942
(80,40)	SC1	0.7833	0.7745	0.6221	0.5896	1.1254	1.0354	0.8899	0.7264
		0.939	0.945	0.954	0.951	0.951	0.949	0.938	0.945
	SC2	0.8377	0.8256	0.6791	0.6337	1.3217	1.2789	0.9347	0.7945
		0.942	0.937	0.941	0.946	0.945	0.941	0.937	0.951
	SC3	0.8799	0.8767	0.7144	0.6823	1.4386	1.3997	1.1243	0.8596
		0.948	0.947	0.951	0.939	0.927	0.928	0.933	0.941
(80,50)	SC1	0.6956	0.7021	0.5792	0.4999	0.9948	0.9275	0.7514	0.6347
		0.936	0.941	0.939	0.942	0.938	0.947	0.954	0.938
	SC2	0.7462	0.7369	0.6247	0.5438	1.1214	1.0998	0.8947	0.7734
		0.941	0.938	0.942	0.956	0.942	0.937	0.951	0.942
	SC3	0.7861	0.7794	0.6695	0.5923	1.2536	1.1974	0.9635	0.8347
		0.944	0.952	0.947	0.949	0.925	0.931	0.929	0.938

Table 5: ALs (first row) and PCs (second row) of 95% ACIs for θ and $S(t=0.4)$.

		θ				$S(t=0.4)$			
(n,m)	CSs	MLE	Bootstrap		MCMC	MLE	Bootstrap		MCMC
		ACIs	boot-p	boot-t	CRIs	ACI	boot-p	boot-t	CRIs
$(30,15)$	SC1	3.5546	3.4478	3.2144	2.6245	0.3365	0.3258	0.2786	0.2247
		0.921	0.919	0.927	0.949	0.929	0.949	0.951	0.939
	SC2	3.6471	3.5481	3.3692	2.7633	0.3564	0.3482	0.2963	0.2495
		0.936	0.942	0.939	0.951	0.939	0.945	0.942	0.939
	SC3	3.7894	3.6892	3.4389	2.8641	0.3741	0.3599	0.3214	0.2763
		0.941	0.933	0.942	0.950	0.942	0.938	0.947	0.951
$(30,20)$	SC1	3.1458	3.0587	2.8762	2.5746	0.2978	0.2964	0.2345	0.1865
		0.947	0.951	0.939	0.939	0.941	0.939	0.951	0.953
	SC2	3.3124	3.2542	2.9945	2.6341	0.3158	0.3214	0.2539	0.1994
		0.943	0.945	0.952	0.942	0.929	0.926	0.937	0.948
	SC3	3.5642	3.4571	3.1255	2.8012	0.3451	0.3399	0.2785	0.2211
		0.938	0.936	0.928	0.947	0.941	0.948	0.958	0.939
$(50,25)$	SC1	2.7944	2.6548	2.3147	1.9948	0.2566	0.2497	0.1998	0.1599
		0.947	0.934	0.937	0.923	0.941	0.919	0.928	0.942
	SC2	2.8974	2.7999	2.5463	2.2657	2.7652	0.2745	0.2098	0.1674
		0.928	0.941	0.933	0.936	0.919	0.921	0.938	0.952
	SC3	3.1247	3.0658	2.6994	2.3996	0.2886	0.2877	0.2258	0.1799
		0.931	0.925	0.943	0.945	0.928	0.918	0.936	0.939
$(50,35)$	SC1	2.4582	2.3044	1.9889	1.7659	0.2158	0.2098	0.1768	0.1397
		0.917	0.919	0.928	0.928	0.942	0.951	0.924	0.938
	SC2	2.5993	2.5576	2.1589	1.8734	0.2369	0.2366	0.1876	0.1496
		0.923	0.941	0.936	0.941	0.943	0.933	0.938	0.952
	SC3	2.6887	2.6542	2.2967	1.9777	0.2499	0.2491	0.2055	0.1594
		0.929	0.927	0.947	0.953	0.942	0.938	0.951	0.947
$(80,40)$	SC1	2.1799	2.1543	1.7983	1.4658	0.1954	0.1866	0.1536	0.1149
		0.941	0.939	0.945	0.951	0.933	0.951	0.935	0.943
	SC2	2.3657	2.3147	1.8576	1.5016	0.2145	0.2095	0.1736	0.1357
		0.938	0.942	0.928	0.934	0.934	0.947	0.939	0.942
	SC3	2.5643	2.4986	1.9654	1.6954	0.2364	0.2287	0.1896	0.1469
		0.941	0.945	0.955	0.958	0.941	0.928	0.939	0.947
$(80,50)$	SC1	1.8653	1.7965	1.5473	1.1369	0.1754	0.1698	0.1394	0.1009
		0.946	0.947	0.939	0.948	0.939	0.947	0.943	0.951
	SC2	1.9965	1.9877	1.6985	1.3012	0.1965	0.1894	0.1547	0.1237
		0.938	0.939	0.941	0.944	0.942	0.941	0.939	0.951
	SC3	2.1354	2.0584	1.7564	1.4583	0.2047	0.1999	0.1756	0.1366
		0.946	0.949	0.951	0.947	0.947	0.939	0.951	0.948

addition, we plot both the empirical survival function (ESF) and the estimated survival functions in Figure 5 and we found that the EGD fits the data very well.

Under the previous data, a progressively censored sample generated from the real data is sample of size $m=40$ taken from sample size $n=88$ with censoring scheme R =(10, 7, 4, 5, 0, 4,5, 0, 3, 0, 0, 2, 0, 2, 0, 0, 1, 0,1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). A progressively censored sample generated from the real data is:

0.04	0.301	0.309	0.557	0.943	1.07	1.124	1.248	1.281	1.281
1.303	1.432	1.48	1.505	1.506	1.568	1.615	1.619	1.652	1.652
1.757	1.795	1.866	1.876	1.899	1.911	2.085	2.097	2.135	2.964
3.000	3.103	3.114	3.117	4.035	4.121	4.167	4.24	4.255	4.694

Table 6: ALs (first row) and PCs (second row) of 95% ACIs for $h(t=0.4)$.

(n,m)	CSs	MLE ACI	Bootstrap boot-p	Bootstrap boot-t	MCMC CRIs
$(30,15)$	SC1	0.2314	0.2305	0.2147	0.1468
		0.939	0.952	0.949	0.951
	SC2	0.2468	0.2463	0.2236	0.1587
		0.943	0.959	0.947	0.948
	SC3	0.2631	0.2597	0.2314	0.1601
		0.929	0.939	0.945	0.941
$(30,20)$	SC1	0.1835	0.1792	0.1347	0.1125
		0.951	0.940	0.942	0.952
	SC2	0.1955	0.1893	0.1462	0.1301
		0.929	0.939	0.953	0.961
	SC3	0.2102	0.2107	0.1536	0.1399
		0.947	0.951	0.948	0.949
$(50,25)$	SC1	0.1534	0.1498	0.1026	0.0956
		0.957	0.952	0.948	0.961
	SC2	0.1674	0.1536	0.1235	0.1128
		0.933	0.929	0.947	0.937
	SC3	0.1801	0.1797	0.1401	0.1265
		0.943	0.924	0.957	0.951
$(50,35)$	SC1	0.1265	0.1199	0.0997	0.0821
		0.953	0.947	0.954	0.956
	SC2	0.1368	0.1294	0.1167	0.0999
		0.946	0.941	0.933	0.942
	SC3	0.1523	0.1475	0.1296	0.1102
		0.947	0.951	0.939	0.938
$(80,40)$	SC1	0.1058	0.1044	0.0814	0.0735
		0.931	0.929	0.927	0.954
	SC2	0.1167	0.1155	0.0934	0.0819
		0.945	0.939	0.947	0.955
	SC3	0.1310	0.1305	0.1099	0.0928
		0.947	0.942	0.957	0.947
$(80,50)$	SC1	0.0987	0.0976	0.0864	0.0699
		0.951	0.941	0.928	0.960
	SC2	0.1025	0.1011	0.0947	0.0754
		0.937	0.939	0.951	0.942
	SC3	0.1201	0.1189	0.1051	0.0835
		0.946	0.937	0.947	0.953

We derive the point estimates (ML, BP, and BT) as well as the corresponding 95% ACIs of λ, β, θ, $S(t)$ and $h(t)$. The results are showed in Table 9 and 10. The MCMC results of the posterior mean, median, mode, standard deviation (SD) and skewness (Ske) of λ, β, θ, $S(t)$ and $h(t)$ are represented in Table 11.

Table 7: Real life data of failure times of windsheild.

0.040	0.301	0.309	0.557	0.943	1.070	1.124	1.248	1.281	1.281	1.303
1.432	1.480	1.505	1.506	1.568	1.615	1.619	1.652	1.652	1.757	1.795
1.866	1.876	1.899	1.911	1.912	1.914	1.981	2.010	2.038	2.085	2.085
2.097	2.135	2.154	2.190	2.194	2.223	2.224	2.229	2.300	2.324	2.349
2.385	2.481	2.610	2.625	2.632	2.646	2.661	2.688	2.823	2.890	2.902
2.934	2.962	2.964	3.000	3.103	3.114	3.117	3.166	3.344	3.376	3.385
3.443	3.467	3.478	3.578	3.595	3.699	3.779	3.924	4.035	4.121	4.167
4.240	4.255	4.278	4.305	4.376	4.449	4.485	4.570	4.602	4.663	4.694

Table 8: The KS, p-value and estimates of the EGD's parameters.

Distribution	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	KS Statistics	p-value
EGD	0.4855	0.0946	1.3577	0.0829	0.5809

Table 9: 95% Confidence intervals for parameters $\lambda, \beta, \theta, S(t)$ and $h(t)$.

Method	λ	β	θ
ML	{0.0235, 9.4821}	{0.0091, 3.3219}	{0.8295, 2.5807}
Boot-p	{0.1598, 2.454}	{0.0248, 0.7469}	{0.9347, 1.9525}
Boot-t	{0.1217, 2.401}	{0.0285, 0.9643}	{0.7798, 2.018}
MCMC	{0.0834, 0.2388}	{0.3974, 0.9522}	{0.7599, 1.3326}

Method	$S(t)$	$h(t)$
ML	{0.8907, 0.9971}	{0.0828, 0.3545}
Boot-p	{0.8837, 0.9788}	{0.1056, 0.3898}
Boot-t	{0.892, 0.9812}	{0.0993, 0.3786}
MCMC	{0.8002, 0.9855}	{0.0543, 0.3142}

Table 10: ML, bootstrap and Bayes estimates

Parameter	MLE	Bootstrap		Bayesian
		BP	BT	SE
λ	0.4716	0.7912	0.7618	0.1506
β	0.1737	0.2163	0.2476	0.6462
θ	1.4631	1.4557	1.432	1.0213
$S(t)$	0.9439	0.94	0.9396	0.9269
$h(t)$	0.2186	0.2235	0.2222	0.1751

Table 11: MCMC results for λ, β, θ, $S(t)$ and $h(t)$.

Parameters	Mean	Median	Mode	SD	Ske
λ	0.1506	0.1467	0.1389	0.0398	0.6028
β	0.6462	0.6418	0.6329	0.1408	0.3372
θ	1.0213	1.0179	1.0112	0.1444	0.2827
$S(t)$	0.9269	0.938	0.96	0.0503	-3.0475
$h(t)$	0.1751	0.1708	0.1621	0.0693	4.1016

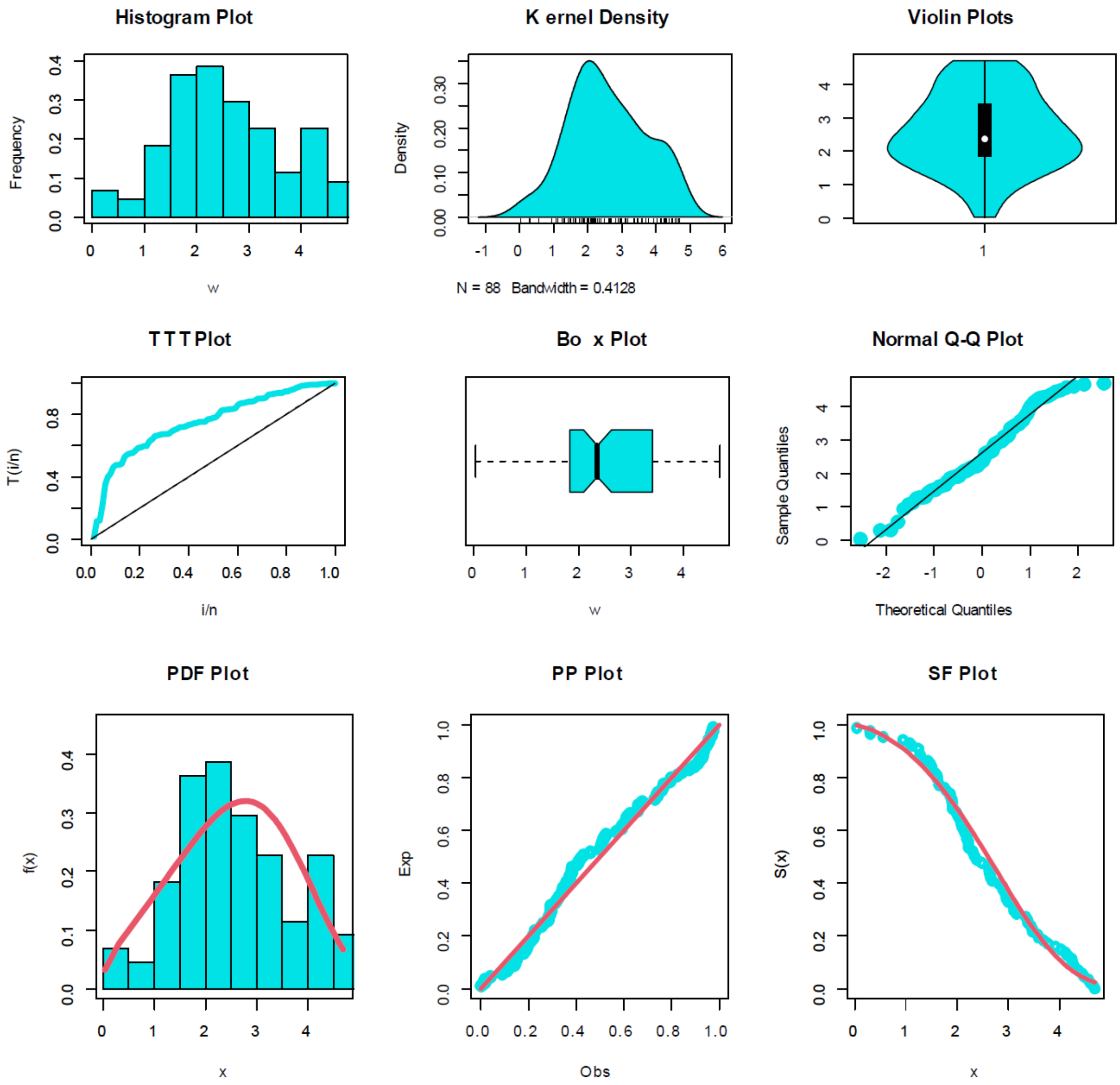

Fig. 5: The empirical and fitted survival functions for the real data.

7 Conclusion

In this study, we developed multiple techniques for estimating and constructing confidence intervals for parameters, the hazard function, and the reliability function of the extended Gompertz distribution under progressively Type-II censored samples. Maximum likelihood estimators are computed for the unknown parameters, and several confidence intervals are proposed using asymptotic distributions and parametric bootstrap methods. Bayesian estimation methods are also explored, revealing that explicit Bayes estimators are challenging to derive but can be obtained via numerical integration methods like Markov chain Monte Carlo . Notably, informative priors significantly enhance the performance of Bayes estimates in this context. Furthermore, balanced loss functions are employed for Bayes estimation. The theoretical findings are exemplified through a numerical example, and a simulation study is conducted to evaluate and compare the methodologies across various sample sizes (n, m) and censoring schemes (I, II, III). The findings show the following:

1. Tables 1, 2, and 3 show that as sample size increases, MSEs decrease, with Bayes estimates having the smallest MSEs for $\lambda, \beta, \theta, S(t)$ and $h(t)$. As a result, Bayesian estimates outperform MLEs and bootstrap methods in all cases.
2. From Tables 1, 2, and 3. Bootstrap-t outperforms percentile bootstrap and MLEs, because bootstrap-t have the MSEs smaller than MSEs in percentile bootstrap and MLEs for $\lambda, \beta, \theta, S(t)$ and $h(t)$.

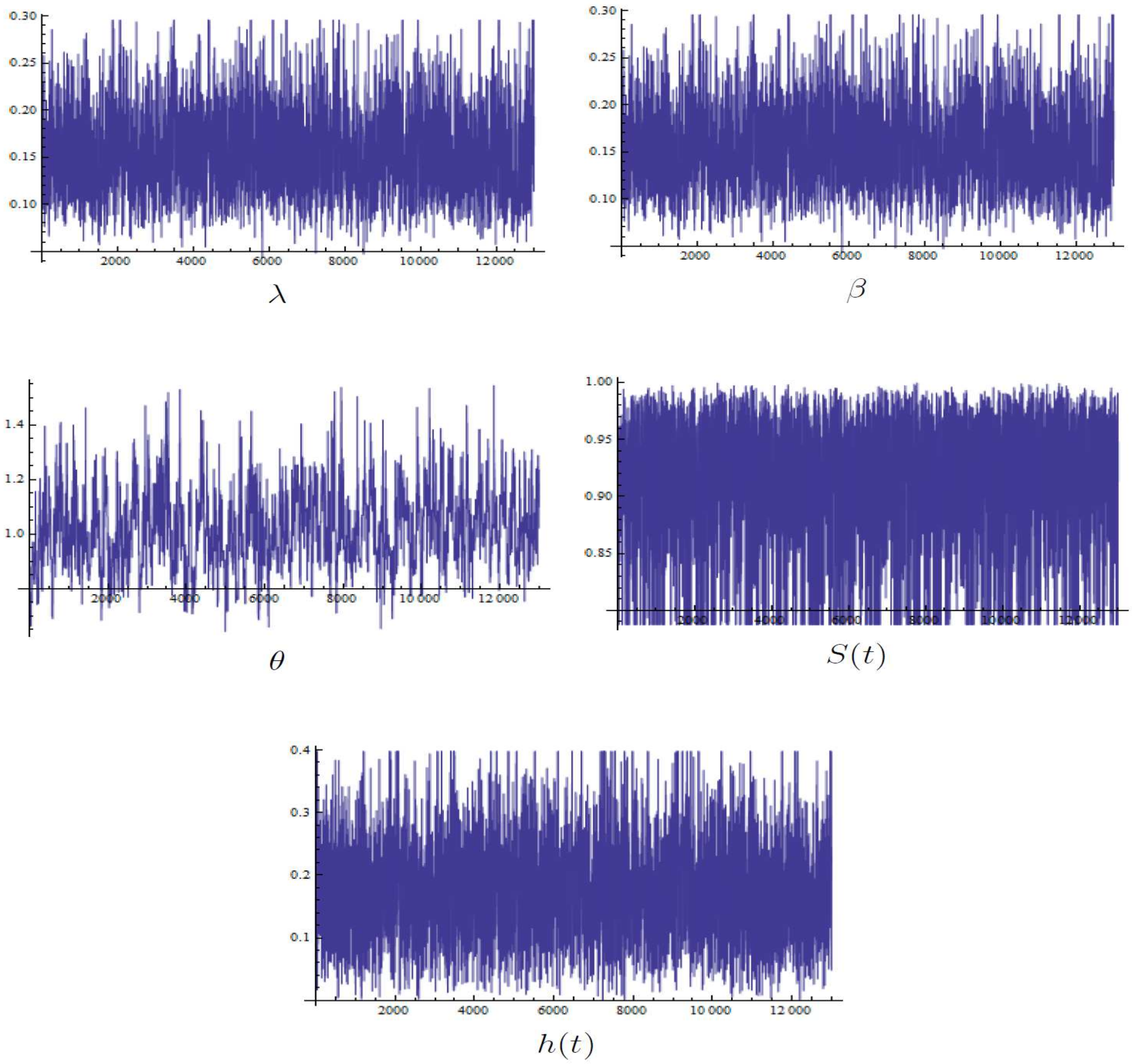

Fig. 6: Traceplots of $\lambda, \beta, \theta, S(t)$ and $h(t)$ obtained from MCMC.

3.For fixed values of the sample n and failure time sizes m, scheme I outperforms schemes II and III in terms of smaller MSEs.
4.From Tables 4, 5, and 6. It can be noticed that the MCMC CRIs produce more accurate results than the approximate CIs and bootstrap CIs since the lengths of the former are less than the lengths of the latter, for various sample sizes, observed failures and schemes.
5.The bootstrap-t CIs is better than the percentile bootstrap CIs and ACIs regarding to have smaller widths.
6.From Table 9, we notice that:
(a)ML estimates have the widest confidence intervals, reflecting high uncertainty.
(b)Bootstrap methods (Boot-p, Boot-t) reduce variability, offering more stable estimates compared to ML.
(c)MCMC provides the narrowest intervals, suggesting Bayesian estimation is more confident and less variable.
(d)For $S(t)$ and $h(t)$, MCMC tends to be more conservative, allowing for lower survival probabilities and hazard rates.
7.From Table 10. It can be noticed that:
(a)MLE and Bootstrap estimates are relatively close, indicating good agreement.
(b)Bayesian estimates tend to be lower than the MLE and bootstrap estimates, particularly for λ, θ, and $h(t)$.
(c)The Bayesian estimate for β is much higher, which may indicate sensitivity to prior assumptions.

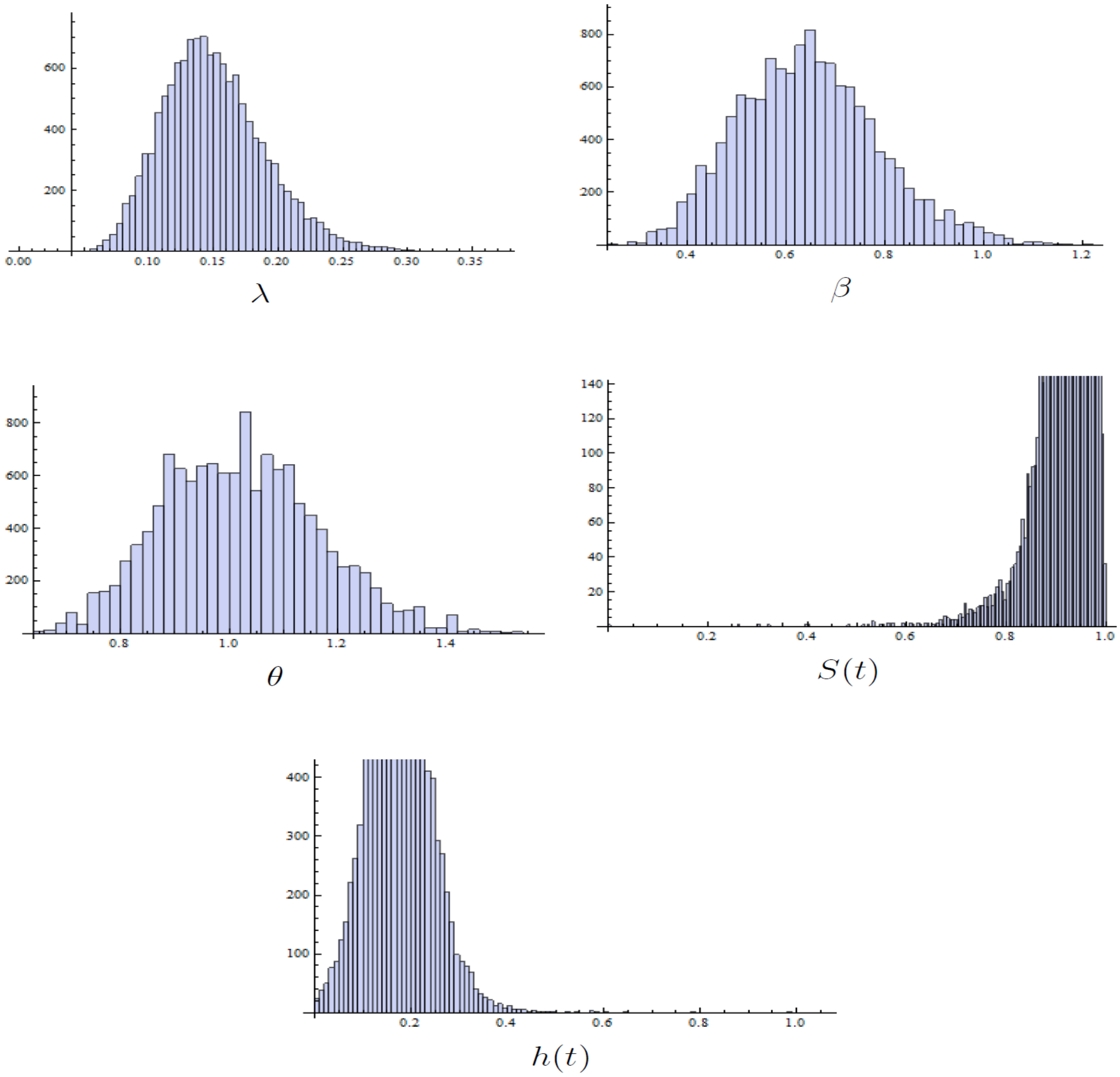

Fig. 7: Histograms of λ, β, θ, $S(t)$ and $h(t)$ obtained from MCMC.

8.Based on Table 11 and Figures 6 and 7, we can conclude the following:

(a)The parameters λ, β, and θ have nearly symmetric distributions, with slight positive skewness and low variability.
(b)$S(t)$ is highly negatively skewed, meaning most estimates are high but with some much lower values.
(c)$h(t)$ is highly positively skewed, meaning most estimates are low but with some much higher values.
(d)The standard deviation is small for all parameters, indicating stable estimates from the MCMC method.

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