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Exploring a New Challenge in Permuted Cordial Labeling of the Corona Product of Two Distinct Graphs

Ashraf Elrokh[1,*], *Khalid A. Alsatami* [2], *Mohamed A. Nouh*[1] *and Yasmin Algrawani*[3]

[1]Mathematics and Computer Since Department, Faculty of Science, Menoufia University, Menoufia, 32511, Egypt
[2]Department of Mathematics, College of Science, Qassim University, Buraydah, KSA
[3]Department of Basic Sciences, Modern Academy for Engineering and Technology, Cairo, Egypt

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Abstract: A graph is said to be Permuted cordial labeling if it has a I, f, g labeling that satisfies certain properties. In this paper, we present a novel application of permuted cordial labeling based on the corona product of two graphs, demonstrating its potential for understanding and studying specific graph structures. The resulting Permuted cordial labeling scheme for the corona product of paths, cycles, second power of paths, cycles Fans , and wheels may provide consideration for the graphs's properties and structures. We can use it to investigate its connectivity, symmetry, and other graph-theoretic properties.

Keywords: Fifth power of graphs, Corona Product, Permuted cordial graph, Path and cycle, Fan and wheel

1 Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operations research and especially computer science. One area of graph theory of considerable recent research is that of graph labeling. In a labeling of a particular type, the vertices area assigned values from a given set, the edges have a prescribed induced labeling, and the labelings must satisfy certain properties [1,2,3]. An excellent reference on this subject is the survey by Gallian [4].

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rose [5] in 1966 and Golomb [6] in 1972, while harmonious labelings were first studied by Graham and Sloane [7] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [8] in 1990. Whereas the label of an edge vw for graceful and harmonious labeling is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced label$(f(v) + f(w))(mod2)$,which of course equals $|f(v) - f(w)|$. Because arithmetic modulo2 is an integral part of computer science, cordial labelings have close connections with that field.

In [9] ELrokh and et.al, proved that each path, cycle and Fan admits permuted cordial labeling. The Wheel graph $W_n, n \geq 3$ admits permuted cordial labeling except $n \equiv 2\ mod\ 3$ and n even. Moreover, they proved that the union of $P_n \cup P_m,\ n,m \geq 2$ admits a permuted cordial labeling for all n,m. The union of $C_n \cup C_m,\ n,m \geq 3$ admits a permuted cordial labeling for all n,m. The union of $P_n \cup C_m,\ n \geq 2, m \geq 3$ admits a permuted cordial labeling for all n,m. In [10] they proved that each $P_n \bigodot P_m$, $n,m \geq 2$ admits permuted cordial labeling, each $C_n \bigodot C_m$, $n,m \geq 3$ admits permuted cordial labeling for all $n,m \neq (1mod3; 2mod3)$. Each $P_n \bigodot C_m$, $n \geq 1, m \geq 3$ admits a permuted cordial labeling. $C_n \bigodot P_m$,$n \geq 3, m \geq 1$ admits permuted cordial labeling. Also, they proved that the corona of $P_n \bigodot P_m^2$, $n,m \geq 2$ admits a permuted cordial labeling. The corona of $P_n \bigodot C_m^2$, $n \geq 2, m > 4$ admits a permuted cordial labeling. For more details about the graph labeling and types of labeling, the reader can refer to [11,12,13,14,15,16,17].We will give brief summary of definitions which are useful for the present investigations.

Definition 1: Second power of graphs, is the graph obtained from graphs by adding edges that join all vertices u and v with $d(u,v) \leq 2$.

* Corresponding author e-mail: ashraf.hefnawy68@yahoo.com

Definition 2: The second power of paths P_n^2, is the graph obtained from the path P_n by adding edges that join all vertices u and v with $d(u,v) \leq 2$. So, the order of the fifth power of paths P_n^2 is n, and the size of the second power of paths P_n^2 is $2n-3$, in particular $P_1^2 = P_1, P_2^2 = P_2$,and $P_3^2 = C_3$.
Definition 3: The second power of cycles C_n^2, is the graph obtained from the path C_n by adding edges that join all vertices u and v with $d(u,v) \leq 2$. So, the order of the second power of cycles C_n^2, is n, and the size of the second power of cycles C_n^2 is $2n-2$, in particular $C_3^2 = C_3$ and $C_4^2 = K_4$.
Definition 4: The second power of a Fans graph is the graph obtained from the joining between N_1 and the second power of path P_n^2. So, the order of the second power of a Fans F_n^2 is $n+1$, and the size of the second power of a Fans F_n^2 is $3n-3$.
Definition 5: The second power of a wheels graph is the graph obtained from the joining between N_1 and the second power of cycle C_n^2. So, the order of the second power of a wheels W_n^2 is $n+1$, and the size of the second power of a wheels W_n^2 is $3n-2$.
Definition 6: The corona $G_1 \odot G_2$ of two graphs G_1 (with n_1 vertices , m_1 edges) and G_2 (with n_2 vertices , m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2 , and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2 . It is easy to see that the corona $G_1 \odot G_2$ that has $n_1 + n_1 n_2$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.

The remainder of this paper is organized as follows: Section 3 presents the permuted cordial labeling for the corona product of paths, cycles, second powers of paths, second powers of cycles, second powers of fans, and second powers of wheels. Section 4 provides examples of the corona product for two graphs. Finally, Section 5 concludes the paper.

2 Materials and Methods

We can use these code of labeling as follows

E_{3r}	$gfI \ldots gfI$ repeated r time	E_{3k}	$gfI \ldots gfI$ repeated k time
S_{3r}	$fIg \ldots fIg$ repeated r time	L_{3k}	$fgI \ldots fgI$ repeated k time
M_{3r}	$Igf \ldots Igf$ repeated r time	Z_{3k}	$Igf \ldots Igf$ repeated k time
O_{3r}	$gIf \ldots gIf$ repeated r time	O_{3k}	$gIf \ldots gIf$ repeated k time
M_{3k}	$Ifg \ldots Ifg$ repeated k time	S_{3k}	$fIg \ldots fIg$ repeated k time

Sometimes, we modify E_{3r}, S_{3r}, and M_{3r} by adding symbols at one end or the other or both; for example, $S_{3r}If$ means the labeling $fIg...fIg(r\text{-times})If$. Similarly, $O_{3r}g$ means the labeling $gIf...gIf(r\text{-time})g$. $v(I)$, $v(g)$, and $v(f)$ represent the number of vertices labeled I, g, and f, respectively. Similarly, we denoted $e(I)$, $e(g)$, and $e(f)$ to be the number of edges labeled I, g, and f, respectively, for the graph G.
A vertex labeling of graph G of h:$V \rightarrow \{I,g,f\}$ with induced edge labeling h^*:$E(G) \rightarrow \{I,g,f\}$ defined by

$\circ$	$v(I)$	$v(f)$	$v(g)$
$u(i)$	I	f	g
$u(f)$	f	g	I
$u(g)$	g	I	f

is called permuted cordial labeling if $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{I, f, g\}$ where $v(x)$ (respectively, $e(x)$) is the number of vertices (respectively, edges) labeled with $x \in \{I, f, g\}$. A graph G is permuted cordial if it admits a permuted labeling. For a given labeling of the corona $G \odot H$, we denote $v(j)$ and $e(j)$ $(j = I, f, g)$ to represent the numbers of vertices and edges labeled by j , respectively. We denote x_j and a_j to be the numbers of vertices and edges labeled by $j \in \{I, f, g\}$, respectively, for the graph G. Also, we let y_j, y'_j and b_j, b'_j be those for H, which are connected to the vertices labeled j of G. It is easy to verify that

$$\left\{\begin{array}{c} v_I = x_I + x_I y_I + x_f y'_I + x_g y''_I \\ v_f = x_f + x_I y_f + x_f y'_f + x_g y''_f \\ v_g = x_g + x_I y_g + x_f y'_g + x_g y''_g \\ e_I = a_I + x_I b_I + x_f b'_I + x_g b''_I + x_I y_I + x_f y'_g + x_g y''_f \\ e_f = a_f + x_I b_f + x_f b'_f + x_g b''_f + x_I y_f + x_f y'_I + x_g y''_g \\ e_g = a_g + x_I b_g + x_f b'_g + x_g b''_g + x_I y_g + x_f y'_f + x_g y''_I \end{array}\right. \quad (1)$$

Finally, for particular labeling A and B that are used for G_1 and G_2 , respectively, we let $[A;B]$ denote the labeling for the corona of G_1 and G_2. Section one contains a brief literary analysis of the topic of this work. Section Two deals with the Materials and Methods employed throughout. Whereas section three is devoted to studying the fifth power of paths, cycles, fans, wheels and lemniscate graphs. Section four investigates the permuted cordial labeling for the corona product of paths, cycles, second power of paths, and second power of cycles, and examples. The last section, is the conclusion which summarize the important points of our finding in this paper.

3 Main Results

In this section, we will demonstrate the permuted cordial labeling for the corona product of paths, cycles, second powers of paths, second powers of cycles, second powers of fans, and second powers of wheels.
Theorem 1. *$P_n \odot F_m^2$ is permuted cordial for all $n \geq 1$ and $m \geq 3$.*
Proof. Let $n = 3r + i$ ($i = 0, 1, 2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0, 1, 2$ and $k \geq 2$), then, we may use the labeling A_i' or A_j for P_n as given in Table 1. For a given value of j with $0 \leq i, j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for F_m^2, where B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of F_m^2 which are connected to the vertices labeled I in P_n, while B_i, $B^{'}_i, B"_i$, C_i, $C'_i$and $C"_i$ are the labeling of F_m^2 which are connected to the vertices labeled f or g in F_m^2 as given in Table 2. Using Table 3 and the formulas (1), we can compute the values shown in the last two columns of Table 3. Since all of these values are 1 or 0, the theorem follows.

Table 1. Labeling of P_n

n=3r+i	P_n	x_I	x_f	x_g	a_I	a_f	a_g
i=0	A_0=E_{3r}	r	r	r	r	r	$r-1$
i=1	A_1=$M_{3r}I$	r+1	r	r	r	r	r
i=1	A'_1=$E_{3r}g$	r	r	r+1	r	r	r
i=1	A''_1=$O_{3r}g$	r	r	r+1	r	r	r
i=2	A_2=$IQ_{3r}g$	r+1	r	r+1	r+1	r	r
i=2	A'_2=$E_{3r}gf$	r	r+1	r+1	r+1	r	r
i=2	A''_2=$O_{3r}gI$	r+1	r	r+1	r	r	r+1

Table 2. Labeling of $F^2{}_m$

m=3k+j	$F^2{}_m$	y_I	y_f	y_g	b_I	b_f	b_g
$i=0$	$B_0 = IZ_{3k}$	$k+1$	k	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$B'_0 = fS_{3k}$	k	$k+1$	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$B''_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k-1$
$i=0$	$C_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k-1$
$i=0$	$C'_0 = IM_{3k}$	$k+1$	k	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$C''_0 = fS_{3k}$	k	$k+1$	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$D_0 = fS_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k-1$
$i=0$	$D'_0 = gE_{3k}$	$k+1$	k	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$D''_0 = IM_{3k}$	k	$k+1$	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$E_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k-1$
$i=0$	$E'_0 = IZ_{3k}$	$k+1$	k	k	$3k-1$	$3k-1$	$3k-1$
$i=0$	$E''_0 = fL_{3k}$	k	$k+1$	k	$3k-1$	$3k-1$	$3k-1$
$i=1$	$B_1 = fS_{3k}g$	$k+1$	k	$k+1$	$3k$	$3k$	$3k$
$i=1$	$B'_1 = IZ_{3k}f$	$k+1$	$k+1$	k	$3k$	$3k$	$3k$
$i=1$	$B''_1 = gE_{3k}I$	k	$k+1$	$k+1$	$3k$	$3k$	$3k$
$i=1$	$C_1 = fS_{3k}g$	k	$k+1$	$k+1$	$3k$	$3k$	$3k$
$i=1$	$C'_1 = gE_{3k}I$	$k+1$	k	$k+1$	$3k$	$3k$	$3k$
$i=1$	$C''_1 = IZ_{3k}f$	$k+1$	$k+1$	k	$3k$	$3k$	$3k$
$i=1$	$D_1 = gE_{3k}I$	$k+1$	k	$k+1$	$3k$	$3k$	$3k$
$i=1$	$D'_1 = IZ_{3k}f$	$k+1$	$k+1$	k	$3k$	$3k$	$3k$
$i=1$	$D''_1 = fS_{3k}g$	k	$k+1$	$k+1$	$3k$	$3k$	$3k$
$i=1$	$E_1 = fS_{3k}g$	k	$k+1$	$k+1$	$3k$	$3k$	$3k$
$i=1$	$E'_1 = IM_{3k}g$	$k+1$	k	$k+1$	$3k$	$3k$	$3k$
$i=1$	$E''_1 = IZ_{3k}f$	$k+1$	$k+1$	k	$3k$	$3k$	$3k$
$i=2$	$B_2 = gS_{3k}fI$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$
$i=2$	$B'_2 = fZ_{3k}Ig$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$
$i=2$	$B''_2 = IE_{3k}gf$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$
$i=2$	$C_2 = fO_{3k}gI$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$
$i=2$	$C'_2 = gM_{3k}If$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$
$i=2$	$C''_2 = IE_{3k}gf$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+1$

Table 3. Labeling of $P_n \odot F_m^2$

$n = 3r+i$	$m = 3r+j$	P_n	F_m^2	$\lvert v(x)-v(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$B''_0, B'_0, B_0, \ldots$	$0,0,0$	$0,1,1$
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	$0,0,0$	$1,1,0$
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	$0,0,0$	$1,1,0$
$i=1$	$j=0$	A_1	$C_0, C''_0, C'_0 \ldots, C_0$	$1,0,1$	$0,1,1$
$i=1$	$j=1$	A'_1	$C''_1, C'_1, C_1, \ldots, C''_1$	$0,0,0$	$1,1,0$
$i=1$	$j=2$	A'_1	$C''_2, C'_2, C_2, \ldots, C''_2$	$1,1,0$	$0,0,0$
$i=2$	$j=0$	A_2	$C''_0, C_0, C'_0, \ldots, C'_0, C''_0$	$0,1,1$	$0,0,0$
$i=2$	$j=1$	A'_2	$D''_1, D'_1, D_1, \ldots, D''_1 D'_1$	$0,0,0$	$0,1,1$
$i=2$	$j=2$	A'_2	$B''_2, B'_2, B_2, \ldots, B''_2, B'_2$	$1,1,0$	$1,1,0$

Theorem 2. *$P_n \odot W_m^2$ is permuted cordial for all $n \geq 1$ and $m \geq 3$*

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for P_n as given in Table 1. For a given value of j with $0 \leq i', j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for W_m^2, where B_i, $B'_i, B"_i$, C_i, $C'_i$and $C"_i$ are the labeling of W_m^2 which are connected to the vertices labeled I in P_n, while B_i, $B^i_i, B"_i$, C_i, $C'_i$and $C"_i$ are the labeling of W_m^2 which are connected to the vertices labeled f or g in W_m^2 as given in Table 4. Using Table 5 and the formulas (1), we can compute the values shown in the last two columns of Table 5. Since all of these values are 1 or 0, the theorem follows.

Table 4. Labeling of $W^2{}_m$

m=3k+j	$W^2{}_m$	y_I	y_f	y_g	b_I	b_f	b_g
$i=0$	$B_0 = fS_{3k}$	k	$k+1$	k	$3k$	$3k-1$	$3k-1$
$i=0$	$B'_0 = IZ_{3k}$	$k+1$	k	k	$3k-1$	$3k$	$3k-1$
$i=0$	$B''_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k$
$i=0$	$C_0 = fO_{3k}$	k	$k+1$	k	$3k$	$3k-1$	$3k-1$
$i=0$	$C'_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k$
$i=0$	$C''_0 = IZ_{3k}$	$k+1$	k	k	$3k-1$	$3k$	$3k-1$
$i=0$	$D_0 = fS_{3k}$	k	$k+1$	k	$3k-1$	$3k$	$3k-1$
$i=0$	$D'_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k$
$i=0$	$D''_0 = IZ_{3k}$	$k+1$	k	k	$3k$	$3k-1$	$3k-1$
$i=0$	$E_0 = gE_{3k}$	k	k	$k+1$	$3k-1$	$3k-1$	$3k$
$i=0$	$E'_0 = IZ_{3k}$	$k+1$	k	k	$3k-1$	$3k$	$3k-1$
$i=0$	$E''_0 = fS_{3k}$	k	$k+1$	k	$3k$	$3k-1$	$3k-1$
$i=1$	$B_1 = IO_{3k}g$	$k+1$	k	$k+1$	$3k$	$3k$	$3k+1$
$i=1$	$B'_1 = fM_{3k}I$	$k+1$	$k+1$	k	$3k$	$3k+1$	$3k$
$i=1$	$B''_1 = fE_{3k}g$	k	$k+1$	$k+1$	$3k+1$	$3k$	$3k$
$i=1$	$C_1 = IO_{3k}g$	$k+1$	k	$k+1$	$3k+1$	$3k$	$3k$
$i=1$	$C'_1 = fM_{3k}I$	$k+1$	$k+1$	k	$3k$	$3k$	$3k+1$
$i=1$	$C''_1 = fE_{3k}g$	k	$k+1$	$k+1$	$3k$	$3k+1$	$3k$
$i=1$	$D_1 = gZ_{3k}I$	$k+1$	k	$k+1$	$3k+1$	$3k$	$3k$
$i=1$	$D'_1 = IS_{3k}f$	$k+1$	$k+1$	k	$3k$	$3k$	$3k+1$
$i=1$	$D''_1 = fE_{3k}g$	k	$k+1$	$k+1$	$3k$	$3k+1$	$3k$
$i=2$	$B_2 = fO_{3k}gI$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+2$
$i=2$	$B'_2 = gM_{3k}If$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+2$	$3k+1$
$i=2$	$B''_2 = IE_{3k}gf$	$k+1$	$k+1$	$k+1$	$3k+2$	$3k+1$	$3k+1$
$i=2$	$C_2 = IE_{3k}gf$	$k+1$	$k+1$	$k+1$	$3k+2$	$3k+1$	$3k+1$
$i=2$	$C'_2 = fO_{3k}gI$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+1$	$3k+2$
$i=2$	$C''_2 = gM_{3k}If$	$k+1$	$k+1$	$k+1$	$3k+1$	$3k+2$	$3k+1$

Table 5. Labeling of $P_n \odot W_m^2$

$n = 3r+i$	$m = 3r+j$	P_n	W_m^2	$\lvert v(x)-v(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$B''_0, B'_0, B_0, \ldots$	$0,0,0$	$0,1,1$
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	$0,0,0$	$1,1,0$
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	$0,0,0$	$1,1,0$
$i=1$	$j=0$	A'_1	$B''_0, B'_0, B_0, \ldots, B''_0$	$1,0,1$	$0,1,1$
$i=1$	$j=1$	A_1	$C_1, C''_1, C'_1 \ldots, C_1$	$0,0,0$	$0,0,0$
$i=1$	$j=2$	A_1	$C_2, C''_2, C'_2, \ldots, C_2$	$1,1,0$	$1,1,0$
$i=2$	$j=0$	A'_2	$C''_0, C_0, C'_0, \ldots, C''_0, C'_0$	$0,1,1$	$1,1,0$
$i=2$	$j=1$	A'_2	$C''_1, C'_1, C_1, \ldots, C''_1 C'_1$	$0,0,0$	$0,1,1$
$i=2$	$j=2$	A'_2	$C''_2, C'_2, C_2, \ldots, C''_2, C'_2$	$1,1,0$	$0,0,0$

Theorem 3. *$C_n \odot F_m^2$ is permuted cordial for all $n,m \geq 1$.*

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 6. For a given value of j with $0 \leq i', j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for F_m^2, where B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of F_m^2 which are connected to the vertices labeled I in P_n, while B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of F_m^2 which are connected to the vertices labeled f or g in F_m^2 as given in Table 2 .Using Table 7 and the formulas (1), we can compute the values shown in the last two columns of Table 7. Since all of these values are 1 or 0, the theorem follows

Table 6. Labeling of C_n

n=3r+i	C_n	x_I	x_f	x_g	a_I	a_f	a_g
$i=0$	$A_0 = E_{3r}$	r	r	r	r	r	r
$i=1$	$A_1 = E_{3r}g$	r	r	$r+1$	r	$r+1$	r
$i=2$	$A_2 = gE_{3r}I$	$r+1$	r	$r+1$	$r+1$	$r+1$	r
	$A'_2 = E_{3r}Ig$	$r+1$	r	$r+1$	$r+1$	$r+1$	r
	$A''_2 = S_{3r}gf$	r	$r+1$	$r+1$	$r+1$	$r+1$	$r+1$

Table 7. Labeling of $C_n \odot F^2{}_m$

$n=3r+i$	$m=3r+j$	C_n	F^2_m	$\lvert v(x)-v(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$D''_0, D'_0, D_0, \ldots$	$0,0,0$	$0,0,0$
	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	$0,0,0$	$0,0,0$
	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	$0,0,0$	$0,0,0$
$i=1$	$j=0$	A_1	$D''_0, D'_0, D_0, \ldots, D''_0$	$1,0,1$	$0,1,1$
	$j=1$	A_1	$E''_1, E'_1, E_1, \ldots, E''_1$	$0,0,0$	$0,0,0$
	$j=2$	A_1	$C''_2, C'_2, C_2, \ldots, C''_2$	$1,1,0$	$1,1,0$
$i=2$	$j=0$	A''_2	$E'_0, E_0, E''_0, \ldots, E''_0, E'_0$	$0,1,1$	$1,1,0$
	$j=1$	A''_2	$C'_1, C_1, C''_1, \ldots, C''_1 C'_1$	$0,0,0$	$0,0,0$
	$j=2$	A'_2	$C''_2, C'_2, C_2, \ldots, C_2, C''_2$	$1,1,0$	$1,1,0$

Theorem 4. *$C_n \odot W^2_m$ is permuted cordial for all $n \geq 3$ and $m \geq 3$*

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 6. For a given value of j with $0 \leq i', j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for W^2_m, where B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of W^2_m which are connected to the vertices labeled I in C_n, while B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of W^2_m which are connected to the vertices labeled f or g in W^2_m as given in Table 4. Using Table 8 and the formulas (1), we can compute the values shown in the last two columns of Table 8. Since all of these values are 1 or 0, the theorem follows.

Table 8. Labeling of $C_n \odot W^2{}_m$

$n=3r+i$	$m=3r+j$	C_n	W^2_m	$\lvert v(x)-v(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$E''_0, E'_0, E_0, \ldots$	$0,0,0$	$0,0,0$
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	$0,0,0$	$0,0,0$
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	$0,0,0$	$0,0,0$
$i=1$	$j=0$	A_1	$D''_0, D'_0, D_0 \ldots, D''_0$	$1,0,1$	$0,0,0$
$i=1$	$j=1$	A_1	$D''_1, D'_1, D_1, \ldots, D''_1$	$0,0,0$	$1,1,0$
$i=1$	$j=2$	A_1	$B''_2, B'_2, B_2, \ldots, \ldots, B''_2$	$1,1,0$	$1,1,0$
$*i=2$	$j=0$	A_2	$C''_0, C''_0, C'_0, C_0 \ldots, C_0$	$0,1,1$	$0,0,0$
$i=2$	$j=1$	A'_2	$D''_1, D'_1, D_1, \ldots, D_1 D''_1$	$0,0,0$	$0,1,1$
$i=2$	$j=2$	A'_2	$B''_2, B'_2, B_2, \ldots, B_2, B''_2$	$1,1,0$	$1,1,0$

Theorem 5. *$C_n \odot P^2_m$ is permuted cordial for all $n \geq 3$ and $m \geq 3$*

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 6. For a given value of j with $0 \leq i', j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for P^2_m, where B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of P^2_m which are connected to the vertices labeled I in C_n, while B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of P^2_m which are connected to the vertices labeledï f or g in P^2_m as given in Table 9. Using Table 10 and the formulas (1), we can compute the values shown in the last two columns of Table 10. Since all of these values are 1 or 0, the theorem follows.

Table 9. Labeling of $P^2{}_m$

m=3k+j	$P^2{}_m$	y_I	y_f	y_g	b_I	b_f	b_g
$i=0$	$B_0 = E_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=0$	$B'{}_0 = S_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=0$	$B''{}_0 = L_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=1$	$B_1 = Z_{3k}f$	k	$k+1$	k	$2k$	$2k-1$	$2k$
$i=1$	$B'{}_1 = Z_{3k}g$	k	k	$k+1$	$2k$	$2k$	$2k-1$
$i=1$	$B''{}_1 = M_{3k}I$	$k+1$	k	k	$2k-1$	$2k$	$2k$
$i=1$	$C_1 = Z_{3k}I$	$k+1$	k	k	$2k-1$	$2k$	$2k$
$i=1$	$C'{}_1 = L_{3k}f$	k	$k+1$	k	$2k$	$2k$	$2k-1$
$i=1$	$C''{}_1 = O_{3k}g$	k	k	$k+1$	$2k$	$2k-1$	$2k$
$i=2$	$B_2 = O_{3k}fg$	k	$k+1$	$k+1$	$2k+1$	$2k$	$2k$
$i=2$	$B'{}_2 = IgL_{3k}$	$k+1$	k	$k+1$	$2k$	$2k+1$	$2k$
$i=2$	$B''{}_2 = IfE_{3k}$	$k+1$	$k+1$	k	$2k$	$2k$	$2k+1$

Table 10. Labeling of $C_n \odot P^2{}_m$

$n=3r+i$	$m=3r+j$	C_n	P^2_m	$\lvert v(x)-v(y)\rvert$ $,x \neq y$ and $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y$ and $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$B''_0, B'_0, B_0, \ldots$	$0,0,0$	$0,0,0$
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	$0,0,0$	$0,0,0$
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	$0,0,0$	$0,0,0$
$i=1$	$j=0$	A_1	$B''_0, B'_0, B_0, .., B''_0$	$1,1,0$	$1,1,0$
$i=1$	$j=1$	A_1	$C''_1, C'_1, C_1, .., C''_1$	$1,1,0$	$1,1,0$
$i=1$	$j=2$	A_1	$B''_2, B'_2, B_2, .., B''_2$	$0,0,0$	$1,1,0$
$i=2$	$j=0$	A'_2	$B''_0, B'_0, B_0, \ldots, B_0, B''_0$	$1,1,0$	$1,1,0$
$i=2$	$j=1$	A'_2	$B''_1, B'_1, B_1, \ldots, B_1, B''_1$	$1,1,0$	$1,1,0$
$i=2$	$j=2$	A'_2	$B''_2, B'_2, B_2, .., B_2, B''_2$	$0,0,0$	$1,1,0$

Theorem 6. *$C_n \odot C^2_m$ is permuted cordial for all $n \geq 3$ and $m \geq 3$*

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$) , and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 6. For a given value of j with $0 \leq i', j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B"_i, C_i, C'_i, C"_i\}$ for C^2_m, where B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of C^2_m which are connected to the vertices labeled I in C_n, while B_i, $B'_i, B"_i$, C_i, C'_i and $C"_i$ are the labeling of C^2_m which are connected to the vertices labeled f or g in C^2_m as given in Table 11. Using Table 12 and the formulas (1), we can compute the values shown in the last two columns of Table 12. Since all of these values are 1 or 0, the theorem follows.

Table 11. Labeling of $C^2{}_m$

$m=3k+j$	C^2_m	y_I	y_f	y_g	b_I	b_f	b_g
$i=0$	$B_0 = L_{3k}$	k	k	k	$2k-1$	$2k$	$2k-1$
$i=0$	$B'_0 = S_{3k}$	k	k	k	$2k$	$2k-1$	$2k-1$
$i=0$	$B''_0 = E_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k$
$i=1$	$B_1 = E_{3k}f$	k	$k+1$	k	$2k$	$2k$	$2k$
$i=1$	$B'_1 = L_{3k}g$	k	k	$k+1$	$2k$	$2k$	$2k$
$i=1$	$B''_1 = M_{3k}I$	$k+1$	k	k	$2k$	$2k$	$2k$
$i=2$	$B_2 = O_{3k}fg$	k	$k+1$	$k+1$	$2k+1$	$2k+1$	$2k$
$i=2$	$B'_2 = M_{3k}gI$	$k+1$	k	$k+1$	$2k+1$	$2k$	$2k+1$
$i=2$	$B''_2 = M_{3k}fI$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k+1$

Table 12. Labeling of $C_n \odot C^2{}_m$

$n=3r+i$	$m=3r+j$	C_n	C^2_m	$\lvert v(x)-v(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$	$\lvert e(x)-e(y)\rvert$ $,x \neq y and$ $x,y \in \{I,f,g\}$
$i=0$	$j=0$	A_0	$B''_0, B'_0, B_0, \ldots$	0,0,0	0,0,0
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	0,0,0	0,0,0
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	0,0,0	0,0,0
$i=1$	$j=0$	A_1	$B''_0, B'_0, B_0, \ldots, B''_0$	1,1,0	1,1,0
$i=1$	$j=1$	A_1	$B''_1, B'_1, B_1, .., B''_1$	1,1,0	1,1,0
$i=1$	$j=2$	A_1	$B''_2, B'_2, B_2, .., B''_2$	0,0,0	1,1,0
$i=2$	$j=0$	A'_2	$B''_0, B'_0, B_0 \ldots, B_0, B''_0$	1,1,0	1,1,0
$i=2$	$j=1$	A'_2	$B''_1, B'_1, B_1 \ldots, B_1, B''_1$	1,1,0	1,1,0
$i=2$	$j=2$	A'_2	$B''_2, B'_2, B_2, \ldots, B_2, B''_2$	0,0,0	1,1,0

4 Examples

The Permuted cordial graph of $C_4 \odot F_6^2$, $C_4 \odot P_6^2$, $C_4 \odot W_7^2$, $P_4 \odot F_6^2$, $P_4 \odot W_7^2$ and $C_4 \odot C_6^2$ are illustrated in Figures (1, ...,6).

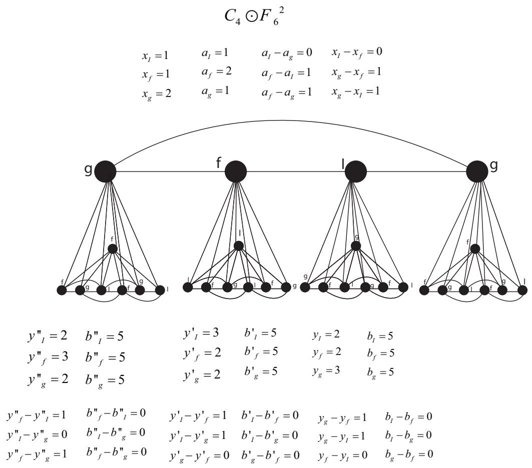

Figure 1. Permuted cordial graph of $C_4 \odot F_6^2$

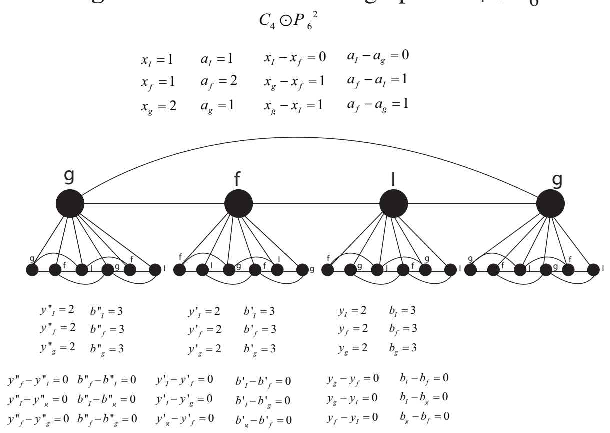

Figure 2. Permuted cordial graph of $C_4 \odot P_6^2$

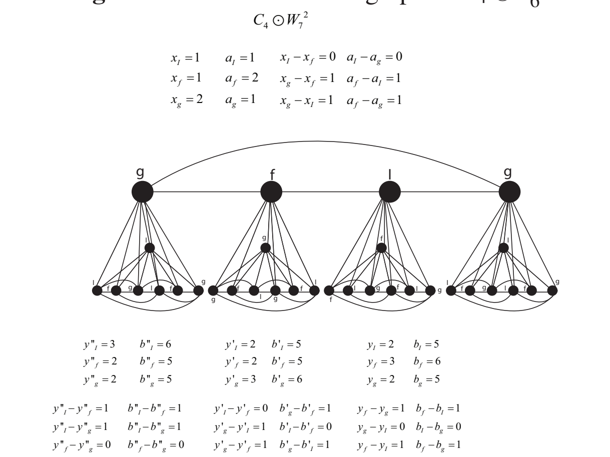

Figure 3. Permuted cordial graph of $C_4 \odot W_7^2$

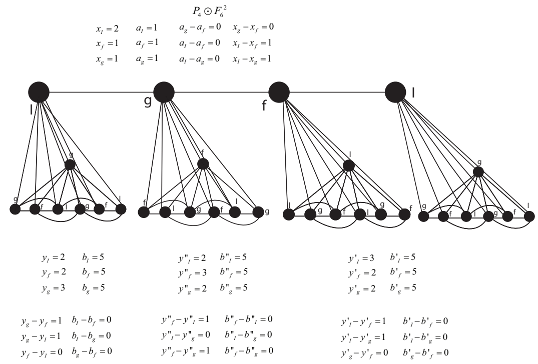

Figure 4. Permuted cordial graph of $P_4 \odot F_6^2$

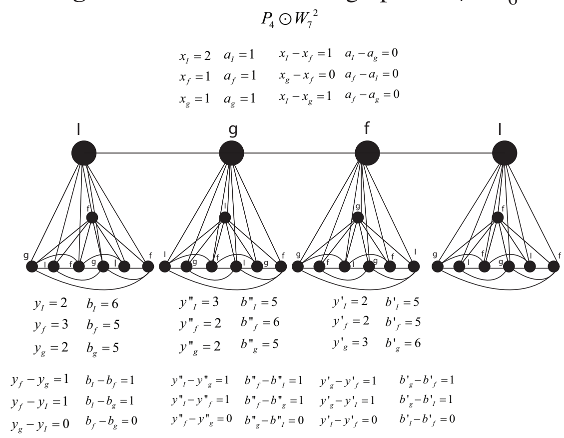

Figure 5. Permuted cordial graph of $P_4 \odot W_7^2$

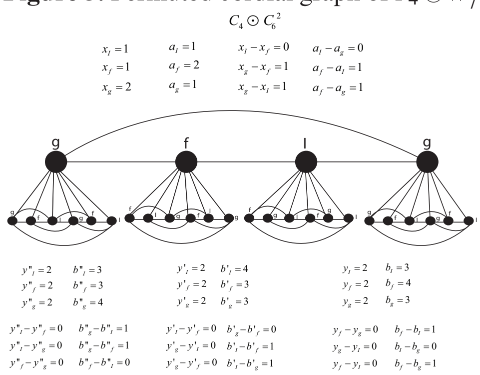

Figure 6. Permuted cordial graph of $C_4 \odot C_6^2$

5 Conclusion

We proved that each $P_n \odot F^2{}_m$, $n \geq 1, m \geq 3$ admits permuted cordial labeling for all $n \geq 1, m \geq 3$. Each $P_n \odot W_m^2$, $n \geq 1, m \geq 3$ admits permuted cordial labeling for all $n \geq 1, m \geq 3$. Each $C_n \odot F_m^2$, $n, m \geq 3$ admits a permuted cordial labeling for all $n, m \geq 3$, and $C_n \odot W_m^2, n, m \geq 3$ admits permuted cordial labeling for all $n, m \geq 3$.
Moreover, we proved that the corona of $C_n \odot P_m^2$, $n, m \geq 3$ admits a permuted cordial labeling for all $n, m \geq 3$. The corona of $C_n \odot C_m^2$, $n, m \geq 3$ admits a permuted cordial labeling for all $n, m \geq 3$. In the future, we will apply permuted cordial labeling to other types of graphs.

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Ashraf Elrokh was born in Melig, Shebin El-Kom, Menoufia, Egypt in 1968. He received the B.S. and M.S. degrees in pure mathematics from faculty of science, Menoufia University, in 1994 and the Ph.D. degree Pure Mathematics (Functional Analysis) from the Faculty of Mathematics and Information, Grethwald University,Germany, in 2004. In addition to over 26 years of teaching and academic experiences. His research interest includes knot theory, graph theory, functional analysis and operation research.

Khalid A. Alsatami received the B.Sc. degree in mathematics from King Saud University, in 2002, the M.Sc. degree in Mathematics and Computational Science from the University of Manchester, UK, in 2008, and the Ph.D. degree in Mathematics West Virginia University, USA, in 2016. In addition, he has 21 years of teaching and academic experience. His research interests include Linear Algebra - numerical analysis - Financial Mathematics -Optimization - Operations Research-Computational Mathematics Discrete Mathematics and graph theory.

Mohamed A. Nouh He is a mathematician from Egypt with a fellowship at Antwerp University from July 1990 to July 1992. His research interests lie in Rings and Algebraic Geometry, particularly rings with large centers, such as those satisfying polynomial identities or birational extensions. He has also explored separable functors and their applications. His work emphasizes the algebraic structures underlying geometric properties.

Yasmin Algrawani received the B.S. in Mathematics and computer science Department from the Faculty of Science, Menoufia University, in 2021. In addition, she has one year of teaching and academic experience. Her research interests include graph theory and, discrete mathematics.