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Exact Posterior Inferences for the Odds Ratio in the Binomial–Kumaraswamy Model

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Abstract: The class of contingency 2×2 tables is an important tool for checking the association between two qualitative variables. Among the several measures of association, the odds ratio is perhaps the most prominent due to its elegant mathematical properties. A common Bayesian model for the odds ratio uses the Beta-Binomial model, which is conjugate, in the sense that the posterior distribution is also a Beta distribution. Although the posterior inferences are exact, the Beta distribution could be replaced by other distributions within the interval (0,1), such as the Kumaraswamy, which has been extensively used in the past few decades. However, the Kumaraswamy-Binomial model is not conjugate, which would require the use of approximate methods. In this work, we show that we can obtain the posterior inferences for the odds ratio in an exact form; that is, we provide explicit and computable forms for the posterior distribution and its quantities. An application is provided comparing cancer screening tests.

Keywords: Exact Bayesian computation, Contingency tables, Beta-Binomial Model, Measures of association.

1 Introduction

The 2×2 contingency tables are important for assessing the relationship between two categorical variables. To quantify this association, there are several measures available: *the odds ratio*, *contingency coefficient*, ϕ *coefficient*, etc. However, the *odds ratio* is one of the most important measures of association due to its friendly mathematical properties. For instance, in logistic regression, it is naturally derived from the regression coefficients. In Bayesian analysis, the most commonly used model for addressing 2×2 tables is the Beta-Binomial model, primarily due to its *conjugacy* property. That is, given a random sample from a Bernoulli distribution with parameter θ, where $\theta \sim \text{Beta}(a,b)$ (with a,b as hyperparameters), the posterior distribution of θ is also a Beta distribution, allowing posterior quantities to be obtained in exact form. Due to this conjugacy, several quantities can be derived. [1] provided the posterior distribution of the *risk ratio*, [11] obtained the exact posterior distribution of the *odds ratio*, and [2] provided estimates for several measures of association.

While the Beta-Binomial model offers great convenience, it comes with the limitation that only the Beta distribution can be used. In cases where this restriction is unsuitable, approximate methods such as MCMC, Laplace approximation, and variational Bayes, among others, must be employed. We should not be constrained to a single family of distributions, as the prior information should represent some relevant and genuine knowledge about the parameter of interest. However, also convenient, the use of exact posterior distributions can make inferences much more straightforward, eliminating the need for simulation algorithms.

An approach that seeks to combine the benefits of exact computation with the flexibility of choosing a wide range of prior distributions was proposed by [6]. They used the theory of *special functions* to derive the exact form of the posterior distribution of a scale parameter in non-conjugate models. In other words, the posterior distribution and its quantities (moments, predictions, etc.) are explicitly written in a computable form. Their work also considered more complex models involving location and location-scale structures ([7,8]). Additionally, [4] provided the exact form of all posterior quantities for the Kumaraswamy-Binomial model, a non-conjugate model. [9] provided exact inferences for the risk ratio of 2×2 contingency tables modeled as Kumaraswamy-Binomial. In this work, we use the same model to provide exact posterior inferences for the odds

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ratio. Note that this introduces additional complexity, as the odds ratio is a ratio of the odds of the parameters, whereas the odds themselves are, in general, probability ratios involving one single parameter.

In Section 2, we provide some preliminary definitions and results useful for the theory. In Section 3, the exact posterior is presented. An illustrative example is given in Section 4, where we compare the results with those obtained by MCMC. A few general comments are made in Section 5.

2 Preliminaries

Consider the Kumaraswamy–Binomial model

$$\begin{cases} y_1,\ldots,y_n|\theta \sim \mathrm{Ber}(\theta)\ iid \\ \theta \sim \mathrm{Kum}(\alpha,\beta), \end{cases}$$

where $0<\theta<1$ is the parameter of interest, α,β are hyperparameters and $\mathrm{Kum}(\alpha,\beta)$ stands for the Kumaraswamydistribution $p(\theta)=\alpha\beta\theta^{\alpha-1}(1-\theta^{\alpha})^{\beta-1}$. [4] gives an exact expression for the *unnormalized posterior moment* of order r, given by

$$I_y(r)=\alpha\beta\Gamma(\beta)\Gamma(n-y+1)\times \sum_{h=0}^{\infty}\frac{(-1)^h\Gamma(r+y+\alpha(h+1))}{h!\Gamma(r+n+\alpha(h+1)+1)\Gamma(\beta-h)}, \tag{1}$$

where $\alpha>1$ and $y=\sum_i y_i$.

An important integral is given by the following lemma.

Lemma 1. *For $\delta\in\mathbb{R}$, $0<t<1$ and $\alpha,\gamma,\beta>0$, the following integral can be expressed in terms of infinity series.*

$$I=\int_0^t \frac{w^{\alpha-1}}{(1+\beta w^{\gamma})^{\delta+1}}dw=\sum_{h=0}^{\infty}\frac{(\delta+1)_h}{h!}\frac{(-\beta)^h t^{\alpha+\gamma h}}{\alpha+\gamma h}, \tag{2}$$

where $(\delta+1)_h$ is the Pochhammer notation $(\delta+1)_h=(\delta+1)(\delta+2)\ldots(\delta+h)=\Gamma(h+\delta+1)/\Gamma(\delta+1)$.

Proof. Let $\theta=-\beta w^{\gamma}$, then

$$I=\frac{1}{\gamma}\left(\frac{-1}{\beta}\right)^{\alpha/\gamma}\int_0^{-\beta t^{\gamma}}\theta^{\frac{\alpha}{\gamma}-1}\frac{1}{(1-\theta)^{\delta+1}}d\theta,$$

applying the Binomial Theorem, we have that $(1-\theta)^{-(\delta+1)}=\sum_{h=0}^{\infty}\theta^h(\delta+1)_h/h!$. The result follows by solving the integral and simplifying the terms.

3 Posterior distribution of the odds ratio

In general, the basic structure of a 2×2 table considers an experiment where n_1 and n_2 subjects are randomly allocated into the groups 1 and 2, say treatment 1 and 2, the responses to the treatments are marked as "Success/Failure" (1/0), thus we have

Group	Response	
	1	0
Treat. 1	θ_1	$1-\theta_1$
Treat. 2	θ_2	$1-\theta_2$

which is modeled as

$$\begin{cases} x_1,\ldots,x_{n_1}|\theta_1 \sim \mathrm{Ber}(\theta_1)\ iid \\ y_1,\ldots,y_{n_2}|\theta_2 \sim \mathrm{Ber}(\theta_2)\ iid \\ \theta_j \sim \mathrm{Kum}(\alpha_j,\beta_j),\ j=1,2 \end{cases}. \tag{3}$$

Considering the probabilities of success within each group are $P(X_i=1|\theta_1)=\theta_1$ $(i=1,\cdots,n_1)$ and $P(Y_j=1|\theta_2)=\theta_2$ $(j=1,\cdots,n_2)$, we model the table above as (3). Note that (X_i,X_j) are independent, hence (θ_1,θ_2) as well. Also, α_j,β_j are hyperparameters for the Kumaraswamydistribution $p_j(\theta_j)=\alpha_j\beta_j\theta^{\alpha_j-1}(1-\theta^{\alpha_j})^{\beta_j-1}$ $(j=1,2)$. The kernel of the posterior distribution is given by

$$\begin{aligned} p(\theta_1,\theta_2|x,y) &\propto L(\theta_1,\theta_2)p_1(\theta_1)p_2(\theta_2) \\ &\propto \theta_1^x(1-\theta_1)^{n_1-x}p_1(\theta_1)\times \\ &\quad\times \theta_2^y(1-\theta_2)^{n_2-y}p_2(\theta_2), \end{aligned} \tag{4}$$

where $x=\sum_i x_i$, $y=\sum_i y_i$ and $L(\theta_1,\theta_2)$ is the likelihood function. Note that the marginal posterior distributions will not be in the Kumaraswamyfamily, as the likelihood and prior distribution will not combine their kernels. The odds ratio is defined as $\psi=\theta_1(1-\theta_2)/\theta_2(1-\theta_1)$, thus the posterior inferences of the odds ratio should be based on the posterior distribution (4).

As shown by [9], the unnormalized moment of order r and s can be used to obtain the normalizing constant, that is

$$\begin{aligned} I_{x,y}(r,s) &= \int_0^1\int_0^1 \theta_1^r\theta_2^s p(\theta_1,\theta_2|x,y)d\theta_1 d\theta_2 \\ &= \int_0^1 \theta_1^{x+r}(1-\theta_1)^{n_1-x}p_1(\theta_1)d\theta_1\times \\ &\quad \int_0^1 \theta_2^{y+s}(1-\theta_2)^{n_2-y}p_2(\theta_2)d\theta_2 =: I_x(r)I_y(s), \end{aligned}$$

where $I_x(r)$ and $I_y(s)$ are given by (1), replacing α and β by α_j, and β_j $(j=1,2)$, respectively. As a consequence, the posterior normalizing constant is $I_{x,y}(0,0)=I_x(0)I_y(0)$, hence the exact posterior distribution is obtained dividing (4) by $I_{x,y}(0,0)$,

$$\begin{aligned} p(\theta_1,\theta_2|x,y) &= \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}\sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\binom{n_1-x}{j}\binom{n_2-y}{\ell}\times \\ &\quad\times(-1)^{j+\ell}\theta_1^{x+\alpha_1+j-1}\theta_2^{y+\alpha_2+\ell-1}\times \\ &\quad\times(1-\theta_1^{\alpha_1})^{\beta_1-1}(1-\theta_2^{\alpha_2})^{\beta_2-1}. \end{aligned} \tag{5}$$

Note that for the convenience of simplifying the forthcoming integrals, we wrote the joint posterior distribution in terms of the *binomial theorem*.

Before obtaining the posterior distribution of ψ, we firstly obtain the joint distributions of the odds $t_i = \theta_i/(1-\theta_i)$ $(i = 1,2)$. This is straightforward by using the *Jacobian theorem* in (5), thus

$$p(t_1,t_2|x,y) = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)} \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y} \binom{n_1-x}{j}\binom{n_2-y}{\ell} \times \times (-1)^{j+\ell}\left(\frac{t_1}{1+t_1}\right)^{x+\alpha_1+j-1} \frac{1}{(1+t_1)^2} \times \times \left(1-\left(\frac{t_1}{1+t_1}\right)^{\alpha_1}\right)^{\beta_1-1} \left(\frac{t_2}{1+t_2}\right)^{y+\alpha_2+\ell-1} \times \times \frac{1}{(1+t_2)^2}\left(1-\left(\frac{t_2}{1+t_2}\right)^{\alpha_2}\right)^{\beta_2-1} =: p(t_1|x)p(t_2|y). \tag{6}$$

It follows that the odds ratio is given by $\psi = t_1/t_2$. As a consequence, we obtain the following posterior quantities.

Theorem 1(Cumulative posterior distribution of the odds ratio). *Let $\psi = t_1/t_2$ be the odds ratio. Considering (6), the cumulative posterior distribution of ψ is given by*
(i) For $0 < \psi_0 < 1$:

$$P(\psi < \psi_0|x,y) = k_p^\star \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\sum_{h=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \binom{n_1-x}{j} \times \times \binom{n_2-y}{\ell} \frac{\psi_0^{x+\alpha_1+j+\alpha_1 h}}{(x+\alpha_1+j+\alpha_1 h)} \times \times (1-\psi_0)^v 2^{-\eta_1} \frac{(\beta_1-1)_h(\beta_2-1)_u}{h!v!u!\eta_1} \times \times (x+\alpha_1+j+\alpha_1 h)_v, \tag{7}$$

where $k_p^\star = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}$ and $\eta_1 = x+\alpha_1+j+\alpha_1 h+y+\alpha_2+\ell+\alpha_2 u+v$.
(ii) for $\psi_0 > 1$:

$$P(\psi < \psi_0|x,y) = 1 - k_p^\star \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\sum_{h=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \binom{n_1-x}{j} \times \times \binom{n_2-y}{\ell} \psi_0^{-(y+\alpha_2+\ell+\alpha_2 h+v)} \times \times \frac{(\beta_2-1)_h(\beta_1-1)_u}{h!u!v!(y+\alpha_2+\ell+\alpha_2 h)} \times \times \frac{(\psi_0-1)^v 2^{\eta_2}(y+\alpha_2+\ell+\alpha_2 h)_v}{\eta_2}, \tag{8}$$

where $k_p^\star = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}$ and $\eta_2 = y+\alpha_2+\ell+\alpha_2 h+x+\alpha_1+j+\alpha_1 u+v$.

Proof.(*i*) For $0 < \psi_0 < 1$: Let $\psi = t_1/t_2$ and $t_2 = t_2$, using (6),

$$P(\psi < \psi_0|x,y) = P(t_1 < \psi_0 t_2|x,y) = \int_0^1 p(t_2|x) \times \times \int_0^{\psi_0 t_2} p(t_1|y)dt_1 dt_2. \tag{9}$$

Considering the integral in t_1, let $w = t_1/(1+t_1)$, then

$$A(t_2) = \int_0^{\psi_0 t_2/(1+\psi_0 t2)} w^{x+\alpha_1+j-1}(1-w^{\alpha_1})^{\beta_1-1}dw = \sum_{h=0}^{\infty} \frac{(\beta_1-1)_h}{h!} \frac{\psi^{x+\alpha_1+j+\alpha_1 h}}{(x+\alpha_1+j+\alpha_1 h)} \left[\frac{t_2}{1+\psi_0 t_2}\right]^{x+\alpha_1+j+\alpha_1 h},$$

the latest identity used the binomial theorem for expanding $(1-w^{\alpha_1})^{\beta_1-1} = \sum_{h=0}^{\infty}(\beta_1-1)_h w^{\alpha_1 h}/h!$.

Thus, replacing $A(t_2)$ in (9) and considering the integral in t_2, we have

$$B = \int_0^1 \left[\frac{t_2}{1+\psi_0 t_2}\right]^{x+\alpha_1+j+\alpha_1 h} \left(\frac{t_2}{1+t_2}\right)^{y+\alpha_2+\ell-1} \times \times \frac{1}{(1+t_2)^2}\left(1-\left(\frac{t_2}{1+t_2}\right)^{\alpha_2}\right)^{\beta_2-1} dt_2 = \int_0^{1/2} \frac{w^{x+\alpha_1+j+\alpha_1 h+y+\alpha_2+\ell-1}(1-w^{\alpha_2})^{\beta_2-1}}{(1-(1-\psi_0)w)^{x+\alpha_1+j+\alpha_1 h}}dw = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{(\beta_2-1)_u}{u!} \frac{(x+\alpha_1+j+\alpha_1 h)_v}{v!}(1-\psi_0)^v \frac{2^{-\eta}}{\eta}$$

The second identity follows by letting $w = t_2/(1+t_2)$, then we expand $(1-w^{\alpha_2})^{\beta_2-1} = \sum_{u=0}^{\infty}(\beta_1-1)_u w^{\alpha_2 u}/u!$, and finally, we use (2).

(*ii*) For $\psi_0 > 1$: Note that $P(\psi < \psi_0|x,y) = 1 - P(\psi > 1/\psi_1|x,y)$, where $\psi_1 = 1/\psi_0$, hence $0 < \psi_1 < 1$, then we consider

$$P(\psi > 1/\psi_1|x,y) = P(t_2 < \psi_1 t_1|x,y) = \int_0^1 p(t_2|y) \times \times \int_0^{\psi_1 t_1} p(t_1|x)dt_2 dt_1. \tag{10}$$

The result follows by the same strategy as before, that is, replacing $\psi_1 = 1/\psi_0$ and using $P(\psi < \psi_0|x,y) = 1 - P(\psi > \psi_0|x,y)$.

Corollary 1(Posterior density of the odds ratio). *Considering Theorem 1, the posterior distribution of the odds ratio is given by*
(i) For $0 < \psi < 1$:

$$p(\psi|x,y) = k_p^\star \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\sum_{h=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \binom{n_1-x}{j}\binom{n_2-y}{\ell} \times \times \frac{(-1)^{j+\ell}}{(x+\alpha_1+j+\alpha_1 h)} \times \times \Big[(x+\alpha_1+j+\alpha_1 h)(\psi-1)^v \psi^{x+\alpha_1+j+\alpha_1 h-1} + + \psi^{x+\alpha_1+j+\alpha_1 h} v(1-\psi)^{v-1}\Big] \times \times \frac{(\beta_1-1)_h(\beta_2-1)_u(x+\alpha_1+j+\alpha_1 h)_v 2^{-\eta_1}}{h!v!u!\eta_1(x+\alpha_1+j+\alpha_1 h+v)},$$

where $k_p^\star = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}$ *and* $\eta_1 = x+\alpha_1+j+\alpha_1 h+y+\alpha_2+\ell+\alpha_2 u+v$.
(ii) For $\psi > 1$*:*

$$p(\psi|x,y) = k_p^\star \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\sum_{h=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{n_1-x}{j}\binom{n_2-y}{\ell}\times \\ \times(-1)^{j+\ell}\times \\ \times\Big[(y+\alpha_2+\ell+\alpha_2 h+v)(1-\psi)^v\times \\ \times\psi^{y+\alpha_2+\ell+\alpha_2 h+v-1}+ \\ +\psi^{-(y+\alpha_2+\ell+\alpha_2 h+v)}(1-\psi)^{v-1}v\Big]\times \\ \times\frac{(-\beta_2+1)_h(-\beta_1+1)_u}{h!u!v!(y+\alpha_2+\ell+\alpha_2 h)}\times \\ \times\frac{2^{\eta_2}(y+\alpha_2+\ell+\alpha_2 h)_v}{\eta_2}. \tag{11}$$

where $k_p^\star = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}$ *and* $\eta_2 = y+\alpha_2+\ell+\alpha_2 h+x+\alpha_1+j+\alpha_1 u+v$.

*Proof.*Straightforward by differentiating the (7) and (8) with respect to ψ_0.

Corollary 2(Posterior moments of the odds ratio). *Considering Corollary (1), the posterior moment of order* $r\in\mathbb{Z}$ *of* ψ *is given by*

$$\mathbb{E}[\psi^r|x,y] = k_p^\star \sum_{j=0}^{n_1-x}\sum_{\ell=0}^{n_2-y}\sum_{h=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\Bigg[\frac{A_{j\ell huv}}{(x+\alpha_1+j+\alpha_1 h)}+ \\ +\frac{B_{j\ell huv}}{(y+\alpha_2+\ell+\alpha_2 h)}\Bigg], \tag{12}$$

where

$$A_{j\ell huv} = \binom{n_1-x}{j}\binom{n_2-y}{\ell}(-1)^{j+\ell}2^{-\eta_1}\times \\ \times\frac{(\beta_1-1)_h(\beta_2-1)_u(x+\alpha_1+j+\alpha_1 h)_v}{h!v!u!\eta_1}\times \\ \times\left[\frac{1}{r}-\frac{\Gamma(x+\alpha_1+j+\alpha_1 h+1)\Gamma(v+1)}{\Gamma(x+\alpha_1+j+\alpha_1 h+v+2)}\right],$$

$$B_{j\ell huv} = \binom{n_1-x}{j}\binom{n_2-y}{\ell}(-1)^{j+\ell}2^{\eta_2}\times \\ \times\frac{(\beta_2-1)_h(\beta_1-1)_u(y+\alpha_2+\ell+\alpha_2 h)_v}{h!u!v!\eta_2}\times \\ \times\frac{\Gamma(y+\alpha_2+\ell+\alpha_2 h+r-2)\Gamma(v+1)}{\Gamma(y+\alpha_2+\ell+\alpha_2 h+r+v-1)},$$

$k_p^\star = \frac{\alpha_1\beta_1\alpha_2\beta_2}{I_x(0)I_y(0)}$, $\eta_1 = x+\alpha_1+j+\alpha_1 h+y+\alpha_2+\ell+\alpha_2 u+v$, *and* $\eta_2 = y+\alpha_2+\ell+\alpha_2 h+x+\alpha_1+j+\alpha_1 u+v$.

*Proof.*Letting $F_1(\psi)$ be (7) and $F_2(\psi)$ be (8), it follows that

$$\mathbb{E}[\psi^r|x,y] = \int_0^1 \psi^{r-1}(1-F_1(\psi))d\psi+ \\ +\int_1^\infty \psi^{r-1}(1-F_2(\psi))d\psi,$$

the result follows by nothing that the first integral is a particular case of the integral of a Beta distribution, as well as the second integral after transforming, say $\xi = 1/\psi$.

4 Application

As an illustration of the theory, we consider a study given by [3], about the Diagnosis of 187 Suspected Tumors by 2D Mammography and 3D Tomosynthesis. In the study the objective is to compare the rate of diagnostic of the two exams, that is, *are the two screening tests similar with respect to the diagnosis?*. The data are summarized in the following 2×2 contingency table, we also provide a table with the respective probabilities.

Mammography	Tomosynthesis	
	Benign	Malignant
Benign	54	68
Malignant	14	51

which gives the structure

Mammography	Tomosynthesis	
	Benign	Malignant
Benign	θ_1	$1-\theta_1$
Malignant	θ_2	$1-\theta_2$

Here $\theta_1 = P(\text{Response} = \text{Yes}|\text{"Benign"})$ and $\theta_2 = P(\text{Response} = \text{Yes}|\text{"Malignant"})$ are the parameters that will be used to estimate the odds ratio. We apply Model (3), noting that $n_1 = 122$, $x = 54$, $n_2 = 65$ and $y = 14$.

It follows that the unnormalized marginal posterior moment of order r of θ_i $(i = 1,2)$ is given by (1). We assigned the same (flat) prior distribution $\theta_i \sim \text{Kum}(\alpha = 1.1, \beta = 1.1)$ $(i = 1,2)$. We compared the results with those obtained by MCMC through the `R2OpenBugs` R package, which took some efforts to achieve convergence, due to highly correlated Markov chains. To overcome this difficulty, we simulated 10 independent chains of 30,000 length, discarding the first 20,000 samples, then discarding 50 values between cycles in every chain.

The exact posterior quantities can be obtained through the `Maplesoft` software (see the code in Appendix A). The expectation and variance of the odds ratio are given by (12) with $r = 1$ and $r = 2$, the summations provided a good precision with 100, this took about 2 minutes; and the cumulative distribution (given by (8)) took a little than 4 minutes to sum up 10 terms. Note that the way the sum

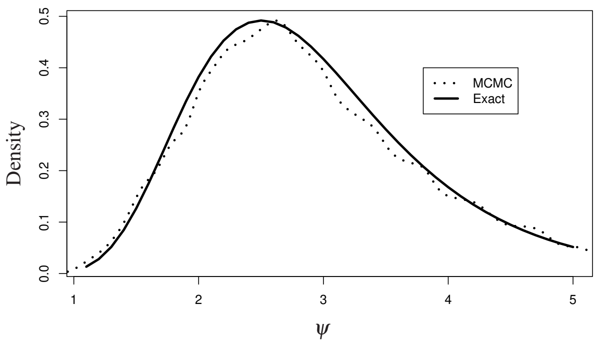

Fig. 1: Exact and approximated posterior density of the odds ratio ψ.

were made can be improved by some different language and computational algorithm. A 95% credible interval is obtained by searching for values (ψ_L,ψ_U) such that $P(\psi < \psi_L|x) = 0.025$ $P(\psi > \psi_U|x) = 0.025$, the search for these values took more than 40 minutes, however the codes for computing the sums could me improved.

In Table 1, note that the these estimates are close to their respective estimates obtained from the MCMC method. Notice that the Mammography test gives $\hat{\psi} = 2.982$ times more benign results than the Tomosynthesis test. In addition, Figure 1 shows the closeness of the posterior densities from the exact and from the MCMC methods, the values $\psi < 1$ were not present in the simulation due to very low probability in this interval.

Table 1: Comparison of the exact posterior estimates with their equivalent obtained by MCMC

Posterior Quantity	Estimates	
	MCMC	Exact
$\mathbb{E}[\psi|x,y]$	2.9820	2.9332
$\mathbb{V}\mathrm{ar}[\psi|x,y]$	1.1578	0.8875
$P(\psi \le 2.0|x,y)$	0.1486	0.1507
95% CI (ψ_L,ψ_U)	(1.434, 5.665)	(1.29,5.3)

Further inference could involve hypothesis testing, although this type of inference is not commonly used in practical Bayesian analysis. However, one can use the posterior CDF to compute probabilities in the context of hypothesis testing. In general, we compare two hypotheses, say $H_0 : \psi \in \Psi_1$ versus $H_1 : \psi \in \Psi_2$, where $\Psi_1 \cap \Psi_2 = \emptyset$. We then compare either the probabilities $P(H_0|\mathbf{x})$ and $P(H_1|\mathbf{x})$, or the posterior expected loss of some suitable loss function. See [10] for details on how to apply hypothesis testing for decision support.

5 Concluding remarks

The most used measures of association for 2×2 contingency tables are the risk ratio and the odds ratio. Considering the Kumaraswamy-Binomial model, [9] provide exact inferences for the risk ratio, now complemented by exact inferences for the odds ratio. This is noteworthy, as the odds ratio is often preferred by analysts due to its favorable mathematical properties.

The Beta and Kumaraswamydistributions are good alternatives for modeling variables within the $(0,1)$ interval. Both allow a wide range of shapes, which is essential when incorporating prior information, as the prior distribution should reflect genuine, relevant prior knowledge, making flexible distributions preferable. The Kumaraswamydistribution, however, has advantageous mathematical properties, as key quantities like the cumulative distribution function (CDF) and moments are analytically tractable. [11] provide exact inferences for the odds ratio in a Beta-Binomial model, which relies on *generalized hypergeometric functions* (see [12]). The Beta-Binomial model is relatively simpler (due to conjugacy) compared to the Kumaraswamy-Binomial model. In this work, we address a more complex structure without conjugacy, yet exact inferences are still obtained.

Future work will focus on improving the computational algorithms for summing up the series, as thedeveloping exact models for the odds ratio, extending the methodology to broader classes of distributions, such as the *unit-gamma* and *triangular* distributions. These more complex models are expected to also enable exact inferences.

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Compliance with Ethical Standards

The author declares that they have no conflicts of interest.

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A Computer code

```
Digits := 50: # number of precision digits:

# Data:
n1 := 122;
x := 54;
n2 := 65;
y := 14;

# Prior hyperparameters:
alpha1 := 1.1;
beta1 := 1.1;
alpha2 := 1.1;
beta2 := 1.1;

# Unnormalized moments:
Ir := (r, z, n, alpha, beta, upto) ->
alpha*beta*GAMMA(beta)*GAMMA(n - z + 1)*
sum(GAMMA(r + z + alpha*(h + 1))*(-1)^h/
(h!*GAMMA(r + n + alpha*(h + 1) + 1)*
GAMMA(beta - h)), h = 0 .. upto);
Ix0 := evalf(Ir(0, x, n1, alpha1, beta1, 100));
Iy0 := evalf(Ir(0, y, n2, alpha1, beta1, 100));
kp_st := alpha1*beta1*alpha2*beta2/(Ix0*Iy0);

# Cumulative posteriori distribution:

## For rho<1:

Sigma1 := (j, l, psi0) -> Sum(Sum(Sum(
psi0^(x + alpha1 + j + alpha1*h)*
pochhammer(beta1 - 1, h)*
pochhammer(beta2 - 1, u)*
pochhammer(x + alpha1 + j + alpha1*h, v)*
(1 - psi0)^v*2^(-x - alpha1 - j - alpha1*
h - y - alpha2 - l - alpha2*u - v)/(h!*
(x + alpha1 + j + alpha1*h)*u!*v!*
(x + alpha1 + j + alpha1*h + y + alpha2 + l
+ alpha2*u + v))
, u = 0 .. 10), h = 0 .. 10), v = 0 .. 10):

F1 := psi0 -> kp_st*Sum(binomial(n1 - x, j)
*Sum(binomial(n2 - y, l)*(-1)^(j+l)*
Sigma1(j, l, psi0), l = 0 .. n2 - y),
j = 0 .. n1 - x):

## for rho>1:

Sigma2 := (j, l, psi0) -> Sum(Sum(Sum(
psi0^(-y - alpha2 - l - alpha2*h - v)*
pochhammer(beta2 - 1, h)*pochhammer(
beta1 - 1, u)*pochhammer(y + alpha2 + l
+ alpha2*h, v)*2^(-y - alpha2 - l -
alpha2*h - x - alpha1 - j - alpha1*u - v)
*(psi0 - 1)^v/(h!*(y + alpha2 + l +
alpha2*h)*u!*v!*(y + alpha2 + l + alpha2*h +
 x + alpha1 + j + alpha1*u + v)),
```

```
u = 0 .. 10), h = 0 .. 10), v = 0 .. 10):
F2 := psi0 -> 1 - kp_st*Sum(binomial(n1 - x, j)*
Sum(binomial(n2 - y, l)*(-1)^(j+l)*
Sigma2(j, l, psi0), l = 0 .. n2 - y),
j = 0 .. n1 - x)

## Odds Ratios posterior moments:

A := (alpha1, beta1, alpha2, beta2, h, j, l)
-> beta1*alpha1*beta2*GAMMA(beta2)*
binomial(n1 - x, j)*binomial(n2 - y, l)*
(-1)^(j + l)*GAMMA(1 - beta1 + h)*
GAMMA((y + l + x + j + alpha1 + alpha2
+ alpha2*h)/alpha2)/(Ix0*Iy0*GAMMA(1 - beta1)*
h!*GAMMA((y + l + x + j + alpha1 +
alpha2 + alpha2*beta2 + alpha1*h)/alpha2)):

B := (alpha1, beta1, alpha2, beta2, h, j, l)
 -> alpha2*beta1*beta2*GAMMA(beta1)*
binomial(n1 - x, j)*binomial(n2 - y, l)
*(-1)^(j + l)*GAMMA(1 - beta2 + h)*
GAMMA((y + l + x + j + alpha2 + alpha1
+ alpha2*h)/alpha1)/(Ix0*Iy0*
GAMMA(1 - beta2)*h!*GAMMA((y + l + x + j
 + alpha1 + alpha2 + beta1*
alpha1 + alpha2*h)/alpha1)):

# this gives the moments of order r,
# "upto" is the number of summations
# terms we want to sum:
mom := (r, alpha1, beta1, alpha2, beta2, upto)
-> sum(sum(sum(A(alpha1, beta1,
alpha2, beta2, h, j, l)/(x + j + alpha1*
h + r + alpha1) + B(alpha1, beta1, alpha2,
beta2, h, j, l)/(y + l + alpha2*r - r + alpha2)
, h = 0 .. upto), j = 0 .. n1 - x),
l = 0 .. n2 - y):
```