Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110211

Voronovskaya Type Asymptotic Expansions for the Approximation by Symmetrized and Perturbed Hyperbolic Tangent Activated Convolution Type Operators, Ordinary and Fractional Approaches

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Received: 14 Sep. 2024, Revised: 22 Nov. 2024, Accepted: 5 Dec. 2024
Published online: 1 Apr. 2025

Abstract: Here we study further the symmetrized and perturbed hyperbolic tangent activated convolution type operators approximation properties. We derive Voronovskaya type asymptotic expansions for the errors of these approximations. We cover the cases of univariate: basic, Kantorovich and quadrature operators. We present ordinary and fractional results. At the end we give related simultaneous Voronovskaya type asymptotic expansions. Shape preservation via these operators is discussed too.

Keywords: symmetrized and perturbed hyperbolic tangent convolution type operator, Caputo fractional derivative, Voronovskaya asymptotic expansion, simultaneous asymptotic expansion.

1 Introduction

We are motivated by [1], Chapters 5, 6, 8, 9, 19, 25, and [2], Chapter 26.

In section 2, we present the needed neural network theory, especially the deformed, parametrized symmetrized theory, and we come up with a symmetrized mixed density function over $\mathbb{R}$.

In section 3, we give our basics and introduce our three activated hyperbolic tangent perturbed and symmetrized convolution type univariate operators: the basic, Kantorovich and quadrature types, and mention their related here properties, see [3].

In section 4, we present our main results, namely Voronovskaya type asymptotic expansions related to the approximations of above mentioned operators to the unit operator. We give ordinary differentiation and fractional differentiation results, as well as their simultaneous asymptotic expansions results.

We also present shape preservation results by our studied operators. Further inspiration here comes from [4]-[13].

2 About q-deformed and $\lambda-$parametrized hyperbolic tangent function $g_{q,\lambda}$

Here, all this initial background comes from Chapter 18, [2].

We use $g_{q,\lambda}$, see (1), and exhibit that it is a sigmoid function and we will present several of its properties related to the approximation by neural network operators.

So, let us consider the hyperbolic tangent activation function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

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We have that

$$g_{q,\lambda}(0)=\frac{1-q}{1+q}.$$

We notice also that

$$g_{q,\lambda}(-x)=\frac{e^{-\lambda x}-qe^{\lambda x}}{e^{-\lambda x}+qe^{\lambda x}}=\frac{\frac{1}{q}e^{-\lambda x}-e^{\lambda x}}{\frac{1}{q}e^{-\lambda x}+e^{\lambda x}}=-\frac{\left(e^{\lambda x}-\frac{1}{q}e^{-\lambda x}\right)}{e^{\lambda x}+\frac{1}{q}e^{-\lambda x}}=-g_{\frac{1}{q},\lambda}(x). \tag{2}$$

That is

$$g_{q,\lambda}(-x)=-g_{\frac{1}{q},\lambda}(x),\ \forall\, x\in\mathbb{R}, \tag{3}$$

and

$$g_{\frac{1}{q},\lambda}(x)=-g_{q,\lambda}(-x),$$

hence

$$g'_{\frac{1}{q},\lambda}(x)=g'_{q,\lambda}(-x). \tag{4}$$

It is

$$g_{q,\lambda}(x)=\frac{e^{2\lambda x}-q}{e^{2\lambda x}+q}=\frac{1-\frac{q}{e^{2lx}}}{1+\frac{q}{e^{2\lambda x}}}\underset{(x\to+\infty)}{\to}1,$$

i.e.

$$g_{q,\lambda}(+\infty)=1, \tag{5}$$

Furthermore

$$g_{q,\lambda}(x)=\frac{e^{2\lambda x}-q}{e^{2\lambda x}+q}\underset{(x\to-\infty)}{\to}\frac{-q}{q}=-1,$$

i.e.

$$g_{q,\lambda}(-\infty)=-1. \tag{6}$$

We find that

$$g'_{q,\lambda}(x)=\frac{4q\lambda e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^2}>0, \tag{7}$$

therefore $g_{q,\lambda}$ is striclty increasing.

Next we obtain ($x\in\mathbb{R}$)

$$g''_{q,\lambda}(x)=8q\lambda^2e^{2\lambda x}\left(\frac{q-e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^3}\right)\in C(\mathbb{R}). \tag{8}$$

We observe that

$$q-e^{2\lambda x}\gtreqless 0\Leftrightarrow q\gtreqless e^{2\lambda x}\Leftrightarrow \ln q\gtreqless 2\lambda x\Leftrightarrow x\lesseqgtr\frac{\ln q}{2\lambda}.$$

So, in case of $x<\frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=0$.

And in case of $x>\frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0)=\frac{1-q}{1+q}$, and $g_{q,\lambda}(-x)=-g_{q^{-1},\lambda}(x)$, (a semi-odd function).

By $1>-1$, $x+1>x-1$, we consider the function

$$M_{q.\lambda}(x):=\frac{1}{4}\left(g_{q,\lambda}(x+1)-g_{q,\lambda}(x-1)\right)>0, \tag{9}$$

$\forall x\in\mathbb{R}$; $q,\lambda>0$. Notice that $M_{q,\lambda}(\pm\infty)=0$, so the x-axis is horizontal asymptote.

We have that

$$M_{q,\lambda}(-x)=\frac{1}{4}\left(g_{q,\lambda}(-x+1)-g_{q,\lambda}(-x-1)\right)=$$

$$\frac{1}{4}\left(g_{q,\lambda}(-(x-1))-g_{q,\lambda}(-(x+1))\right)=$$

$$\frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1)+g_{\frac{1}{q},\lambda}(x+1)\right)= \tag{10}$$
$$\frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1)-g_{\frac{1}{q},\lambda}(x-1)\right)=M_{\frac{1}{q},\lambda}(x),\ \ \forall\, x\in\mathbb{R}.$$

Thus

$$M_{q,\lambda}(-x)=M_{\frac{1}{q},\lambda}(x),\ \forall\, x\in\mathbb{R};\ q,\lambda>0, \tag{11}$$

a deformed symmetry.

Next, we have that

$$M'_{q,\lambda}(x)=\frac{1}{4}\left(g'_{q,\lambda}(x+1)-g'_{q,\lambda}(x-1)\right),\ \ \forall\, x\in\mathbb{R}. \tag{12}$$

Let $x<\frac{\ln q}{2\lambda}-1$, then $x-1<x+1<\frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1)>g'_{q,\lambda}(x-1)$ (by $g_{q,\lambda}$ being strictly concave up for $x<\frac{\ln q}{2\lambda}$), that is $M'_{q,\lambda}(x)>0$. Hence $M_{q,\lambda}$ is striclty increasing over $\left(-\infty,\frac{\ln q}{2\lambda}-1\right)$.

Let now $x-1>\frac{\ln q}{2\lambda}$, then $x+1>x-1>\frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1)<g'_{q,\lambda}(x-1)$, that is $M'_{q,\lambda}(x)<0$.

Therefore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda}+1,+\infty\right)$.

Let us next consider, $\frac{\ln q}{2\lambda}-1\le x\le\frac{\ln q}{2\lambda}+1$. We have that

$$M''_{q,\lambda}(x)=\frac{1}{4}\left(g''_{q,\lambda}(x+1)-g''_{q,\lambda}(x-1)\right)=$$
$$2q\lambda^{2}\left[e^{2\lambda(x+1)}\left(\frac{q-e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)}+q\right)^{3}}\right)-e^{2\lambda(x-1)}\left(\frac{q-e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)}+q\right)^{3}}\right)\right]. \tag{13}$$

By $\frac{\ln q}{2\lambda}-1\le x\Leftrightarrow\frac{\ln q}{2\lambda}\le x+1\Leftrightarrow\ln q\le 2\lambda(x+1)\Leftrightarrow q\le e^{2\lambda(x+1)}\Leftrightarrow q-e^{2\lambda(x+1)}\le 0$.

By $x\le\frac{\ln q}{2\lambda}+1\Leftrightarrow x-1\le\frac{\ln q}{2\lambda}\Leftrightarrow 2\lambda(x-1)\le\ln q\Leftrightarrow e^{2\lambda(x-1)}\le q\Leftrightarrow q-e^{2\lambda\beta(x-1)}\ge 0$.

Clearly by (13) we get that $M''_{q,\lambda}(x)\le 0$, for $x\in\left[\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right]$.

More precisely $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right)$.

Consequently $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

Of course it holds $M''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)<0$.

At $x=\frac{\ln q}{2\lambda}$, we have

$$M'_{q,\lambda}(x)=\frac{1}{4}\left(g'_{q,\lambda}(x+1)-g'_{q,\lambda}(x-1)\right)=$$
$$q\lambda\left(\frac{e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)}+q\right)^{2}}-\frac{e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)}+q\right)^{2}}\right). \tag{14}$$

Thus

$$M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=q\lambda\left(\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}+q\right)^{2}}-\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}+q\right)^{2}}\right)=$$
$$\lambda\left(\frac{e^{2\lambda}\left(e^{-2\lambda}+1\right)^{2}-e^{-2\lambda}\left(e^{2\lambda}+1\right)^{2}}{\left(e^{2\lambda}+1\right)^{2}\left(e^{-2\lambda}+1\right)^{2}}\right)=0. \tag{15}$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence at $x=\frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{1}{4}\left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda}+1\right)-g_{q,\lambda}\left(\frac{\ln q}{2\lambda}-1\right)\right]= \tag{16}$$

$$\frac{1}{4}\left[\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)-\left(\frac{e^{-\lambda}-e^{\lambda}}{e^{-\lambda}+e^{\lambda}}\right)\right]=$$

$$\frac{1}{4}\left[\frac{2\left(e^{\lambda}-e^{-\lambda}\right)}{e^{\lambda}+e^{-\lambda}}\right]=\frac{1}{2}\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)=\frac{\tanh(\lambda)}{2}.$$

Conclusion: The maximum value of $M_{q,\lambda}$ is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{\tanh(\lambda)}{2},\quad \lambda>0. \tag{17}$$

We mention

Theorem 1.(*[2], Ch. 18, p. 458) We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i)=1,\ \ \forall\, x\in\mathbb{R},\ \forall\, \lambda,q>0. \tag{18}$$

Also it holds

Theorem 2.(*[2], Ch. 18, p. 459) It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx=1,\ \ \lambda,q>0. \tag{19}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda,q>0$.
Similarly we get that

$$\int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx=1,\ \ \lambda,q>0, \tag{20}$$

so that $M_{\frac{1}{q},\lambda}$ is a density function.

Furthermore, we observe the symmetry

$$\left(M_{q,\lambda}+M_{\frac{1}{q},\lambda}\right)(-x)=\left(M_{q,\lambda}+M_{\frac{1}{q},\lambda}\right)(x),\ \ \forall\, x\in\mathbb{R}. \tag{21}$$

Furthermore

$$\varphi=\frac{M_{q,\lambda}+M_{\frac{1}{q},\lambda}}{2} \tag{22}$$

is a new density function over $\mathbb{R}$, i.e.

$$\int_{-\infty}^{\infty}\varphi(x)\,dx=1.$$

Clearly, then

$$\int_{-\infty}^{\infty}\varphi(nx-u)\,du=1,\ \ \forall\, n\in\mathbb{N},\, x\in\mathbb{R}. \tag{23}$$

3 Basics

We give

Definition 1.*Let $f\in C_B(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$), $n\in\mathbb{N}$. We define the following basic activated hyperbolic tangent perturbed convolution type operators*

$$A_n(f)(x):=\int_{-\infty}^{\infty} f\left(\frac{u}{n}\right)\varphi(nx-u)\,du,\ \ \forall\, x\in\mathbb{R}. \tag{24}$$

In this work we examine the quantitative convergence of A_n to the unit operator.

We study similarly the activated Kantorovich type operators,

$$A_n^*(f)(x) := n\int_{-\infty}^{\infty}\left(\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t)\,dt\right)\varphi(nx-u)\,du$$

$$= n\int_{-\infty}^{\infty}\left(\int_0^{\frac{1}{n}} f\left(t+\frac{u}{n}\right)dt\right)\varphi(nx-u)\,du, \tag{25}$$

where $f \in C_B(\mathbb{R})$, $n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}$,

and the activated Quadrature operators

$$\overline{A_n}(f)(x) := \int_{-\infty}^{\infty}\left(\sum_{i=1}^{r} w_i f\left(\frac{u}{n}+\frac{i}{nr}\right)\right)\varphi(nx-u)\,du, \tag{26}$$

where $w_i \geq 0$, $\sum_{i=1}^{r} w_i = 1$; $f \in C_B(\mathbb{R})$, $n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}$.

An essential property follows:

Theorem 3.([3]) *Let* $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. *Then*

$$\int_{\left\{u\in\mathbb{R}:|nx-u|\geq n^{1-\alpha}\right\}} \varphi(nx-u)\,du < \frac{\left(q+\frac{1}{q}\right)}{e^{2\lambda\left(n^{1-\alpha}-1\right)}},\quad q,\lambda > 0. \tag{27}$$

By [3], w eobtain that

$$\varphi(x) < \left(q+\frac{1}{q}\right)\lambda e^{-2\lambda(x-1)},\quad \forall\, x \geq 1. \tag{28}$$

We make

*Remark.*Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}(\mathbb{R})$, with $f^{(j)} \in C_B(\mathbb{R})$, for $j = 0, 1, ..., i$.

We have that

$$A_n(f)(x) = \int_{-\infty}^{\infty} f\left(x-\frac{z}{n}\right)\varphi(z)\,dz.$$

By applying repeatedly Leibnitz's rule we get

$$\frac{\partial^i A_n(f)(x)}{\partial x^i} = \int_{-\infty}^{\infty} f^{(i)}\left(x-\frac{z}{n}\right)\varphi(z)\,dz =$$

$$\int_{-\infty}^{\infty} f^{(i)}\left(\frac{u}{n}\right)\varphi(nx-u)\,du = A_n\left(f^{(i)}\right)(x),\quad \forall\, x \in \mathbb{R}. \tag{29}$$

Clearly, it is valid that

$$\frac{\partial^i A_n^*(f)(x)}{\partial x^i} = A_n^*\left(f^{(i)}\right)(x), \tag{30}$$

and

$$\frac{\partial^i \overline{A_n}(f)(x)}{\partial x^i} = \overline{A_n}\left(f^{(i)}\right)(x), \tag{31}$$

$\forall\, x \in \mathbb{R}$.

For $f \in C_B(\mathbb{R})$, we obtain $A_n(f), A_n^*(f), \overline{A}_n(f) \in C_B(\mathbb{R})$, see [3].

4 Main Results

Next, we give Voronovskaya type asymptotic expansions for the error of approximation.

We present about the basic operator.

Theorem 4. *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ large enough; $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, with $f^{(N)} \in C_B(\mathbb{R})$, $0 < \varepsilon \leq N$. Then*

1)

$$A_n(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\alpha(N-\varepsilon)}}\right), \tag{32}$$

the last (32) implies

$$n^{\alpha(N-\varepsilon)}\left[A_n(f)(x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x)\right] \to 0, \tag{33}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

2) when $f^{(j)}(x) = 0$, $j = 1, \ldots, N$, we derive that

$$n^{\alpha(N-\varepsilon)}\left[A_n(f)(x) - f(x)\right] \to 0, \text{ as } n \to \infty, \ 0 < \varepsilon \leq N. \tag{34}$$

Of interest is the case $\alpha = \frac{1}{2}$.

Proof. We have that

$$f\left(\frac{u}{n}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}\left(\frac{u}{n} - x\right)^j + \int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt. \tag{35}$$

Then

$$f\left(\frac{u}{n}\right)\varphi(nx-u) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \varphi(nx-u)\left(\frac{u}{n} - x\right)^j +$$

$$\varphi(nx-u)\int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt. \tag{36}$$

Hence

$$A_n(f)(x) = \int_{-\infty}^{\infty} f\left(\frac{u}{n}\right)\varphi(nx-u)\,du =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \int_{-\infty}^{\infty} \varphi(nx-u)\left(\frac{u}{n} - x\right)^j du +$$

$$\int_{-\infty}^{\infty} \varphi(nx-u)\left(\int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt\right) du. \tag{37}$$

Thus, it holds

$$A_n(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \int_{-\infty}^{\infty} \varphi(nx-u)\left(\frac{u}{n} - x\right)^j du$$

$$+ \int_{-\infty}^{\infty} \varphi(nx-u)\left(\int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt\right) du. \tag{38}$$

Call

$$R_n(x) := \int_{-\infty}^{\infty} \varphi(nx-u)\left(\int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt\right) du.$$

Call

$$\gamma(u) := \int_x^{\frac{u}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt. \tag{39}$$

Let $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}$.
i) Case of $\frac{u}{n}\geq x$. Then

$$|\gamma(u)|\leq\int_{x}^{\frac{u}{n}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{x}^{\frac{u}{n}}\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt=$$

$$2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\frac{u}{n}-t\right)^{N}}{N!}\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}. \tag{40}$$

ii) Case of $\frac{u}{n}<x$. Then

$$|\gamma(u)|=\left|\int_{\frac{u}{n}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_{\frac{u}{n}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\frac{u}{n}\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{\frac{u}{n}}^{x}\frac{\left(t-\frac{u}{n}\right)^{N-1}}{(N-1)!}dt= \tag{41}$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(x-\frac{u}{n}\right)^{N}\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}.$$

Therefore, it holds

$$|\gamma(u)|\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}, \tag{42}$$

when $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}$.
Consequently, we get that

$$\left|\int_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt\right)du\right|\leq$$

$$\int_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}}\varphi(nx-u)|\gamma(u)|du\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}. \tag{43}$$

Next, we treat

$$\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt\right)du\right|\leq \tag{44}$$

$$\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\varphi(nx-u)\left|\int_{x}^{\frac{u}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{(N-1)!}dt\right|du=$$

$$\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\varphi(nx-u)|\gamma(u)|du=:(\xi_{1}).$$

Let $\frac{u}{n}\geq x$, then

$$|\gamma(u)|\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\frac{u}{n}-x\right)^{N}}{N!}.$$

If $\frac{u}{n} < x$, then

$$|\gamma(u)| = \left| \int_{\frac{u}{n}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{u}{n} - t\right)^{N-1}}{(N-1)!} dt \right| \leq$$

$$2 \left\| f^{(N)} \right\|_{\infty} \int_{\frac{u}{n}}^{x} \frac{\left(t - \frac{u}{n}\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left(x - \frac{u}{n}\right)^{N}}{N!}. \tag{45}$$

Consequently, we get that

$$|\gamma(u)| \leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left|x - \frac{u}{n}\right|^{N}}{N!}. \tag{46}$$

Furthermore, we obtain

$$(\xi_1) \leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{N!} \int\limits_{\left|\frac{u}{n} - x\right| \geq \frac{1}{n^{\alpha}}} \varphi(nx - u) \left| x - \frac{u}{n} \right|^{N} du =$$

$$\frac{2 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \int\limits_{|nx-u| \geq n^{1-\alpha}} \varphi(|nx - u|) |nx - u|^{N} du \overset{(28)}{\leq}$$

(here it is $\Delta := \left\{ u \in \mathbb{R} : |nx - u| \geq n^{1-\alpha} \right\}$)

$$\frac{2 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \lambda \int_{\Delta} e^{2\lambda(|nx-u|-1)} |nx - u|^{N} du =$$

$$\frac{4 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \lambda \int_{n^{1-\alpha}}^{\infty} e^{-2\lambda(x-1)} x^N dx =$$

$$\frac{4 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \lambda e^{2\lambda} \int_{n^{1-\alpha}}^{\infty} e^{-2\lambda x} x^N dx =$$

$$\frac{4 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \lambda \frac{e^{2\lambda}}{(2\lambda)^{N+1}} \int_{n^{1-\alpha}}^{\infty} e^{-2\lambda x} (2\lambda x)^N d(2\lambda x) = \tag{47}$$

$$\frac{2 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{(2\lambda)^N} \int_{2\lambda n^{1-\alpha}}^{\infty} e^{-y} y^N dy \leq$$

$$\frac{2 \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{(2\lambda)^N} 2^N N! \int_{2\lambda n^{1-\alpha}}^{\infty} e^{-y} e^{\frac{y}{2}} dy =$$

$$\frac{2^{N+1} \left\| f^{(N)} \right\|_{\infty}}{n^N} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{(2\lambda)^N} \int_{2\lambda n^{1-\alpha}}^{\infty} e^{-\frac{y}{2}} dy =$$

$$\frac{2 \left\| f^{(N)} \right\|_{\infty}}{n^N} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{\lambda^N} (-2) \left(e^{-\frac{y}{2}} \Big|_{2\lambda n^{1-\alpha}}^{\infty} \right) =$$

$$\frac{4 \left\| f^{(N)} \right\|_{\infty}}{n^N} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{\lambda^N} \left(e^{-\frac{y}{2}} \Big|_{\infty}^{2\lambda n^{1-\alpha}} \right) = \tag{48}$$

$$\frac{4 \left\| f^{(N)} \right\|_{\infty}}{n^N} \left(q + \frac{1}{q} \right) \frac{e^{2\lambda}}{\lambda^N} e^{-\lambda n^{1-\alpha}}.$$

We have proved that

$$\left| \int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}} \varphi\left(nx-u\right)\left(\int_{x}^{\frac{u}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{\left(N-1\right)!}dt\right)du\right| \leq \frac{4\left\|f^{(N)}\right\|_{\infty}}{n^{N}}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{\lambda^{N}}e^{-\lambda n^{1-\alpha}} \to 0, \text{ as } n\to\infty. \tag{49}$$

Finally, we have that

$$|R_n(x)| \leq \left| \int_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}} \varphi\left(nx-u\right)\left(\int_{x}^{\frac{u}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{\left(N-1\right)!}dt\right)du\right|$$
$$+\left| \int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}} \varphi\left(nx-u\right)\left(\int_{x}^{\frac{u}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{u}{n}-t\right)^{N-1}}{\left(N-1\right)!}dt\right)du\right| \leq$$
$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}+\frac{4\left\|f^{(N)}\right\|_{\infty}}{n^{N}}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{\lambda^{N}}e^{-\lambda n^{1-\alpha}} \leq \frac{4\left\|f^{(N)}\right\|_{\infty}}{N!n^{\alpha N}}, \tag{50}$$

for large enough $n\in\mathbb{N}$.

Hence it holds

$$|R_n(x)| = O\left(\frac{1}{n^{\alpha N}}\right), \tag{51}$$

and

$$|R_n(x)| = o(1). \tag{52}$$

And, letting $0<\varepsilon\leq N$, we derive

$$\frac{|R_n(x)|}{\left(\frac{1}{n^{\alpha(N-\varepsilon)}}\right)} \leq \frac{4\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n^{\alpha\varepsilon}}\right) \to 0, \text{ as } n\to\infty. \tag{53}$$

I.e.

$$|R_n(x)| = o\left(\frac{1}{n^{\alpha(N-\varepsilon)}}\right), \tag{54}$$

proving the claim.

We continue with the Kantorovich operator.

Theorem 5. *Let $0<\alpha<1$, $n\in\mathbb{N}$ large enough; $x\in\mathbb{R}$, $f\in C^{N}(\mathbb{R})$, $N\in\mathbb{N}$, with $f^{(N)}\in C_B(\mathbb{R})$, $0<\varepsilon\leq N$. Then 1)*

$$A_n^{*}(f)(x)-f(x)=\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}A_n^{*}\left((\cdot-x)^{j}\right)(x)+o\left(\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N-\varepsilon}\right), \tag{55}$$

the last (55) implies

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N-\varepsilon}}\left[A_n^{*}(f)(x)-f(x)-\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}A_n^{*}\left((\cdot-x)^{j}\right)(x)\right]\to 0, \tag{56}$$

as $n\to\infty$, $0<\varepsilon\leq N$.

2) when $f^{(j)}(x)=0$, $j=1,\ldots,N$, we derive that

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N-\varepsilon}}\left[A_n^{*}(f)(x)-f(x)\right]\to 0, \tag{57}$$

as $n\to\infty$, $0<\varepsilon\leq N$. Of interest is the case $\alpha=\frac{1}{2}$.

*Proof.*We have that

$$f\left(t+\frac{u}{n}\right)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(t+\frac{u}{n}-x\right)^{j}+$$

$$\int_{x}^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds, \tag{58}$$

and

$$\int_{0}^{\frac{1}{n}}f\left(t+\frac{u}{n}\right)dt=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j}dt+ \tag{59}$$

$$\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt.$$

Hence

$$n\int_{-\infty}^{\infty}\left(\int_{0}^{\frac{1}{n}}f\left(t+\frac{u}{n}\right)dt\right)\varphi(nx-u)\,du= \tag{60}$$

$$\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}n\int_{-\infty}^{\infty}\left(\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j}dt\right)\varphi(nx-u)\,du+$$

$$n\int_{-\infty}^{\infty}\left(\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\varphi(nx-u)\,du.$$

Furthermore, it holds

$$A_{n}^{*}(f)(x)-f(x)=\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}A_{n}^{*}\left((\cdot-x)^{j}\right)(x)+R_{n}(x), \tag{61}$$

where

$$R_{n}(x):=n$$

$$\int_{-\infty}^{\infty}\left(\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\varphi(nx-u)\,du. \tag{62}$$

Call

$$\lambda(u):=n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt. \tag{63}$$

Let $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}$, $(0<\alpha<1)$.
i) Case of $t+\frac{u}{n}\geq x$. Then

$$|\lambda(u)|\leq n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left|f^{(N)}(s)-f^{(N)}(x)\right|\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}n\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}n\int_{0}^{\frac{1}{n}}\left(|t|+\left|\frac{u}{n}-x\right|\right)^{N}dt\leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}. \tag{64}$$

So that

$$|\lambda(u)| \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}. \tag{65}$$

ii) Case of $t+\frac{u}{n}<x$. Then

$$|\lambda(u)| = n\left|\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right| =$$

$$n\left|\int_0^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\left(f^{(N)}(x)-f^{(N)}(s)\right)\frac{\left(s-\left(t+\frac{u}{n}\right)\right)^{N-1}}{(N-1)!}ds\right)dt\right| \leq \tag{66}$$

$$n\int_0^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\left|f^{(N)}(x)-f^{(N)}(s)\right|\frac{\left(s-\left(t+\frac{u}{n}\right)\right)^{N-1}}{(N-1)!}ds\right)dt \leq$$

$$2\left\|f^{(N)}\right\|_{\infty} n\int_0^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\frac{\left(s-\left(t+\frac{u}{n}\right)\right)^{N-1}}{(N-1)!}ds\right)dt =$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} n\int_0^{\frac{1}{n}}\left(x-\left(t+\frac{u}{n}\right)\right)^{N} dt \leq \tag{67}$$

(by $x-\left(t+\frac{u}{n}\right)=\left|x-t-\frac{u}{n}\right| \leq \left|\frac{u}{n}-x\right|+|t|$)

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} n\int_0^{\frac{1}{n}}\left(\left|\frac{u}{n}-x\right|+|t|\right)^{N} dt \leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}.$$

Consequently, it holds

$$|\lambda(u)| \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}, \tag{68}$$

when $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}$.

Therefore, we obtain

$$\left|n\int_{\substack{-\infty \\ \left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}}}^{\infty}\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\varphi(nx-u)\,du\right|$$

$$\leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}. \tag{69}$$

Next, we treat

$$\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}} n\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\varphi(nx-u)\,du\right|$$

$$\leq \int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\left|n\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\right| \tag{70}$$

$$\varphi(nx-u)\,du =: (\psi_1).$$

We also have that

$$|\lambda(u)| = \left| n\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\right| \leq$$

$$n\left(\int_0^{\frac{1}{n}}\left|\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right|dt\right) \leq \tag{71}$$

(let $t+\frac{u}{n}\geq x$)

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right) =$$

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\frac{\left(t+\frac{u}{n}-x\right)^{N}}{N!}dt\right) \leq$$

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\frac{\left(|t|+\left|\frac{u}{n}-x\right|\right)^{N}}{N!}dt\right) \leq$$

$$\frac{2\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^N.$$

So, when $t+\frac{u}{n}\geq x$, we obtain

$$|\lambda(u)| \leq 2\left\|f^{(N)}\right\|_\infty \frac{\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^N}{N!}. \tag{72}$$

If $t+\frac{u}{n}<x$, then

$$|\lambda(u)| = n\left(\int_0^{\frac{1}{n}}\left|\int_{t+\frac{u}{n}}^{x}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(s-\left(t+\frac{u}{n}\right)\right)^{N-1}}{(N-1)!}ds\right|dt\right) \leq$$

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\frac{\left(s-\left(t+\frac{u}{n}\right)\right)^{N-1}}{(N-1)!}ds\right)dt\right) = \tag{73}$$

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\frac{\left(x-\left(t+\frac{u}{n}\right)\right)^{N}}{N!}dt\right) \leq$$

$$2\left\|f^{(N)}\right\|_\infty n\left(\int_0^{\frac{1}{n}}\frac{\left(|t|+\left|\frac{u}{n}-x\right|\right)^{N}}{N!}dt\right) \leq$$

$$\frac{2\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^N.$$

Consequently, we get that

$$|\lambda(u)| \leq \frac{2\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^N. \tag{74}$$

Furthermore, we obtain

$$(\psi_1) \leq \left(\int\limits_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^\alpha}}\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^N\varphi(nx-u)\,du\right)\frac{2\left\|f^{(N)}\right\|_\infty}{N!} =$$

$$\frac{2^N\left\|f^{(N)}\right\|_\infty}{N!}\int\limits_{|nx-u|\geq n^{1-\alpha}}\left(\frac{1}{n^N}+\frac{|nx-u|^N}{n^N}\right)\varphi(nx-u)\,du\leq \tag{75}$$

$$\frac{2^N\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(\int\limits_{|nx-u|\geq n^{1-\alpha}}\left(1+|nx-u|^N\right)\varphi(|nx-u|)\,du\right)\overset{(28)}{\leq}$$

(here it is $\Delta:=\left\{u\in\mathbb{R}:|nx-u|\geq n^{1-\alpha}\right\}$)

$$\frac{2^N\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(\left(q+\frac{1}{q}\right)\lambda\int_\Delta\left(1+|nx-u|^N\right)e^{-2\lambda(|nx-u|-1)}du\right)=$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda\int_{n^{1-\alpha}}^\infty e^{-2\lambda(x-1)}\left(1+x^N\right)dx=$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\int_{n^{1-\alpha}}^\infty e^{-2\lambda x}\left(1+x^N\right)dx=$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\int_{n^{1-\alpha}}^\infty e^{-2\lambda x}dx+\int_{n^{1-\alpha}}^\infty e^{-2\lambda x}x^N dx\right]= \tag{76}$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{e^{-2\lambda n^{1-\alpha}}}{2\lambda}+\int_{n^{1-\alpha}}^\infty e^{-2\lambda x}x^N dx\right]=$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{e^{-2\lambda n^{1-\alpha}}}{2\lambda}+\frac{1}{(2\lambda)^{N+1}}\int_{2\lambda n^{1-\alpha}}^\infty e^{-y}y^N dy\right]\leq$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{e^{-2\lambda n^{1-\alpha}}}{2\lambda}+\frac{2^N N!}{(2\lambda)^{N+1}}\int_{2\lambda n^{1-\alpha}}^\infty e^{-\frac{y}{2}}dy\right]= \tag{77}$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{e^{-2\lambda n^{1-\alpha}}}{2\lambda}+\frac{2^{N+1}N!}{(2\lambda)^{N+1}}e^{-\lambda n^{1-\alpha}}\right]=$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{e^{-2\lambda n^{1-\alpha}}}{2\lambda}+\frac{N!}{\lambda^{N+1}}e^{-\lambda n^{1-\alpha}}\right]\leq \tag{78}$$

$$\leq\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^N}\right]e^{-\lambda n^{1-\alpha}}.$$

We have proved that

$$\left|\int\limits_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^\alpha}}n\left(\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{u}{n}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{n}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)\varphi(nx-u)\,du\right| \tag{79}$$

$$\leq\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^N}\right]e^{-\lambda n^{1-\alpha}}\to 0,\ \text{ as } n\to\infty.$$

Finally, we have that

$$|R_n(x)| \leq \frac{2\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^N +$$

$$\frac{2^{N+1}\left\|f^{(N)}\right\|_\infty}{n^N N!}\left(q+\frac{1}{q}\right)e^{2\lambda}\left(\frac{1}{2}+\frac{N!}{\lambda^N}\right)e^{-\lambda n^{1-\alpha}} \leq \frac{4\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^N, \tag{80}$$

for large enough $n \in \mathbb{N}$.

Hence it holds

$$|R_n(x)| = O\left(\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^N\right), \tag{81}$$

and

$$|R_n(x)| = o(1). \tag{82}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R_n(x)|}{\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^{N-\varepsilon}} \leq \frac{4\left\|f^{(N)}\right\|_\infty}{N!}\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^\varepsilon \to 0, \quad \text{as } n \to \infty. \tag{83}$$

I.e.

$$|R_n(x)| = o\left(\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^{N-\varepsilon}\right), \tag{84}$$

proving the claim.

We continue with the Quadrature operator.

Theorem 6. *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ large enough; $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, with $f^{(N)} \in C_B(\mathbb{R})$, $0 < \varepsilon \leq N$. Then*

1)

$$\overline{A_n}(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!}\overline{A_n}\left((\cdot - x)^j\right)(x) + o\left(\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^{N-\varepsilon}\right), \tag{85}$$

the last (85) implies

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^{N-\varepsilon}}\left[\overline{A_n}(f)(x) - f(x) - \sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}\overline{A_n}\left((\cdot - x)^j\right)(x)\right] \to 0, \tag{86}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

2) when $f^{(j)}(x) = 0$, $j = 1, \ldots, N$, we derive that

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^\alpha}\right)^{N-\varepsilon}}\left[\overline{A_n}(f)(x) - f(x)\right] \to 0, \ \text{as } n \to \infty, \ 0 < \varepsilon \leq N. \tag{87}$$

Of interest is the case $\alpha = \frac{1}{2}$.

Proof. We have that

$$f\left(\frac{u}{n}+\frac{i}{nr}\right) = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(\frac{u}{n}+\frac{i}{nr}-x\right)^j +$$

$$\int_x^{\frac{u}{n}+\frac{i}{nr}}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt, \tag{88}$$

and

$$\sum_{i=1}^{r} w_i f\left(\frac{u}{n}+\frac{i}{nr}\right) = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\sum_{i=1}^{r} w_i\left(\frac{u}{n}+\frac{i}{nr}-x\right)^j + \tag{89}$$

$$\sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt.$$

Furthermore it holds

$$\overline{A_n}(f)(x) = \int_{-\infty}^{\infty} \left(\sum_{i=1}^{r} w_i f\left(\frac{u}{n}+\frac{i}{nr}\right)\right) \varphi(nx-u)\, du =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\int_{-\infty}^{\infty} \sum_{i=1}^{r} w_i \left(\frac{u}{n}+\frac{i}{nr}-x\right)^j\right) \varphi(nx-u)\, du \tag{90}$$

$$+\int_{-\infty}^{\infty} \left(\sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt\right) \varphi(nx-u)\, du,$$

and it is

$$\overline{A_n}(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \left(\overline{A_n}\left((\cdot - x)^j\right)(x)\right) + R_n(x), \tag{91}$$

where

$$R_n(x) :=$$

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt\right) \varphi(nx-u)\, du. \tag{92}$$

Let

$$\delta(u) := \sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt.$$

Let $\left|\frac{u}{n} - x\right| < \frac{1}{n^{\alpha}}$.

i) Case of $\frac{u}{n} + \frac{i}{nr} \geq x$. Then

$$|\delta(u)| \leq 2\left\|f^{(N)}\right\|_{\infty} \sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt =$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} \sum_{i=1}^{r} w_i \left(\frac{u}{n}+\frac{i}{nr}-x\right)^N \leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} \sum_{i=1}^{r} w_i \left(\left|\frac{u}{n}-x\right|+\frac{i}{nr}\right)^N \leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} \left(\frac{1}{n^{\alpha}}+\frac{1}{n}\right)^N. \tag{93}$$

So that

$$|\delta(u)| \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{N!} \left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^N. \tag{94}$$

ii) Case of $\frac{u}{n} + \frac{i}{nr} < x$. Then

$$|\delta(u)| = \left|\sum_{i=1}^{r} w_i \int_{\frac{u}{n}+\frac{i}{nr}}^{x} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(t-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!} dt\right| \leq$$

$$\sum_{i=1}^{r} w_i \int_{\frac{u}{n}+\frac{i}{nr}}^{x} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left(t-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!} dt \leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\frac{\left(t-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt= \tag{95}$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\sum_{i=1}^{r}w_i\left(x-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N}\leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\sum_{i=1}^{r}w_i\left(\left|x-\frac{u}{n}\right|+\frac{i}{nr}\right)^{N}\leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n^{\alpha}}+\frac{1}{n}\right)^{N}.$$

So, when $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}$, we get that

$$|\delta(u)|\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n^{\alpha}}+\frac{1}{n}\right)^{N}. \tag{96}$$

Consequently, we have that

$$\left|\int_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}}\left(\sum_{i=1}^{r}w_i\int_{x}^{\frac{u}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt\right)\varphi(nx-u)\,du\right|$$

$$\leq\int_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\alpha}}}\varphi(nx-u)\,|\delta(u)|\,du\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\frac{1}{n}+\frac{1}{n^{\alpha}}\right)^{N}. \tag{97}$$

Next, we treat

$$\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\varphi(nx-u)\left(\sum_{i=1}^{r}w_i\int_{x}^{\frac{u}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{u}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt\right)du\right| \tag{98}$$

$$\leq\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}}\varphi(nx-u)\,|\delta(u)|\,du=:(\varphi_1).$$

Let $\frac{u}{n}+\frac{i}{nr}\geq x$, then

$$|\delta(u)|\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\sum_{i=1}^{r}w_i\left(\frac{u}{n}+\frac{i}{nr}-x\right)^{N}. \tag{99}$$

Let $\frac{u}{n}+\frac{i}{nr}<x$, then

$$|\delta(u)|=\left|\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(t-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt\leq$$

$$\frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\sum_{i=1}^{r}w_i\left(x-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{N}. \tag{100}$$

Consequently, we get that

$$\left|\delta\left(u\right)\right| \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{N!}\left(\left|x-\frac{u}{n}\right|+\frac{1}{n}\right)^{N} \tag{101}$$

(see also (74)).

The quantity

$$\left|\int\limits_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\alpha}}} \varphi\left(nx-u\right)\delta\left(u\right)du\right| \leq \left(\varphi_{1}\right) \leq \ldots \tag{102}$$

is estimated as in Theorem 5 based on (101).

We finish as in the proof of Theorem 5.

We need the following.

Definition 2. *A function $f:\mathbb{R}\rightarrow\mathbb{R}$ is absolutely continuous over $\mathbb{R}$, iff $f|_{[a,b]}$ is absolutely continuous, for every $[a,b]\subset\mathbb{R}$. We write $f\in AC^{n}\left(\mathbb{R}\right)$, iff $f^{(n-1)}\in AC\left(R\right)$ (absolutely continuous functions over $\mathbb{R}$), $n\in\mathbb{N}$.*

Definition 3. *Let $\nu\geq 0$, $n=\lceil\nu\rceil$ ($\lceil\cdot\rceil$ is the ceiling of the number), $f\in AC^{n}\left(\mathbb{R}\right)$. We call left Caputo fractional derivative ([18], [19], [16], pp. 49-52) the function*

$$D_{*a}^{\nu}f\left(x\right)=\frac{1}{\Gamma\left(n-\nu\right)}\int_{a}^{x}\left(x-t\right)^{n-\nu-1}f^{(n)}\left(t\right)dt, \tag{103}$$

$\forall\, x\in[a,\infty)$, $a\in\mathbb{R}$, where Γ is the gamma function.

Notice $D_{*a}^{\nu}f\in L_{1}\left([a,b]\right)$ and $D_{*a}^{\nu}f$ exists a.e. on $[a,b]$, $\forall\,[a,b]\subset\mathbb{R}$.

We set $D_{*a}^{0}f\left(x\right)=f\left(x\right)$, $\forall\, x\in[a,\infty)$.

We need

Lemma 1. *(see also [15]) Let $\nu>0$, $\nu\notin\mathbb{N}$, $n=\lceil\nu\rceil$, $f\in C^{n-1}\left(\mathbb{R}\right)$ and $f^{(n)}\in L_{\infty}\left(\mathbb{R}\right)$. Then $D_{*a}^{\nu}f\left(a\right)=0$ for any $a\in\mathbb{R}$.*

Definition 4. *(see also [14,17,18]) Let $f\in AC^{m}\left(\mathbb{R}\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$. The right Caputo fractional derivative of order $\alpha>0$ is given by*

$$D_{b-}^{\alpha}f\left(x\right)=\frac{\left(-1\right)^{m}}{\Gamma\left(m-\alpha\right)}\int_{x}^{b}\left(z-x\right)^{m-\alpha-1}f^{(m)}\left(z\right)dz, \tag{104}$$

$\forall\, x\in(-\infty,b]$, $b\in\mathbb{R}$. We set $D_{b-}^{0}f\left(x\right)=f\left(x\right)$.

Notice $D_{b-}^{\alpha}f\in L_{1}\left([a,b]\right)$ and $D_{b-}^{\alpha}f$ exists a.e. on $[a,b]$, $\forall\,[a,b]\subset\mathbb{R}$.

Lemma 2. *(see also [15]) Let $f\in C^{m-1}\left(\mathbb{R}\right)$, $f^{(m)}\in L_{\infty}\left(\mathbb{R}\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$. Then $D_{b-}^{\alpha}f\left(b\right)=0$, for any $b\in\mathbb{R}$.*

We assume that

$$\begin{array}{l} D_{*x_0}^{\alpha}f\left(x\right)=0,\text{ for } x<x_0,\\ \text{and}\\ D_{x_0-}^{\alpha}f\left(x\right)=0,\text{ for } x>x_0. \end{array} \tag{105}$$

We mention

Proposition 1. *(see also [15]) Let $f\in C^{n}\left(\mathbb{R}\right)$, $n=\lceil\nu\rceil$, $\nu>0$. Then $D_{*a}^{\nu}f\left(x\right)$ is continuous in $x\in[a,\infty)$, $a\in\mathbb{R}$.*

Also we have

Proposition 2. *(see also [15]) Let $f\in C^{m}\left(\mathbb{R}\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$. Then $D_{b-}^{\alpha}f\left(x\right)$ is continuous in $x\in(-\infty,b]$, $b\in\mathbb{R}$.*

We further mention

Proposition 3. *(see also [15]) Let $f\in C^{m-1}\left(\mathbb{R}\right)$, $f^{(m)}\in L_{\infty}\left(\mathbb{R}\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$ and let $x,x_0\in\mathbb{R}:x\geq x_0$. Then $D_{*x_0}^{\alpha}f\left(x\right)$ is continuous in x_0.*

Proposition 4.*(see also [15]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and let $x, x_0 \in \mathbb{R} : x \leq x_0$. Then $D^\alpha_{x_0-} f(x)$ is continuous in x_0.*

Proposition 5.*(see also [15]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$; $x, x_0 \in \mathbb{R}$. Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $R^2 \to \mathbb{R}$.*

Next we treat the fractional case of Voronovsakaya type asymptotic expansions. We start with the basic operators.

Theorem 7.*Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N(\mathbb{R})$, $f^{(N)} \in L_\infty(\mathbb{R})$, $0 < \beta < 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ large enough. Assume that both $\|D^\alpha_{*x} f\|_{\infty,[x,\infty)}$, $\left\|D^\alpha_{x-} f\right\|_{\infty,(-\infty,x]}$ are finite, $0 < \varepsilon \leq \alpha$. Then*

1)

$$A_n(f)(x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{106}$$

the last (106) implies

$$n^{\beta(\alpha-\varepsilon)} \left[A_n(f)(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x)\right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

2) when $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, \ldots, N-1$, we get that

$$n^{\beta(\alpha-\varepsilon)} \left[A_n(f)(x) - f(x)\right] \to 0, \text{ as } n \to \infty, \; 0 < \varepsilon \leq \alpha. \tag{107}$$

Of interest is the case $\beta = \frac{1}{2}$.

*Proof.*Let $x \in \mathbb{R}$. We have that $D^\alpha_{x-} f(x) = D^\alpha_{*x} f(x) = 0$.

From [16], p. 54, we get the left Caputo fractional Taylor formula that

$$f\left(\frac{u}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{u}{n} - x\right)^j + \tag{108}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^{\frac{u}{n}} \left(\frac{u}{n} - s\right)^{\alpha-1} \left(D^\alpha_{*x} f(s) - D^\alpha_{*x} f(x)\right) ds,$$

for all $x \leq \frac{u}{n} < \infty$.

Also from [14], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{u}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{u}{n} - x\right)^j + \tag{109}$$

$$\frac{1}{\Gamma(\alpha)} \int_{\frac{u}{n}}^x \left(s - \frac{u}{n}\right)^{\alpha-1} \left(D^\alpha_{x-} f(s) - D^\alpha_{x-} f(x)\right) ds,$$

for all $-\infty < \frac{u}{n} \leq x$.

Hence

$$f\left(\frac{u}{n}\right) \varphi(nx - u) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \varphi(nx - u) \left(\frac{u}{n} - x\right)^j + \tag{110}$$

$$\frac{\varphi(nx-u)}{\Gamma(\alpha)} \int_x^{\frac{u}{n}} \left(\frac{u}{n} - s\right)^{\alpha-1} \left(D^\alpha_{*x} f(s) - D^\alpha_{*x} f(x)\right) ds,$$

for all $x \leq \frac{u}{n} < \infty$, and

$$f\left(\frac{u}{n}\right) \varphi(nx - u) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \varphi(nx - u) \left(\frac{u}{n} - x\right)^j + \tag{111}$$

$$\frac{\varphi(nx-u)}{\Gamma(\alpha)} \int_{\frac{u}{n}}^x \left(s - \frac{u}{n}\right)^{\alpha-1} \left(D^\alpha_{x-} f(s) - D^\alpha_{x-} f(x)\right) ds,$$

for all $-\infty < \frac{u}{n} \leq x$.

Therefore we have

$$\int_{nx}^{\infty} f\left(\frac{u}{n}\right)\varphi(nx-u)\,du = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\int_{nx}^{\infty}\varphi(nx-u)\left(\frac{u}{n}-x\right)^{j}du+ \tag{112}$$

$$\frac{1}{\Gamma(\alpha)}\int_{nx}^{\infty}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du,$$

and

$$\int_{-\infty}^{nx} f\left(\frac{u}{n}\right)\varphi(nx-u)\,du = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\int_{-\infty}^{nx}\varphi(nx-u)\left(\frac{u}{n}-x\right)^{j}du+ \tag{113}$$

$$\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{nx}\varphi(nx-u)\left(\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du.$$

Adding the last two equalities (112), (113) we have

$$A_n(f)(x) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^{j}\right)(x)+$$

$$\frac{1}{\Gamma(\alpha)}\left[\int_{nx}^{\infty}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du\right. \tag{114}$$

$$\left.+\int_{-\infty}^{nx}\varphi(nx-u)\left(\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du\right].$$

Consequently, it holds

$$A_n(f)(x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^{j}\right)(x) = R_n(x), \tag{115}$$

where

$$R_n(x) := \frac{1}{\Gamma(\alpha)}$$

$$\left[\int_{-\infty}^{nx}\varphi(nx-u)\left(\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du\right.$$

$$\left.+\int_{nx}^{\infty}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du\right], \tag{116}$$

$\forall\, x\in\mathbb{R}$.

Denote by

$$R_{n1}(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{nx}\varphi(nx-u)\left(\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du, \tag{117}$$

$\forall\, x\in\mathbb{R} : x \geq \frac{u}{n} > -\infty$,

and

$$R_{n2}(x) = \frac{1}{\Gamma(\alpha)}\int_{nx}^{\infty}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du, \tag{118}$$

$\forall\, x\in\mathbb{R} : \infty > \frac{u}{n} \geq x$.

That is

$$R_n(x) = R_{n1}(x)+R_{n2}(x),\ \ \forall\, x\in\mathbb{R}. \tag{119}$$

Let first $\left|\frac{u}{n}-x\right| < \frac{1}{n^{\beta}}$.

Call

$$\gamma_{n1}(x) := \frac{1}{\Gamma(\alpha)}\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds, \tag{120}$$

and

$$\gamma_{n2}(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\frac{u}{n}} \left(\frac{u}{n} - s\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds. \tag{121}$$

Let assume that $x \geq \frac{u}{n}$, then

$$|\gamma_{n1}(x)| \leq \frac{1}{\Gamma(\alpha)} \int_{\frac{u}{n}}^{x} \left(s - \frac{u}{n}\right)^{\alpha-1} \left|D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right| ds \leq$$

$$\frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)} \left(x - \frac{u}{n}\right)^{\alpha} \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{122}$$

Hence

$$|\gamma_{n1}(x)| \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}, \tag{123}$$

for $x \geq \frac{u}{n}$ and $\left|\frac{u}{n} - x\right| < \frac{1}{n^{\beta}}$.

Let us assume that $x \leq \frac{u}{n}$, then

$$|\gamma_{n2}(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^{\frac{u}{n}} \left(\frac{u}{n} - s\right)^{\alpha-1} \left|D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right| ds$$

$$\leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha)} \int_x^{\frac{u}{n}} \left(\frac{u}{n} - s\right)^{\alpha-1} ds =$$

$$\frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)} \left(\frac{u}{n} - x\right)^{\alpha} \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{124}$$

We have proved that

$$|\gamma_{n2}(x)| \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1) n^{\alpha\beta}}, \tag{125}$$

for $x \leq \frac{u}{n}$ and $\left|\frac{u}{n} - x\right| < \frac{1}{n^{\beta}}$.

Consequently we get

$$|R_{n1}(x)| \Big|_{\left|\frac{u}{n} - x\right| < \frac{1}{n^{\beta}}} \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}, \tag{126}$$

and

$$|R_{n2}(x)| \Big|_{\left|\frac{u}{n} - x\right| < \frac{1}{n^{\beta}}} \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{127}$$

Furthermore we have that

$$|\gamma_{n1}(x)| \leq \left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]} \frac{\left(x - \frac{u}{n}\right)^{\alpha}}{\Gamma(\alpha+1)}, \tag{128}$$

where $x \geq \frac{u}{n} > -\infty$,

and

$$|\gamma_{n2}(x)| \leq \|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \frac{\left(\frac{u}{n} - x\right)^{\alpha}}{\Gamma(\alpha+1)}, \tag{129}$$

where $\infty > \frac{u}{n} \geq x$.

Next, we see that

$$\frac{1}{\Gamma(\alpha)} \left| \int_{\left|\frac{u}{n} - x\right| \geq \frac{1}{n^{\beta}}} \varphi(nx - u) \left(\int_{\frac{u}{n}}^{x} \left(s - \frac{u}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f\right)(s) ds \right) du \right| \leq$$

$$\frac{1}{\Gamma(\alpha)} \int_{\left|\frac{u}{n} - x\right| \geq \frac{1}{n^{\beta}}} \varphi(nx - u) \left| \int_{\frac{u}{n}}^{x} \left(s - \frac{u}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f\right)(s) ds \right| du \leq \tag{130}$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)}\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(x-\frac{u}{n}\right)^{\alpha}du=$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\int_{|u-nx|\geq n^{1-\beta}}\varphi(|nx-u|)\left|nx-u\right|^{\alpha}du\overset{(28)}{\leq}$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda\int_{|u-nx|\geq n^{1-\beta}}e^{-2\lambda(|nx-u|-1)}\left|nx-u\right|^{\alpha}du=$$

$$\frac{2\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda\int_{n^{1-\beta}}^{\infty}e^{-2\lambda(x-1)}x^{\alpha}dx= \tag{131}$$

$$\frac{2\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda\frac{e^{2\lambda}}{(2\lambda)^{\alpha+1}}\int_{n^{1-\beta}}^{\infty}e^{-2\lambda x}(2\lambda x)^{\alpha}d(x2\lambda)=$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}\int_{2\lambda n^{1-\beta}}^{\infty}e^{-y}y^{\alpha}dy\leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}\int_{2\lambda n^{1-\beta}}^{\infty}e^{-y}y^{N}dy\leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}2^{N}N!\int_{2\lambda n^{1-\beta}}^{\infty}e^{-y}e^{\frac{y}{2}}dy=$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}2^{N}N!\int_{2\lambda n^{1-\beta}}^{\infty}e^{-\frac{y}{2}}dy= \tag{132}$$

$$\frac{2^{N+1}N!\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}e^{-\lambda n^{1-\beta}}.$$

We have proved that

$$\frac{1}{\Gamma(\alpha)}\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(\int_{\frac{u}{n}}^{x}\left(s-\frac{u}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f\right)(s)\,ds\right)du\right|\leq \tag{133}$$

$$\frac{2^{N+1}N!\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}e^{-\lambda n^{1-\beta}}.$$

Next, we see that

$$\frac{1}{\Gamma(\alpha)}\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f\right)(s)\,ds\right)du\right|\leq$$

$$\frac{1}{\Gamma(\alpha)}\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left|\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f\right)(s)\,ds\right|du\leq$$

$$\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)}\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(\frac{u}{n}-x\right)^{\alpha}du=$$

$$\frac{\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\int_{|u-nx|\geq n^{1-\beta}}\varphi(|nx-u|)\left|nx-u\right|^{\alpha}du\leq \tag{134}$$

(as before)

$$\frac{2^{N+1}N!\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}e^{-\lambda n^{1-\beta}}.$$

Therefore it holds

$$\frac{1}{\Gamma(\alpha)}\left|\int_{\left|\frac{u}{n}-x\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(\int_{x}^{\frac{u}{n}}\left(\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f\right)(s)\,ds\right)du\right|\leq \tag{135}$$

$$\frac{2^{N+1}N!\,\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\frac{e^{2\lambda}}{(2\lambda)^{\alpha}}e^{-\lambda n^{1-\beta}}.$$

Consequently, we derive

$$|R_n(x)| \leq \frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}\right]+$$

$$\left\{\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}\right]\frac{2^{N+1}N!\left(q+\frac{1}{q}\right)e^{2\lambda}}{\Gamma(\alpha+1)(2\lambda)^{\alpha}n^{\alpha}e^{\lambda n^{1-\beta}}}\right\}. \tag{136}$$

We need $2\lambda n^{1-\beta}\geq 1$, iff $n^{1-\beta}\geq\frac{1}{2\lambda}$, iff $n\geq\frac{1}{(2\lambda)^{\frac{1}{1-\beta}}}$.

(i) In the case of $0<2\lambda\leq 1$ (i.e. $0<\lambda\leq\frac{1}{2}$), then $\frac{1}{2\lambda}\geq 1$, and $\frac{1}{(2\lambda)^{\frac{1}{1-\beta}}}\geq 1$, so for large enough $n\in\mathbb{N}$ we can have $2\lambda n^{1-\beta}\geq 1$.

(ii) If $2\lambda>1$ (i.e. $\lambda>\frac{1}{2}$), then $\frac{1}{2\lambda}<1$ and $\frac{1}{(2\lambda)^{\frac{1}{1-\beta}}}<1$. So for any $n\in\mathbb{N}$ we have that $2\lambda n^{1-\beta}\geq 1$.

For large enough $n\in\mathbb{N}$, we derive that

$$|R_n(x)|\leq\frac{2\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)n^{\alpha\beta}}. \tag{137}$$

That is

$$|R_n(x)|=O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{138}$$

and

$$|R_n(x)|=o(1). \tag{139}$$

And, letting $0<\varepsilon\leq\alpha$, we obtain

$$\frac{|R_n(x)|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)}\leq\frac{2\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\|D_{*x}^{\alpha}f\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)}\left(\frac{1}{n^{\beta\varepsilon}}\right)\to 0,\quad\text{as } n\to\infty. \tag{140}$$

I.e.

$$|R_n(x)|=o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{141}$$

proving the claim.

We continue with the Kantorovich operators.

Theorem 8. *Here all are as in Theorem 7. Then*

1)

$$A_n^*(f)(x)-f(x)=\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot-x)^j\right)(x)+o\left(\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha-\varepsilon}\right), \tag{142}$$

the last (142) implies

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha-\varepsilon}}\left[A_n^*(f)(x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot-x)^j\right)(x)\right]\to 0, \tag{143}$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$.

2) when $N=1$, or $f^{(j)}(x)=0$, $j=1,\ldots,N-1$, we get that

$$\frac{1}{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha-\varepsilon}}\left[A_n^*(f)(x)-f(x)\right]\to 0,\ \text{as } n\to\infty,\ 0<\varepsilon\leq\alpha. \tag{144}$$

Of interest is the case of $\beta=\frac{1}{2}$.

Proof. Let $x \in \mathbb{R}$. We have that $D_{x-}^{\alpha} f(x) = D_{*x}^{\alpha} f(x) = 0$.

We can write

$$f\left(t+\frac{u}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(t+\frac{u}{n}-x\right)^{j} + \frac{1}{\Gamma(\alpha)}\int_{x}^{t+\frac{u}{n}}\left(t+\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds, \tag{145}$$

for all $x \leq t+\frac{u}{n} < \infty$.

Also it holds

$$f\left(t+\frac{u}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(t+\frac{u}{n}-x\right)^{j} + \frac{1}{\Gamma(\alpha)}\int_{t+\frac{u}{n}}^{x}\left(s-\left(t+\frac{u}{n}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds, \tag{146}$$

for all $-\infty < t+\frac{u}{n} \leq x$.

Hence we see

$$\int_{0}^{\frac{1}{n}} f\left(t+\frac{u}{n}\right) dt = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j} dt + \frac{1}{\Gamma(\alpha)}\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(t+\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds\right) dt, \tag{147}$$

for all $x \leq t+\frac{u}{n} < \infty$, iff $nx \leq nt+u < \infty$.

And, also it holds

$$\int_{0}^{\frac{1}{n}} f\left(t+\frac{u}{n}\right) dt = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j} dt + \frac{1}{\Gamma(\alpha)}\int_{0}^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\left(s-\left(t+\frac{u}{n}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds\right) dt, \tag{148}$$

for all $-\infty < t+\frac{u}{n} \leq x$, iff $-\infty < nt+u \leq nx$.

Therefore we obtain

$$n\int_{nx}^{\infty}\left(\int_{0}^{\frac{1}{n}} f\left(t+\frac{u}{n}\right) dt\right)\varphi(nx-u)\, du = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} n\int_{nx}^{\infty}\left(\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j} dt\right)\varphi(nx-u)\, du + \frac{1}{\Gamma(\alpha)} n\int_{nx}^{\infty}\varphi(nx-u) \left(\int_{0}^{\frac{1}{n}}\left(\int_{x}^{t+\frac{u}{n}}\left(t+\frac{u}{n}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds\right) dt\right) du, \tag{149}$$

and

$$n\int_{-\infty}^{nx}\left(\int_{0}^{\frac{1}{n}} f\left(t+\frac{u}{n}\right) dt\right)\varphi(nx-u)\, du = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} n\int_{-\infty}^{nx}\left(\int_{0}^{\frac{1}{n}}\left(t+\frac{u}{n}-x\right)^{j} dt\right)\varphi(nx-u)\, du + \frac{1}{\Gamma(\alpha)} n\int_{-\infty}^{nx}\varphi(nx-u) \left(\int_{0}^{\frac{1}{n}}\left(\int_{t+\frac{u}{n}}^{x}\left(s-\left(t+\frac{u}{n}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds\right) dt\right) du. \tag{150}$$

Adding the last two (149) and (150), we derive

$$A_n^*(f)(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j\right)(x) = R_n^*(x), \tag{151}$$

where

$$R_n^*(x) = R_{n1}^*(x) + R_{n2}^*(x), \tag{152}$$

with

$$R_{n2}^*(x) := \frac{n}{\Gamma(\alpha)} \int_{nx}^{\infty} \varphi(nx-u)$$
$$\left(\int_0^{\frac{1}{n}} \left(\int_x^{t+\frac{u}{n}} \left(t + \frac{u}{n} - s\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds\right) dt\right) du, \tag{153}$$

$nx \leq nt + u < \infty$,
and

$$R_{n1}^*(x) := \frac{n}{\Gamma(\alpha)} \int_{-\infty}^{nx} \varphi(nx-u)$$
$$\left(\int_0^{\frac{1}{n}} \left(\int_{t+\frac{u}{n}}^{x} \left(s - \left(t + \frac{u}{n}\right)\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds\right) dt\right) du, \tag{154}$$

$-\infty < nt + u \leq nx$.
Let $\left|\frac{u}{n} - x\right| < \frac{1}{n^{\beta}}$.
Call

$$\delta_{n1}(x) := \frac{n}{\Gamma(\alpha)} \int_0^{\frac{1}{n}} \left(\int_{t+\frac{u}{n}}^{x} \left(s - \left(t + \frac{u}{n}\right)\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds\right) dt, \tag{155}$$

and

$$\delta_{n2}(x) := \frac{n}{\Gamma(\alpha)} \int_0^{\frac{1}{n}} \left(\int_x^{t+\frac{u}{n}} \left(t + \frac{u}{n} - s\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds\right) dt. \tag{156}$$

We have that

$$|\delta_{n1}(x)| \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]} n}{\Gamma(\alpha)} \int_0^{\frac{1}{n}} \frac{\left(x - t - \frac{u}{n}\right)^{\alpha}}{\alpha} dt =$$
$$\frac{1}{\Gamma(\alpha+1)} \left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}, \tag{157}$$

and

$$|\delta_{n2}(x)| \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} n}{\Gamma(\alpha)} \int_0^{\frac{1}{n}} \frac{\left(t + \frac{u}{n} - x\right)^{\alpha}}{\alpha} dt =$$
$$\frac{1}{\Gamma(\alpha+1)} \|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}. \tag{158}$$

Consequently, we get that

$$|R_{n1}^*(x)|\,|_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}}{\Gamma(\alpha+1)}, \tag{159}$$

and

$$|R_{n2}^*(x)|\,|_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}}{\Gamma(\alpha+1)}. \tag{160}$$

We continue as follows: it is

$$|\delta_{n1}(x)| \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha)} n \int_0^{\frac{1}{n}} \frac{\left(x - t - \frac{u}{n}\right)^{\alpha}}{\alpha} dt \leq \tag{161}$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)}\left(\left|x-\frac{u}{n}\right|+\frac{1}{n}\right)^{\alpha},$$

and

$$|\delta_{n2}(x)|\leq\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)}\left(\left|x-\frac{u}{n}\right|+\frac{1}{n}\right)^{\alpha}. \tag{162}$$

We have that

$$|R_{n1}^{*}(x)|\,|_{\left|x-\frac{u}{n}\right|\geq\frac{1}{n^{\beta}}}\leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)}\int\limits_{(-\infty,nx]\cap\left|x-\frac{u}{n}\right|\geq\frac{1}{n^{\beta}}}\varphi(nx-u)\left(\left|x-\frac{u}{n}\right|+\frac{1}{n}\right)^{\alpha}du=$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\int\limits_{(-\infty,nx]\cap|nx-u|\geq n^{1-\beta}}\varphi(nx-u)(|nx-u|+1)^{\alpha}du\leq \tag{163}$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\int\limits_{|nx-u|\geq n^{1-\beta}}\varphi(|nx-u|)(|nx-u|+1)^{N}du\leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}2^{N-1}\int\limits_{|nx-u|\geq n^{1-\beta}}\varphi(|nx-u|)(1+|nx-u|)^{N}du\leq \tag{164}$$

$$\frac{2^{N-1}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}$$

$$\left[\int\limits_{|nx-u|\geq n^{1-\beta}}\varphi(|nx-u|)du+\int\limits_{|nx-u|\geq n^{1-\beta}}\varphi(|nx-u|)|nx-u|^{N}du\right]\overset{(28)}{\leq}$$

$$\frac{2^{N-1}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda$$

$$\left[\int\limits_{|nx-u|\geq n^{1-\beta}}e^{-2\lambda(|nx-u|-1)}du+\int\limits_{|nx-u|\geq n^{1-\beta}}e^{-2\lambda(|nx-u|-1)}|nx-u|^{N}du\right]= \tag{165}$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda\left[\int_{n^{1-\beta}}^{\infty}e^{-2\lambda(x-1)}dx+\int_{n^{1-\beta}}^{\infty}e^{-2\lambda(x-1)}x^{N}dx\right]=$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\int_{n^{1-\beta}}^{\infty}e^{-2\lambda x}dx+\int_{n^{1-\beta}}^{\infty}e^{-2\lambda x}x^{N}dx\right]=$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{1}{2\lambda}\int_{2\lambda n^{1-\beta}}^{\infty}e^{-y}dy+\frac{1}{(2\lambda)^{N+1}}\int_{2\lambda n^{1-\beta}}^{\infty}e^{-y}y^{N}dy\right]\leq \tag{166}$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{1}{2\lambda}e^{-2\lambda n^{1-\beta}}+\frac{2^{N}N!}{(2\lambda)^{N+1}}\int_{2\lambda n^{1-\beta}}^{\infty}e^{-\frac{y}{2}}dy\right]=$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{1}{2\lambda}e^{-2\lambda n^{1-\beta}}+\frac{2^{N+1}N!}{(2\lambda)^{N+1}}e^{-\lambda n^{1-\beta}}\right]\leq \tag{167}$$

$$\frac{2^{N}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{1}{2\lambda}+\frac{2^{N+1}N!}{(2\lambda)^{N+1}}\right]e^{-\lambda n^{1-\beta}}=$$

$$\frac{2^N \left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda}\left[\frac{1}{2\lambda}+\frac{N!}{\lambda^{N+1}}\right]e^{-\lambda n^{1-\beta}} = \tag{168}$$

$$\frac{2^N \left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^{N}}\right]e^{-\lambda n^{1-\beta}} \to 0, \text{ as } n\to\infty.$$

So that it holds

$$|R^*_{n1}(x)|\,|_{\left|x-\frac{u}{n}\right|\geq\frac{1}{n^{\beta}}} \leq \tag{169}$$

$$\frac{2^N \left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^{N}}\right]e^{-\lambda n^{1-\beta}} \to 0, \text{ as } n\to\infty.$$

Similarly, we obtain that

$$|R^*_{n2}(x)|\,|_{\left|x-\frac{u}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\frac{2^N \left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^{N}}\right]e^{-\lambda n^{1-\beta}}. \tag{170}$$

At the end we get

$$|R^*_n(x)| \leq \frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha}}{\Gamma(\alpha+1)}\left[\left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,\infty)}\right]+ \tag{171}$$

$$\left[\left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,\infty)}\right]$$

$$\frac{2^N}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)e^{2\lambda}\left[\frac{1}{2}+\frac{N!}{\lambda^{N}}\right]e^{-\lambda n^{1-\beta}}.$$

So for large enough $n\in\mathbb{N}$, we obtain

$$|R^*_n(x)| \leq \frac{2\left[\left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)}\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha}. \tag{172}$$

Hence it holds

$$|R^*_n(x)| = O\left(\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha}\right), \tag{173}$$

and

$$|R^*_n(x)| = o(1). \tag{174}$$

And, letting $0<\varepsilon\leq\alpha$, we obtain

$$\frac{|R^*_n(x)|}{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha-\varepsilon}} \leq \frac{2\left[\left\|D^{\alpha}_{x-}f\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,\infty)}\right]}{\Gamma(\alpha+1)}\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\varepsilon} \to 0, \tag{175}$$

as $n\to\infty$.

I.e.

$$|R^*_n(x)| = o\left(\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha-\varepsilon}\right), \tag{176}$$

proving the theorem.

We continue with the quadrature operators.

Theorem 9. *Here all are as in Theorem 7. Then*

1)

$$\overline{A_n}(f)(x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \overline{A_n}\left((\cdot - x)^j\right)(x) + o\left(\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{\alpha-\varepsilon}\right), \tag{177}$$

the last (177) implies

$$\frac{1}{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{\alpha-\varepsilon}} \left[\overline{A_n}(f)(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \overline{A_n}\left((\cdot - x)^j\right)(x)\right] \to 0, \tag{178}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

2) when $N = 1$, *or* $f^{(j)}(x) = 0$, $j = 1, \ldots, N-1$, *we get that*

$$\frac{1}{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{\alpha-\varepsilon}} \left[\overline{A_n}(f)(x) - f(x)\right] \to 0, \text{ as } n \to \infty,\ 0 < \varepsilon \leq \alpha. \tag{179}$$

Of interest is the case of $\beta = \frac{1}{2}$.

Proof. Let $x \in \mathbb{R}$. We have that $D_{x-}^{\alpha} f(x) = D_{*x}^{\alpha} f(x) = 0$.

We can write

$$f\left(\frac{u}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{u}{n} + \frac{i}{nr} - x\right)^j + \tag{180}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(\frac{u}{n} + \frac{i}{nr} - s\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds,$$

for all $x \leq \frac{u}{n} + \frac{i}{nr} < \infty$, $i = 1, \ldots, r$.

Also, it holds

$$f\left(\frac{u}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{u}{n} + \frac{i}{nr} - x\right)^j + \tag{181}$$

$$\frac{1}{\Gamma(\alpha)} \int_{\frac{u}{n}+\frac{i}{nr}}^{x} \left(s - \left(\frac{u}{n} + \frac{i}{nr}\right)\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds,$$

for all $-\infty < \frac{u}{n} + \frac{i}{nr} \leq x$, $i = 1, \ldots, r$.

Hence

$$\sum_{i=1}^{r} w_i f\left(\frac{u}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{i=1}^{r} w_i \left(\frac{u}{n} + \frac{i}{nr} - x\right)^j + \tag{182}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} w_i \int_x^{\frac{u}{n}+\frac{i}{nr}} \left(\frac{u}{n} + \frac{i}{nr} - s\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(s) - D_{*x}^{\alpha} f(x)\right) ds,$$

and

$$\sum_{i=1}^{r} w_i f\left(\frac{u}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{i=1}^{r} w_i \left(\frac{u}{n} + \frac{i}{nr} - x\right)^j + \tag{183}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} w_i \int_{\frac{u}{n}+\frac{i}{nr}}^{x} \left(s - \left(\frac{u}{n} + \frac{i}{nr}\right)\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(s) - D_{x-}^{\alpha} f(x)\right) ds.$$

Furthermore it holds

$$\int_{nx}^{\infty} \left(\sum_{i=1}^{r} w_i f\left(\frac{u}{n} + \frac{i}{nr}\right)\right) \varphi(nx - u)\, du = \tag{184}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \int_{nx}^{\infty} \left(\sum_{i=1}^{r} w_i \left(\frac{u}{n} + \frac{i}{nr} - x\right)^j\right) \varphi(nx - u)\, du +$$

$$\frac{1}{\Gamma(\alpha)}\int_{nx}^{\infty}\varphi(nx-u)$$
$$\left(\sum_{i=1}^{r}w_i\int_{x}^{\frac{u}{n}+\frac{i}{nr}}\left(\frac{u}{n}+\frac{i}{nr}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du,$$

and

$$\int_{-\infty}^{nx}\left(\sum_{i=1}^{r}w_i f\left(\frac{u}{n}+\frac{i}{nr}\right)\right)\varphi(nx-u)\,du= \tag{185}$$
$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\int_{-\infty}^{nx}\left(\sum_{i=1}^{r}w_i\left(\frac{u}{n}+\frac{i}{nr}-x\right)^{j}\right)\varphi(nx-u)\,du+$$
$$\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{nx}\varphi(nx-u)$$
$$\left(\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\left(s-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du.$$

Adding the last two (184), (185), we obtain

$$\overline{A_n}(f)(x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\overline{A_n}\left((\cdot-x)^{j}\right)(x)=\overline{R_n}(x), \tag{186}$$

where

$$\overline{R_n}(x)=\overline{R_{n1}}(x)+\overline{R_{n2}}(x), \tag{187}$$

with

$$\overline{R_{n1}}(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{nx}\varphi(nx-u)$$
$$\left(\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\left(s-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds\right)du, \tag{188}$$

and

$$\overline{R_{n2}}(x)=\frac{1}{\Gamma(\alpha)}\int_{nx}^{\infty}\varphi(nx-u)$$
$$\left(\sum_{i=1}^{r}w_i\int_{x}^{\frac{u}{n}+\frac{i}{nr}}\left(\frac{u}{n}+\frac{i}{nr}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds\right)du.$$

Let $\left|\frac{u}{n}-x\right|<\frac{1}{n^{\beta}}$.
Call

$$\varepsilon_{n1}(x):=\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}w_i\int_{\frac{u}{n}+\frac{i}{nr}}^{x}\left(s-\left(\frac{u}{n}+\frac{i}{nr}\right)\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(s)-D_{x-}^{\alpha}f(x)\right)ds, \tag{189}$$

and

$$\varepsilon_{n2}(x):=\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}w_i\int_{x}^{\frac{u}{n}+\frac{i}{nr}}\left(\frac{u}{n}+\frac{i}{nr}-s\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(s)-D_{*x}^{\alpha}f(x)\right)ds. \tag{190}$$

Then

$$|\varepsilon_{n1}(x)|\leq\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}w_i\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}\frac{\left(x-\frac{u}{n}-\frac{i}{nr}\right)^{\alpha}}{\alpha}\leq$$
$$\frac{1}{\Gamma(\alpha+1)}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{\alpha}. \tag{191}$$

Therefore it holds

$$|\varepsilon_{n1}(x)|\leq\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)}\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha}. \tag{192}$$

Furthermore wee see that

$$|\varepsilon_{n2}(x)| \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} w_i \|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \frac{\left(\frac{u}{n} + \frac{i}{nr} - x\right)^{\alpha}}{\alpha} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \left(\frac{1}{n^{\beta}} + \frac{1}{n}\right)^{\alpha}. \tag{193}$$

That is

$$|\varepsilon_{n2}(x)| \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}. \tag{194}$$

Consequently, we get that

$$\left|\overline{R_{n1}}(x)\right| |_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}}{\Gamma(\alpha+1)}, \tag{195}$$

and

$$\left|\overline{R_{n2}}(x)\right| |_{\left|\frac{u}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)} \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{\alpha}}{\Gamma(\alpha+1)}. \tag{196}$$

Furthermore it holds

$$|\varepsilon_{n1}(x)| \leq \frac{\left\|D_{x-}^{\alpha} f\right\|_{\infty,(-\infty,x]}}{\Gamma(\alpha+1)} \left(\left|x - \frac{u}{n}\right| + \frac{1}{n}\right)^{\alpha}, \tag{197}$$

and

$$|\varepsilon_{n2}(x)| \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)} \left(\left|x - \frac{u}{n}\right| + \frac{1}{n}\right)^{\alpha}. \tag{198}$$

The rest of the proof is similar to Theorem 8. As such it is omitted.

We make

*Remark.*Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}(\mathbb{R})$, with $f^{(j)} \in C_B(\mathbb{R})$, for $j = 0, 1, ..., i$. We derive from Remark 3 that

$$(A_n(f))^{(j)}(x) = A_n\left(f^{(j)}\right)(x),$$

$$(A_n^*(f))^{(j)}(x) = A_n^*\left(f^{(j)}\right)(x), \tag{199}$$

$$\left(\overline{A_n}(f)\right)^{(j)}(x) = \overline{A_n}\left(f^{(j)}\right)(x), \quad \forall\, x \in \mathbb{R},$$

for all $j = 1, ..., i$.

Let $f^{(j)} \underset{(<0)}{\geq} 0$, then $(A_n(f))^{(j)}$, $(A_n^*(f))^{(j)}$, $\left(\overline{A_n}(f)\right)^{(j)} \underset{(<0)}{\geq} 0$, respectively. So we have preservation of monotonicity, and convexity-concavity, etc.

Next, we finish with simultaneous Voronovskaya type asymptotic expansions.

Theorem 10.*Let $0 < \alpha < 1$, $n \in \mathbb{N}$ large enough, $x \in \mathbb{R}, j = 0, 1, ..., i \in \mathbb{N}$, $i \in \mathbb{N}$ is fixed, $f^{(j)} \in C^N(\mathbb{R})$, $N \in \mathbb{N}$; for $f^{(j)} \in C_B(\mathbb{R})$, with $f^{(N+j)} \in C_B(\mathbb{R})$; $0 < \varepsilon \leq N$. Then*

1)

$$(A_n(f))^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N} \frac{f^{(j+\lambda)}(x)}{\lambda!} A_n\left((\cdot - x)^{\lambda}\right)(x) = o\left(\frac{1}{n^{\alpha(N-\varepsilon)}}\right), \tag{200}$$

2)

$$(A_n^*(f))^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N} \frac{f^{(j+\lambda)}(x)}{\lambda!} A_n^*\left((\cdot - x)^{\lambda}\right)(x) = o\left(\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^{N-\varepsilon}\right), \tag{201}$$

and

3)

$$\left(\overline{A_n}(f)\right)^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N} \frac{f^{(j+\lambda)}(x)}{\lambda!} \overline{A_n}\left((\cdot - x)^{\lambda}\right)(x) = o\left(\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^{N-\varepsilon}\right). \tag{202}$$

Proof. By Theorems 4- 6 and Remark 4.

Theorem 11. *Let $x \in \mathbb{R}$, $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $0 < \beta < 1$, $n \in \mathbb{N}$ large enough. Here $i \in \mathbb{N}$ is fixed and $j = 0, 1, ..., i$; $f^{(j)} \in C_B(\mathbb{R})$, $f^{(j)} \in AC^N(\mathbb{R})$, $f^{(j+N)} \in L_\infty(\mathbb{R})$. Assume that $\|D^\alpha_{*x} f\|_{\infty,[x,\infty)}$, $\left\|D^\alpha_{x-} f\right\|_{\infty,(-\infty,x]}$ are finite, $0 < \varepsilon \leq \alpha$. Then*

1)
$$\left(A_n(f)\right)^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N-1} \frac{f^{(j+\lambda)}(x)}{\lambda!} A_n\left((\cdot - x)^\lambda\right)(x) = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{203}$$

2)
$$\left(A_n^*(f)\right)^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N-1} \frac{f^{(j+\lambda)}(x)}{\lambda!} A_n^*\left((\cdot - x)^\lambda\right)(x) = o\left(\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{\alpha-\varepsilon}\right), \tag{204}$$

and

3)
$$\left(\overline{A_n}(f)\right)^{(j)}(x) - f^{(j)}(x) - \sum_{\lambda=1}^{N-1} \frac{f^{(j+\lambda)}(x)}{\lambda!} \overline{A_n}\left((\cdot - x)^\lambda\right)(x) = o\left(\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^{\alpha-\varepsilon}\right). \tag{205}$$

Proof. By Theorems 7-9 and Remark 4.

Conclusion: Here we presented the new idea of going from the neural networks main tools, the activation functions, to convolution integrals approximation Voronovsaka type asymptotic expansions. That is the rare case of employing applied mathematics to theoretical ones.

References

[1] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Heidelberg, New York, 2016.
[2] G.A. Anastassiou, *Parametrized, Deformed and General Neural Networks*, Springer, Heidelberg, New York, 2023.
[3] G.A. Anastassiou, *Approximation by symmetrized and perturbed hyperbolic tangent activated convolution-type operators*, MDPI/Mathematics, 2024, 12, 3302; https://doi.org.10.3390.math12203302.
[4] Wang, Yuxuan; Wang, Tongke; Liam, Huan, *The series expansions and blow-up time estimation for the solutions of convolution Volterra-Hammerstein integral equations*. Numer. Algorithms 95 (2024), no. 2, 637-663.
[5] Askhabov, S.N. *A system of inhomogeneous integral equations of convolution type with power nonlinearity*. Syb. Math. J. 64 (2023), no. 3, 691-698.
[6] Wang, Mengjie; Dai, Xinjie; Yu, Yanyan; Xiao, Aiguo, *Fast θ-Maruyama scheme for stochastic Volterra integral equations of convolution type: mean-square stability and strong convergence analysis*. Comput. Appl. Math. 42 (2023), no. 3, Paper no. 108, 36 pp.
[7] Angeloni, Laura; Merentes, Nelson J.; Valera-López, Maira A. *Convolution integral operators in variable bounded variation spaces*. Mediterr. J. Math. 20 (2023), no. 3, Paper no. 141, 20 pp.
[8] Askhabov, S.N. *System of inhomogeneous integral equations of convolution type with power nonlinearity*. (Russian) Vladikavkaz. Mat. Zh. 24 (2022), no. 1, 5-14.
[9] Khachatryan, Kh. A.; Petrosyan, A.S. *Alternating bounded solutions of a class of nonlinear two-dimensional convolution-type integral equations*. trans. Moscow Math. Soc. 82 (2021), 259-271.
[10] Askhabov, S.N. *Nonlinear convolution type integral equations in complex spaces*. Ufa Math. J. 13 (2021), no. 1, 17-30.
[11] Safaryan, M.H. *On generalizations of Fatou's theorem in L_p for convolution integrals with general kernels*. J. Geom. Anal. 31 (2021), no. 4, 3280-3299.
[12] Karchevskii, A.L. *Solution of the convolution type Volterra integral equations of the first kind by the quadtrature-sum method*. (Russian) Sib. Zh. Ind. Mat. 23 (2020), no. 3, 40-52; translation in J. Appl. Ind. Math. 14 (2020), no. 3, 503-512.
[13] Karapetrović, Boban; Mashreghi, Javad, *Hadamard convolution and area integral means in Bergman spaces*. Results Math. 75 (2020), no. 2, Paper no. 70, 11 pp.
[14] G.A. Anastassiou, *On Right Fractional Calculus*, Chaos, solitons and fractals, 42 (2009), 365-376.
[15] G.A. Anastassiou, *Fractional Korovkin theory*, Chaos, Solitons & Fractals, Vol. 42, No. 4 (2009), 2080-2094.
[16] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer-Verlag, Berlin, Heidelberg, 2010.
[17] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, Vol. 3, no. 12 (2006), 81-95.
[18] G.S. Frederico and D.F.M. Torres, *Fractional Optimal Control in the sense of Caputo and the fractional Noether's theorem*, International Mathematical Forum, Vol. 3, no. 10 (2008), 479-493.
[19] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, (Gordon and Breach, Amsterdam, 1993) [English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications (Nauka i Tekhnika, Minsk, 1987)].