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Exploring Multiple and Singular Solutions for Three equations of Fractional Space-Time KdV Models

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Abstract: This study investigates the fractional Hirota bilinear technique in the context of nonlinear fractional differential models. We focus on the core properties of bilinear fractional differential operators and perform calculations for a variety of fractional differential equations (FDEs), particularly the fractional time-space KdV, KP, and (3+1) KdV models. Utilizing an efficient implementation of Hirota's technique, we leverage symbolic computation to develop solutions. For each equation, we identify both multiple singular soliton and soliton solutions. Importantly, as the fractional order approaches one, our findings converge to the familiar soliton solutions of the KdV, KP, and (3+1) KdV equations. Our results not only build upon existing literature but also contribute to a broader understanding of nonlinear wave dynamics across multiple scientific domains.

Keywords: Multiple soliton solution, singular soliton solution, KdV equation, fractional derivatives.

1 Introduction

Nonlinear evolution equations (NEEs) are prevalent across various scientific fields, such as physics, chemistry, and engineering [1-5]. The pursuit of exact solutions to these equations has spurred significant research efforts [6-8], resulting in the creation of a variety of methods that enhance the discipline [9,10], uncover new solutions, and streamline calculations [11-26]. Notable methods encompass the Backland transformations, Lie group analysis, Darboux transformations variable separation technique, Painlevé properties, Exp-function technique, sine-cosine procedure, and a range of strategies utilizing hyperbolic and geometric functions..

Among the fundamental analytical methods are three primary techniques: the inverse scattering method [1], the Bäcklund transformation method [2, 3], and the Hirota bilinear method [4, 5], which allow us to derive abundant soliton solutions of integrable NEEs. Hirota's bilinear method is particularly distinguished by its intuitive framework, offering significant advantages for directly obtaining several soliton solutions across a broad spectrum of nonlinear evolution models. Moreover, mathematical software systems for example Maple and Mathematica have gained popularity for their ability to efficiently handle complex and tedious calculations.

Converting NEEs into Hirota bilinear formula and deriving its several soliton solution are fundamental processes in the analysis of soliton solutions in modern soliton theory. Hirota bilinear structure, after expressing NEEs in bilinear form through a change of the dependent variable, the N-soliton solution can generally be obtained using the perturbation technique. This process entails expanding the functions in the bilinear equations in terms of a small parameter and truncating them to a finite number of terms. However, using the perturbation technique on bilinear equations can be quite laborious. To overcome this challenge, the bilinear techniques have been enhanced to simplify the derivation of several soliton solutions for soliton equations [27-32].

The fractional space-time KdV model is the generalization of the classical KdV model that incorporates both time and space fractional differentials. This equation can describe more complex wave dynamics in various physical systems. The time-space fractional KdV model typically has the form

$$D_t^{\alpha}\psi + 6\psi D_x^{\alpha}\psi + D_x^{\alpha\alpha\alpha}\psi = 0. \tag{1}$$

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The time-space fractional KdV model offers a more comprehensive framework for understanding wave phenomena, capturing effects that classical equations may overlook. Its solutions provide valuable insights into the dynamics of complex systems, modeling intricate wave behaviors in fluids with memory effects or nonlocal interactions, such as in shallow water waves and ocean dynamics. This equation is also useful in studies of solitons and wave propagation in nonlinear systems, enhancing our understanding of wave stability and interactions. Furthermore, it helps describe materials with non-local response behaviors, particularly in viscoelasticity and diffusion processes. Additionally, it has been employed to model phenomena like population dynamics, where both spatial and temporal processes exhibit fractional characteristics in biological systems. It can also be applied to model certain phenomena in financial markets, where both time and spatial effects are relevant, especially in option pricing.

The fractional space time KP model is the general form of classical KP model that incorporates fractional differentials in both space and time. This model designates the propagation of waves in two-dimensional media and accounts for more complex behaviors due to nonlocal effects

$$D_x^{\alpha}\left(D_t^{\alpha}\psi + 6\psi D_x^{\alpha}\psi + D_x^{\alpha\alpha\alpha}\psi\right) + D_y^{\alpha\alpha}\psi = 0. \tag{2}$$

The space-time fractional KP equation offers a comprehensive framework for understanding wave phenomena, capturing behaviors that classical equations may not fully address. Its solutions provide valuable insights into the dynamics of complex systems. This equation can model wave propagation in shallow water and other fluid systems, effectively representing intricate dynamics. Additionally, it describes light propagation in nonlinear media, incorporating both spatial and temporal effects in nonlinear optics. In plasma physics, it is applicable to the study of nonlinear wave phenomena, including solitons. Furthermore, it can model biological phenomena where spatial and temporal processes exhibit fractional characteristics, particularly in mathematical biology. Lastly, it is useful for describing behaviors in materials that display non-local responses in materials science.

The (3+1) fractional time- space KdV model extend the classical (3+1) KdV model by incorporating fractional differentials in both time and spatial dimensions. This formulation allows for a more general representation of wave phenomena, capturing nonlocal and memory effects in three spatial dimensions and one time dimension. The (3+1) time-space fractional KdV model can be expressed as

$$D_x^{\alpha}\left(D_t^{\alpha}\psi + 6\psi D_x^{\alpha}\psi + D_x^{\alpha\alpha\alpha}\psi\right) + D_x^{\alpha\alpha}\psi + D_z^{\alpha\alpha}\psi = 0. \tag{3}$$

The (3+1) fractional space-time KdV model enhance our understanding of wave phenomena by accommodating complex dynamics that classical equations may not fully capture. Solutions to these equations provide valuable insights into the behavior of various physical systems. They model intricate wave behaviors in three-dimensional fluid systems, such as ocean currents and wave interactions in fluid dynamics. Additionally, they describe the propagation of light in nonlinear media, incorporating spatial and temporal effects in nonlinear optics. These equations are also useful for studying nonlinear wave phenomena in plasmas, including the dynamics of solitons in plasma physics. Furthermore, they model biological processes where spatial and temporal dynamics exhibit fractional characteristics in mathematical biology. Lastly, they apply to the behavior of materials that exhibit non-local responses, particularly in viscoelastic materials within materials science.

Newly, there has been an increasing interest in FDEs because of their wide-ranging applications in engineering and physics. Numerous significant phenomena in fields such as acoustics, electromagnetics, electrochemistry, viscoelasticity, and materials science are effectively modeled by FDEs. However, solving these FDEs can be quite challenging. Typically, there is no universal techniques that produces explicit and numerical solution for nonlinear FDEs [15-26].

The purpose of this manuscript is to extend the Hirota bilinear technique to tackle the fractional space-time KdV model, the fractional space-time KP model, and the fractional (3+1) space-time KdV model. In this framework, fractional differentials are considered in the conformable sense. Non-integer calculus encompasses the idea of differentials and integrals of random order, effectively unifying and generalizing the principles of integer-order derivatives and repeated integrals. Numerous books have been published on non-integer calculus, discussing numerous classifications of non-integer differentiation and integration, including those by Riemann–Liouville, Grünwald–Letnikov, Caputo, and the modified Riemann–Liouville. For the purposes of this study, we will employ the conformable fractional differential (CFD).

Khalil et al. in [24] presented the CFD in the limit form as

$$D^{\alpha}\psi(s) = \lim_{\varepsilon \to o} \frac{\psi(s+\varepsilon s^{1-\alpha}) - \psi(s)}{\varepsilon}, \quad \psi^{(\alpha)}(0) = \lim_{s \to o^{+}} \psi^{(\alpha)}(s),$$

with $s > 0$ and $\alpha\,(0,1]$ since $\psi^{(\alpha)}(0)$ is not determined. This fractional differential goes back to the famous integer differential at $\alpha = 1$, The corresponding CFD satisfies the will know axioms of differentiation see [20- 26].

The structure of this research leave is constructed as follows: Section 2 presents the techniques of Hirota's bilinear approach as applied to nonlinear FDEs. Section 3 discusses the several soliton solutions of the fractional space-time KdV model. In Section 4, we explore the several soliton solutions of the fractional space-time KP model, while Section 5 addresses the several soliton solutions of the (3+1) fractional space-time KdV model. Finally, we conclude the paper in Section 6.

2 The fractional Hirota bilinear technique formalism

This approach is referred to Hirota bilinear formalism or the Hirota direct method. A crucial requirement for utilizing this technique is the ability to transform the FDE into a bilinear form. Hirota showed that soliton solutions can be represented as polynomials of exponentials within this bilinear framework. However, finding bilinear formulaes of nonlinear FDEs—when they exist—can be quite difficult. The Hirota bilinear approach is mainly effective for handling several soliton solutions across various NEEs [27-32]. The standard explanation of the fractional Hirota bilinear formulae is provided by

$$D_x^{\alpha\alpha\ldots\alpha(n\,times)} D_t^{\alpha\alpha\ldots\alpha(m\,times)}(P.Q) = (D_x^\alpha - D_{x'}^\alpha)^n (D_t^\alpha - D_{t'}^\alpha)^m P(t^\alpha, x^\alpha) Q(t^\alpha, x^\alpha)|_{x=x', t=t'}. \tag{4}$$

In the following, we will present some of the bilinear fractional differential operators

$$\begin{aligned}
&D_x^\alpha(P.Q) = (D_x^\alpha P)Q - PD_x^\alpha(Q),\\
&D_x^{\alpha\alpha}(P.Q) = (D_x^{\alpha\alpha}P)Q - 2D_x^\alpha P D_x^\alpha Q + PD_x^{\alpha\alpha}Q,\\
&D_x^{\alpha\alpha\ldots\alpha(n\,times)}(Q.Q) = 0, \quad for\ n\ odd,\\
&D_x^\alpha D_t^\alpha(P.Q) = D_x^\alpha((D_t^\alpha P)Q - PD_t^\alpha(Q))\\
&\qquad = (D_x^\alpha D_t^\alpha P)Q - D_x^\alpha P D_t^\alpha Q - D_t^\alpha P D_x^\alpha Q + P(D_x^\alpha D_t^\alpha Q),\\
&D_x^\alpha D_t^\alpha(Q.Q) = 2(QD_x^\alpha D_t^\alpha Q - D_x^\alpha Q D_t^\alpha Q),\\
&D_x^{\alpha\alpha\alpha\alpha}(P.Q) = (D_x^{\alpha\alpha\alpha\alpha}P)Q - 4D_x^{\alpha\alpha\alpha}PD_x^\alpha Q + 6D_x^{\alpha\alpha}PD_x^{\alpha\alpha}Q - 4D_x^\alpha PD_x^{\alpha\alpha\alpha}Q + PD_x^{\alpha\alpha\alpha\alpha}Q,
\end{aligned} \tag{5}$$

Moreover, the properties of the fractional operators are as follows

$$\begin{aligned}
&\frac{D_x^{\alpha\alpha}(\varphi.\varphi)}{\varphi^2} = \psi,\\
&\frac{D_t^{\alpha\alpha}(\varphi.\varphi)}{\varphi^2} = \int\int D_t^{\alpha\alpha}\psi dx^\alpha dx^\alpha,\\
&\frac{D_t^\alpha D_x^\alpha(\varphi.\varphi)}{\varphi^2} = D_t^\alpha D_x^\alpha(\ln\varphi^2),\\
&\frac{D_t^\alpha D_x^{\alpha\alpha\alpha}(\varphi.\varphi)}{\varphi^2} = D_t^\alpha D_x^\alpha\psi + 3\psi\int xD_t^\alpha\psi dx'^\alpha,\\
&\frac{D_x^{\alpha\alpha\alpha\alpha}(\varphi.\varphi)}{\varphi^2} = D_x^{\alpha\alpha}\psi + 3\psi^2,\\
&\frac{D_x^{\alpha\alpha\alpha\alpha\alpha\alpha}(\varphi.\varphi)}{\varphi^2} = D_x^{\alpha\alpha\alpha\alpha}\psi + 15\psi^3 + 15\psi D_x^{\alpha\alpha}\psi,
\end{aligned} \tag{6}$$

with $\psi(t^\alpha, x^\alpha) = 2D_x^{\alpha\alpha}(\ln\varphi(t^\alpha, x^\alpha))$ For example, the fractional space time KdV model

$$D_t^\alpha\psi + 6\psi D_x^\alpha\psi + D_t^{\alpha\alpha\alpha}\psi = 0, \tag{7}$$

Can be converted into the fractional bilinear form as

$$(D_x^\alpha D_t^\alpha + D_x^{\alpha\alpha\alpha\alpha})(\varphi.\varphi) = 0, \qquad \psi = 2D_x^{\alpha\alpha}(\ln\varphi). \tag{8}$$

Similarly, the space time fractional KP equation

$$D_x^\alpha(D_t^\alpha\psi + 6\psi D_x^\alpha\psi + D_t^{\alpha\alpha\alpha}\psi) + D_y^{\alpha\alpha}\psi = 0. \tag{9}$$

Can be converted into the fractional bilinear form as

$$(D_x^\alpha D_t^\alpha + D_x^{\alpha\alpha\alpha\alpha} + 3D_y^{\alpha\alpha})(\varphi.\varphi) = 0, \qquad \psi = 2D_x^{\alpha\alpha}(\ln\varphi). \tag{10}$$

Also, the space time fractional (3+1) KdV equation

$$D_x^\alpha(D_t^\alpha\psi + 6\psi D_x^\alpha\psi + D_t^{\alpha\alpha\alpha}\psi) + D_y^{\alpha\alpha}\psi + D_z^{\alpha\alpha}\psi = 0. \tag{11}$$

Can be converted into the fractional bilinear form as

$$(D_x^\alpha D_t^\alpha + D_x^{\alpha\alpha\alpha\alpha} + 3D_y^{\alpha\alpha} + 3D_z^{\alpha\alpha})(\varphi.\varphi) = 0, \qquad \psi = 2D_x^{\alpha\alpha}(\ln\varphi). \tag{12}$$

3 The fractional Hirota Technique

The simplified fractional Hirota technique explores soliton solutions that can be represented as polynomial expressions of exponentials. This approach eliminates the need to create the bilinear forms typically required by the original Hirota technique, allowing us to tackle FDE more directly. Below, we outline the key steps of the simplified fractional method.

The basic version of the fractional Hirota technique discusses the soliton solutions which can be expressed as polynomials of exponentials. By means of the simplified fractional Hirota technique, there is no requirement to make bilinear forms suggested by Hirota procedure. Instead, we can approach the FDE in a straightforward fashion. In what follows we tilt the little steps of the simplified fractional technique:
First, we put

$$\psi = e^{(kx^{\alpha}-\omega t^{\alpha})/\alpha}, \tag{13}$$

in the linear components of the model at hand to derive the dispersion relative between k and ω. Next, we supernumerary the one-soliton solution

$$\begin{aligned} &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha}(\ln\varphi(x^{\alpha},t^{\alpha})) = R\frac{D_x^{\alpha}\varphi}{\varphi},\\ or\ \ &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha\alpha}(\ln\varphi(x^{\alpha},t^{\alpha})) = R\frac{\varphi D_x^{\alpha\alpha}\varphi-(D_x^{\alpha})^2}{\varphi^2},\\ or\ \ &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha}(\arctan(\frac{\varphi(x^{\alpha},t^{\alpha})}{\phi(x^{\alpha},t^{\alpha})}) = R\frac{\phi D_x^{\alpha}\varphi-\varphi D_x^{\alpha}\phi}{\phi^2+\varphi^2} \end{aligned} \tag{14}$$

into the equation currently being analyzed, where the auxiliary function φ is definite as

$$\varphi = 1+\varphi_1 = 1+e^{\theta_1}, \qquad \theta_1 = (k_1x^{\alpha}-\omega_1t^{\alpha})/\alpha. \tag{15}$$

We proceed to resolve the subsequent equation to calculate the arithmetical value of R. It's worth mentioning that N-soliton solutions for the model can be achieved by applying the successive steps:
1- To obtain the dispersion relative, we usage

$$\psi = e^{\theta_i}, \quad \theta_i = (k_ix^{\alpha}-\omega_t^{\alpha})/\alpha, \qquad i = 1,2,3,...N, \tag{16}$$

2- To discuss 1-soliton, we put

$$\phi = 1+e^{\theta_1}, \tag{17}$$

3- To establish 2-soliton, we usage

$$\phi = 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \tag{18}$$

4- To construct 3-soliton, we usage

$$\phi = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3} \tag{19}$$

Observe that equation (18) is used to generalize the phase shift a_{12} for the extra elements, a_{ij} , and ultimately we apply equation (19) to find b_{123} , defined as $b_{123} = a_{12}a_{23}a_{31}$ for the FDE. The identification of 3-soliton supports the existence of N-soliton for any order.

Conversely, to construct several singular soliton, we usage

$$\psi = e^{(kx^{\alpha}-\omega t^{\alpha})/\alpha}, \tag{20}$$

in the linear components of the model at hand to derive the dispersion relative between k and ω. Substituting into the 1-soliton

$$\begin{aligned} &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha}(\ln\varphi(x^{\alpha},t^{\alpha})) = R\frac{D_x^{\alpha}\varphi}{\varphi},\\ or\ \ &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha\alpha}(\ln\varphi(x^{\alpha},t^{\alpha})) = R\frac{\varphi D_x^{\alpha\alpha}\varphi-(D_x^{\alpha}\varphi)^2}{\varphi^2},\\ or\ \ &\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha}(\arctan(\frac{\varphi(x^{\alpha},t^{\alpha})}{\phi(x^{\alpha},t^{\alpha})}) = R\frac{\phi D_x^{\alpha}\varphi-\varphi D_x^{\alpha}\phi}{\phi^2+\varphi^2} \end{aligned} \tag{21}$$

in the model being analyzed, with the auxiliary function φ is specified as

$$\varphi = 1-\varphi_1 = 1-e^{\theta_1}, \qquad \theta_1 = (k_1x^{\alpha}-\omega_1t^{\alpha})/\alpha. \tag{22}$$

We next resolve the subsequent outcomes to find the mathematical value of R. It is important to footnote that the N-singular soliton for the equation can be derived using the following steps:
1- To establish the dispersion relative, we apply

$$\psi = e^{\theta_i}, \quad \theta_i = (k_ix^{\alpha}-\omega_t^{\alpha})/\alpha, \qquad i = 1,2,3,...N, \tag{23}$$

2- To gain the singular 1-soliton, we utilize

$$\phi = 1 - e^{\theta_1}, \tag{24}$$

3- To establish 2-soliton, we have

$$\phi = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{25}$$

4- To construct the singular 3-soliton, we utilize

$$\phi = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3} \tag{26}$$

The primary benefit of the simplified version of the Hirota technique is its ability to address a wide range of problems without relying on bilinear formulae or preventive assumptions that could alter the physical characteristics of the resolution. The arena of dynamical integrable models has produced numerous valuable advancements, stemming from the productive interplay between mathematics and physics. As previously mentioned, this simplified approach enables the derivation of multiple-soliton solutions without the need for predetermined conditions. In contrast, traditional methods often involve complex calculations to obtain these solutions. Our examples illustrate that a diverse array of multiple-soliton solutions can be generated, rather than being limited to a single set. This promising finding warrants further investigation to determine its applicability to other integrable models, and it may also inspire the use of alternative methods to explore this new insight.

4 Several-soliton of the fractional space time KdV model

The fractional space-time KdV model generalizes the classical KdV model by incorporating fractional derivatives in both time and spatial dimensions. This formulation describes complex interactions among waveforms and captures nonlocal effects, making it applicable to a range of physical phenomena. The space-time fractional KdV equation provides a more comprehensive framework for understanding wave interactions, accommodating dynamics that classical models might overlook. Solutions to this equation can yield valuable insights into complex systems across various scientific disciplines. The fractional space-time KdV model have the form

$$D_t^{\alpha}\psi + 6\psi D_x^{\alpha}\psi + D_x^{\alpha\alpha\alpha}\psi = 0. \tag{27}$$

with $D_t^{\alpha}, D_x^{\alpha}$ is CFD w.r.t t and x respectively, and $D_x^{\alpha\alpha\alpha}\psi = D_x^{\alpha}D_x^{\alpha}D_x^{\alpha}\psi$. Replacing

$$\psi = e^{(kx^{\alpha}-\omega t^{\alpha})/\alpha}, \tag{28}$$

in the linear aspects of model (27) to derive the dispersion relative between k and ω, which take the form

$$\omega = k^3, \tag{29}$$

this involves the variables for the next stage

$$\theta_i = (k_i x^{\alpha} - k_i^3 t^{\alpha})/\alpha, \quad i = 1,2,3,...,N, \tag{30}$$

we derive the multi-soliton of the fractional space-time KdV model (27) using Hirota's fractional approach, assuming the form of

$$\psi(x^{\alpha},t^{\alpha}) = RD_x^{\alpha\alpha}(\ln\varphi(x^{\alpha},t^{\alpha})) = R\frac{\varphi D_x^{\alpha\alpha}\varphi - (D_x^{\alpha}\varphi)^2}{\varphi^2}, \tag{31}$$

define the auxiliary function for the 1-soliton as

$$\varphi = 1 + e^{\theta_1}, \quad \theta_1 = (k_i x^{\alpha} - k_i^3 t^{\alpha})/\alpha, \tag{32}$$

By substituting (31) in (27), we can determine a solution for R using the formula

$$R = 2. \tag{33}$$

Thus, the 1-soliton is expressed as

$$\psi = \frac{2k_1^2 e^{(k_1x^{\alpha}-k_1^3t^{\alpha})/\alpha}}{(1+e^{(k_1x^{\alpha}-k_1^3t^{\alpha})/\alpha})^2}. \tag{34}$$

To deduce two solitons, we define the auxiliary function as

$$\varphi = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \quad \theta_1 = (k_i x^\alpha - k_i^3 t^\alpha)/\alpha, \quad i = 1,2. \tag{35}$$

Using (36) in (31) and then inserting the result into (27), the coefficient phase shift can be written in the form

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{36}$$

and hence we set the coefficient phase shifts as

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i \le j \le 3, \tag{37}$$

By utilizing the results from (36) and (35) and inserting them into (31), we derive the two-soliton solutions. For the 3-soliton, we define

$$\begin{aligned} \varphi &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ \theta_1 &= (k_i x^\alpha - k_i^3 t^\alpha)/\alpha, \quad i = 1,2,3. \end{aligned} \tag{38}$$

Following the same approach, we discovery that

$$b_{123} = a_{12}a_{23}a_{31}, \tag{39}$$

This demonstrates that 3-soliton can be gotten. The presence of these outcomes often suggests the integrability of the model under study. Conversely, integrability must be verified through additional approaches.

Doing the same manner as above we can construct the singular 1-soliton as

$$\psi = \frac{2k_1^2\, e^{(k_1 x^\alpha - k_1^3 t^\alpha)/\alpha}}{(1 - e^{(k_1 x^\alpha - k_1^3 t^\alpha)/\alpha})^2}. \tag{40}$$

To deduce singular 2-soliton, we define the function φ as

$$\varphi = 1 - e^{\theta_1} - e^{\theta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\theta_1+\theta_2}, \quad \theta_1 = (k_i x^\alpha - k_i^3 t^\alpha)/\alpha, \quad i = 1,2. \tag{41}$$

By utilizing the results from (41) into (31) and inserting them into (31), we derive the singular 2-soliton. For the singular 3-soliton, we define

$$\begin{aligned} &\varphi = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ &a_{ij} = \tfrac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad b_{123} = a_{12}a_{23}a_{31}, \quad 1 \le i \le j \le 3, \quad \theta_i = (k_i x^\alpha - k_i^3 t^\alpha)/\alpha, \quad i = 1,2,3. \end{aligned} \tag{42}$$

Substituting the equation (42) into (31), we obtain the singular three-soliton.

5 Multiple-soliton solutions of the fractional space time KP model

The fractional space-time KP model generalized the classical KP model by incorporating fractional derivatives in both time and spatial dimensions. This equation is essential for modeling wave propagation in two-dimensional media, particularly when accounting for nonlocal effects and memory. The space-time fractional KP equation provides a comprehensive framework for understanding multi-dimensional wave phenomena, accommodating complex dynamics that classical equation may overlook. Solution to this equation can yield insights into the behavior of various physical systems and interactions. The space time fractional KP model can be presented in the formula

$$D_x^\alpha\,(D_t^\alpha\,\psi + 6\,\psi D_x^\alpha\,\psi + D_x^{\alpha\alpha\alpha}\,\psi) + D_y^{\alpha\alpha}\,\psi = 0. \tag{43}$$

Replacing

$$\psi = e^{(kx^\alpha + ry^\alpha - \omega t^\alpha)/\alpha}, \tag{44}$$

in the linear aspects of (43) to derive the dispersion relative between k,r and ω, which take the form

$$\omega = \frac{k^4+r^2}{k}, \tag{45}$$

this involves the variables for the next stage

$$\theta_i = (k_i x^\alpha + r_i y^\alpha - (k_i^4+r_i^2)t^\alpha/k_i)/\alpha, \quad i=1,2,3,...,N, \tag{46}$$

we derive the multi-soliton of the fractional space-time KP model (43) using Hirota's fractional technique, assuming the form of

$$\psi(x^\alpha,y^\alpha,t^\alpha) = RD_x^{\alpha\alpha}(\ln\varphi(x^\alpha,y^\alpha,t^\alpha)) = R\frac{\varphi D_x^{\alpha\alpha}\varphi-(D_x^\alpha\varphi)^2}{\varphi^2}, \tag{47}$$

define the function φ for the 1-soliton take the following formula

$$\psi = 1+e^{\theta_1}, \quad \theta_1 = (k_1 x^\alpha + r_1 y^\alpha - (k_1^4+r_1^2)t^\alpha/k_1)\alpha, \tag{48}$$

By substituting (47) in (43), we can determine a solution for R using the formula

$$R=2. \tag{49}$$

Thus, the 1-soliton is expressed as

$$\psi = \frac{2k_1^2\, e^{(k_1x^\alpha+r_1y^\alpha-(k_1^4+r_1^2)t^\alpha/k_1)/\alpha}}{(1+e^{(k_1x^\alpha+r_1y^\alpha-(k_1^4+r_1^2)t^\alpha/k_1)/\alpha})^2}. \tag{50}$$

To deduce 2-soliton, we define the function φ in this form

$$\varphi = 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \quad \theta_i = (k_i x^\alpha + r_i y^\alpha - (k_i^4+r_i^2)t^\alpha/k_i)\alpha, \quad i=1,2. \tag{51}$$

Using (51) in (47) and then inserting the result into (43), the coefficient phase shift can be written in the form

$$a_{12} = \frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}, \tag{52}$$

and hence we set the coefficient phase shifts as

$$a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_ir_j-k_jr_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_ir_j-k_jr_i)^2}, \quad 1\le i\le j\le 3, \tag{53}$$

By utilizing the results from (52) and (51) and inserting them into (43), we derive the 2-soliton. For the 3-soliton, we can define

$$\begin{aligned} &\varphi = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{23}e^{\theta_2+\theta_3}+a_{13}e^{\theta_1+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3},\\ &\theta_i = (k_i x^\alpha + r_i y^\alpha - (k_i^4+r_i^2)t^\alpha/k_i)\alpha, \quad i=1,2,3, \end{aligned} \tag{54}$$

Following the same approach, we discovery that

$$b_{123} = a_{12}a_{23}a_{31}, \tag{55}$$

This demonstrates that the 3-soliton can be gained. The presence of these results often suggests the integrability of the model under study. Though, integrability must be verified through additional approaches.

Doing the same manner as above we can construct the singular one soliton n the following form

$$\psi = \frac{2k_1^2\, e^{(k_1x^\alpha+r_1y^\alpha-(k_1^4+r_1^2)t^\alpha/k_1)/\alpha}}{(1-e^{(k_1x^\alpha+r_1y^\alpha-(k_1^4+r_1^2)t^\alpha/k_1)/\alpha})^2}. \tag{56}$$

To deduce singular two-soliton, we define the function in the formula

$$\begin{aligned} &\varphi = 1-e^{\theta_1}-e^{\theta_2}+\tfrac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}e^{\theta_1+\theta_2},\\ &\theta_i = (k_i x^\alpha + r_i y^\alpha - (k_i^4+r_i^2)t^\alpha/k_i)\alpha, \quad i=1,2. \end{aligned} \tag{57}$$

By utilizing the results from (57) into (47) and inserting them into (43), we derive the singular two-soliton. For the singular three-soliton, we define

$$\begin{aligned} &\varphi = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ &a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_ir_j-k_jr_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_ir_j-k_jr_i)^2}, \quad 1 \le i \le j \le 3, \\ &\theta_i = (k_ix^\alpha + r_iy^\alpha - (k_i^4+r_i^2)t^\alpha/k_i)\alpha, \quad i=1,2,3, \quad b_{123} = a_{12}a_{23}a_{31}. \end{aligned} \tag{58}$$

Substituting the equation (58) into (43), we gain the singular three-soliton.

6 Several-soliton of the (3+1) fractional space time KdV model

The (3+1) fractional space-time KdV model is an extension of the classical (3+1) KdV model that incorporates fractional derivatives in both space and time across three spatial dimensions and one time dimension. This equation is particularly useful for modeling complex wave interactions in higher-dimensional systems, capturing nonlocal effects and memory characteristics. The (3+1) fractional space-time KdV model provide a more sophisticated framework for analyzing multi-dimensional wave phenomena, accommodating complex dynamics that classical models may not fully capture. Solutions to this equation can offer deeper insights into the behavior of various physical systems and their interactions. The (3+1) fractional space-time KdV model can be written as

$$D_x^\alpha\,(D_t^\alpha\,\psi\,+6\,\psi D_x^\alpha\,\psi + D_x^{\alpha\alpha\alpha}\,\psi) + D_y^{\alpha\alpha}\,\psi + D_z^{\alpha\alpha}\,\psi = 0. \tag{59}$$

Replacing

$$\psi = e^{(kx^\alpha + ry^\alpha + iiz^\alpha - \omega t^\alpha)/\alpha}, \tag{60}$$

in the linear aspects of (43) to derive the dispersion relative between k, r, l and ω, which take the form

$$\omega = \frac{k^4+r^2+l^2}{k}, \tag{61}$$

this involves the variables for the next stage

$$\theta_i = (k_ix^\alpha + r_iy^\alpha + l_iz^\alpha - (k_i^4+r_i^2+l_i^2)t^\alpha/k_i)/\alpha, \quad i = 1,2,3,\ldots,N, \tag{62}$$

we derive the multi-soliton of the (3+1) fractional space-time KdV model (99) using Hirota's fractional technique, assuming the form of

$$\psi(x^\alpha,y^\alpha,z^\alpha,t^\alpha) = RD_x^{\alpha\alpha}(\ln\varphi(x^\alpha,y^\alpha,z^\alpha,t^\alpha)) = R\frac{\varphi D_x^{\alpha\alpha}\varphi - (D_x^\alpha\varphi)^2}{\varphi^2}, \tag{63}$$

define the function φ for the 1-soliton take the following formula

$$\psi = 1 + e^{\theta_1}, \quad \theta_i = (k_1x^\alpha + r_1y^\alpha + l_1z^\alpha - (k_1^4+r_1^2+l_1^2)t^\alpha/k_1)/\alpha, \tag{64}$$

By substituting (64) in (59), we can determine a solution for R using the formula

$$R = 2. \tag{65}$$

Thus, the 1-soliton is expressed as

$$\psi = \frac{2\,k_1^2\,e^{(k_1x^\alpha + r_1y^\alpha + l_1z^\alpha - (k_1^4+r_1^2+l_1^2)t^\alpha/k_1)/\alpha}}{(1+e^{(k_1x^\alpha + r_1y^\alpha + l_1z^\alpha - (k_1^4+r_1^2+l_1^2)t^\alpha/k_1)/\alpha})^2}. \tag{66}$$

To deduce 2-soliton, we define the function φ in this form

$$\varphi = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \quad \theta_i = (k_ix^\alpha + r_iy^\alpha + l_iz^\alpha - (k_i^4+r_i^2+l_i^2)t^\alpha/k_i)/\alpha, \quad i=1,2, \tag{67}$$

Using (67) in (63) and then inserting the result into (59), the coefficient phase shift can be written in the form

$$a_{12} = \frac{3k_1^2k_2^2(k_1-k_2)^2 - (k_1r_2-k_2r_1)^2 - (k_1l_2-k_2l_1)^2}{3k_1^2k_2^2(k_1+k_2)^2 - (k_1r_2-k_2r_1)^2 - (k_1l_2-k_2l_1)^2}, \tag{68}$$

and hence we set the coefficient phase shifts as

$$a_{ij} = \frac{3k_i^2k_j^2(k_i - k_j)^2 - (k_ir_j - k_jr_i)^2 - (k_il_j - k_jl_i)^2}{3k_i^2k_j^2(k_i + k_j)^2 - (k_ir_j - k_jr_i)^2 - (k_il_j - k_jl_i)^2}, \quad 1 \le i \le j \le 3, \tag{69}$$

By utilizing the results from (68) into (63) and inserting them into (59), we derive the two- soliton solutions. For the 3-soliton, we define

$$\begin{aligned} \varphi &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ \theta_i &= (k_ix^\alpha + r_iy^\alpha + l_iz^\alpha - (k_i^4 + r_i^2 + l_i^2)t^\alpha/k_i)/\alpha, \quad i = 1,2,3, \end{aligned} \tag{70}$$

Following the same approach, we find that

$$b_{123} = a_{12}a_{23}a_{31}, \tag{71}$$

Substituting the equation (71), (70) into (63), we gain the three-soliton. Similarly, the singular soliton can be gained using the approach outlined earlier

7 Conclusion

We have successfully derived 1-soliton, 2-soliton, 3-soliton, and general N-soliton for three space-time FDEs using fractional Hirota's bilinear technique. The equations analyzed include the fractional space-time KdV, KP, and (3+1) KdV models. This paper illustrates that the fractional Hirota technique can effectively serve as a mathematical tool for constructing multi-soliton solutions of nonlinear FDEs. Our discussions reveal several important points: firstly, soliton solutions can be represented as polynomials of exponentials, as highlighted by Hirota, Hietarinta, and others. Furthermore, the three-soliton and those of higher orders do not introduce any new free parameters beyond those aij resulting from the two-soliton. Any NEEs and FDEs that possesses generate $N = 3$ soliton will also have multi-soliton for any $N \ge 4$. One of the primary advantages of the simplified style of the fractional Hirota technique is its ability to tackle problems without requiring bilinear formulas or imposing obstructive expectations that could alter the physical behavior of the solutions. The arena of dynamical integrable models has yielded much valuable advancement, largely owing to the successful interplay between physics and mathematics. As previously mentioned, the simplified fractional Hirota technique produces multiple-soliton solutions without the need for predetermined conditions. Existing techniques often involve tedious calculations to derive these solutions. Our examples demonstrate that a diversity of multi-soliton can be gotten, rather than yielding just a single set. This newly conventional result warrants further investigation to determine its applicability to other NEEs and FDEs, and other present procedures may also be employed to explore this innovative finding.

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