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Conformable Semigroups Via Compact Operators

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Abstract: Let Y be a Banach space and $\{\mathscr{S}(s)\}_{s\geq 0}$ be a one parameter conformable semigroup of bounded linear operators on Y. In this paper, we give an answer to the question: "As $\mathscr{S}(s)$ is a bounded linear operator under what conditions a strongly continuous conformable α-semigroup is compact". Further we show that under certain conditions the resolvent operator $\mathscr{R}(\mu:\mathscr{A})$ of the generator of $\{\mathscr{S}(s)\}_{s\geq 0}$ can be compact.

Keywords: conformable operator; α-semigroup; compact operator.

1 Introduction

In [15], the authors gave an interesting definition an operator that uses limit approach as an extension of the usual definition of the first order derivative as follows:

Let $f:[0,\infty)\to\mathbb{R}$. Then for all $s>0$, $\alpha\in(0,1)$,

$$D_\alpha(f)(s)=f^{(\alpha)}(s)=\begin{cases}\lim\limits_{\varepsilon\to 0}\frac{f(s+\varepsilon s^{1-\alpha})-f(s)}{\varepsilon}, & s>0\\ \lim\limits_{s\to 0^+} f^{(\alpha)}(s), & s=0\end{cases}$$

provided that $D_\alpha(f)(s)$ exists in $(0,b)$, $b>0$. It is known, [5,15], that conformable operator obeys all the classical properties of the usual first derivative except the chain role, see [5] .

The α-conformable integral of a function f starting from $a\geq 0$ is defined as

$$I_\alpha^a(f)(s)=\int_a^s f(t)d_\alpha t=\int_a^s f(t)t^{\alpha-1}dt.$$

It is known, [5,15], that for a continuous function f, the integral $I_\alpha^a f$ exists and

$$D_\alpha(I_\alpha^a f)(s)=f(s),\quad for\ s\geq a\geq 0.$$

Foremore on conformable operators and their applications we refer the reader to [13,14,16,18,22], [1]- [11] and references therin

For a Banach space Y, a family $\{\mathscr{S}(s)\}_{t\geq 0}\subseteq\mathscr{L}(Y)$, where $\mathscr{L}(Y)$ is the space of all bounded linear operators on Y, is called a semigroup of operators if:

(1) $\mathscr{S}(0)=I$ (identity operator) ;

(2) $\mathscr{S}(s+v)=\mathscr{S}(s)\mathscr{S}(v)$ for all $s,v\geq 0$.

If for each fixed $y\in Y$, $\mathscr{S}(s)y\to y$ as $s\to 0^+$, then the semigroup $\{\mathscr{S}(s)\}_{s\geq 0}$ is called a c_0-semigroup or strongly continuous semigroup.

The theory of semigroups had immediate applications in partial differential equations, Markov processes, and ergodic theory, etc. see,[19,21,23].

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Regarding the α-Abstract Cauchy problem

$$\begin{aligned} \mathrm{u}^{(\alpha)}(s) &= \mathscr{B}\mathrm{u}(s), \quad s>0, \\ \mathrm{u}(0) &= \mathrm{u}_0, \end{aligned}$$

where $\mathscr{B}:\mathscr{D}(\mathscr{B})\subset Y\longrightarrow Y$ is a linear operator of an appropriate type, $u_0\in Y$ is given and $\mathrm{u}:[0,\infty)\longrightarrow Y$ is the unknown function, that is not easy to be obtained a new definition of a conformable α-semigroup of bounded linear operators has been given by Alhorani, Khalil and abdeljawad in 2015,[2] as follows:

A family $\{\mathscr{S}(s)\}_{s\geq 0}\subseteq\mathscr{L}(Y)$ is called α-semigroup if

(1) $\mathscr{S}(0)=I$ (identity operator);

(2) $\mathscr{S}\left((s+v)^{\frac{1}{\alpha}}\right)=\mathscr{S}(s^{\frac{1}{\alpha}})\mathscr{S}(v^{\frac{1}{\alpha}})$ for all $s,v\geq 0$.

If $\alpha=1$, then the α-semigroup turns to be the usual semigroups. If for any fixed $y\in Y$, $\mathscr{S}(s)y\to y$ as $s\to 0^+$, then $\{\mathscr{S}(s)\}_{s\geq 0}$ is called strongly continuous α-semigroup or c_0-α-semigroup.

*Example 1.*If $\mathscr{A}$ is a bounded linear operator on a Banach space Y, then the family $\{\mathscr{S}(s)\}_{s\geq 0},\mathscr{S}(s)=e^{\frac{s^\alpha}{\alpha}\mathscr{A}}$ is a c_0-α-semigroup. In fact $\mathscr{S}(0)=e^{0\mathscr{A}}=I$ and

$$\mathscr{S}\left((s+v)^{\frac{1}{\alpha}}\right)=e^{\frac{(s+v)}{\alpha}\mathscr{A}}=e^{\frac{s}{\alpha}\mathscr{A}}e^{\frac{v}{\alpha}\mathscr{A}}=\mathscr{S}(s^{\frac{1}{\alpha}})\mathscr{S}(v^{\frac{1}{\alpha}})$$

.

The linear operator $\mathscr{A}$ defined by

$$\mathscr{A}y=\lim_{s\to 0^+}\lim_{\varepsilon\to 0}\frac{\mathscr{S}\left(s+\varepsilon s^{1-\alpha}\right)y-\mathscr{S}(s)y}{\varepsilon}$$

with domain

$$\mathscr{D}(A)=\left\{y\in Y:\mathscr{A}y=\lim_{s\to 0^+}\lim_{\varepsilon\to 0}\frac{\mathscr{S}\left(s+\varepsilon s^{1-\alpha}\right)y-\mathscr{S}(s)y}{\varepsilon}\text{ exists}\right\}$$

is called the infinitesimal generator of the α-semigroup. More precisely $\mathscr{A}y$ is value of the conformable α-operator of $\mathscr{S}(s)y$ at $s=0$. It is known [2], that for $y\in\mathscr{D}(A)$,

$$\mathscr{S}^\alpha(s)y=\mathscr{A}\mathscr{S}(s)y=\mathscr{S}(s)\mathscr{A}y.$$

For a complex number $\mu\in\rho(\mathscr{A})$, the resolvent set of the operator $\mathscr{A}$, provided $\mu-\mathscr{A}$ is injective let $\mathscr{R}(\mu:\mathscr{A})=(\mu-\mathscr{A})^{-1}$.

In this paper, we give an answer to the question: "As $\mathscr{S}(s)$ is a bounded linear operator under what conditions a strongly continuous conformable α-semigroup is compact. Moreover we show that if the semigroup $\mathscr{S}(s)$ satisfies certain conditions then the resolvent operator $\mathscr{R}(\mu:\mathscr{A})$ of the generator of an exponentially bounded conformable semigroup is compact.

Throughout this paper $\mathscr{L}(Y)$ will be the space of all bounded linear operators on the Banach space Y. For a densely defined linear operator $\mathscr{A}$ on Y, by $\sigma(\mathscr{A}),\rho(\mathscr{A})$ we denote the spectrum and resolvent set of $\mathscr{A}$ respectively.

2 Conformable Compact Semigroup

In this section sufficient and necessary conditions for a c_0-α-semigroup to be compact have been obtained. We begin by the following theorem

Theorem 1.*Let $\{\mathscr{S}(s)\}_{s\geq 0}$ be a c_0-α-semigroup on Y. Then there exist constants $\omega\geq 0$ and $\mathscr{M}\geq 1$ such that*

$$\|\mathscr{S}(s)\|\leq\mathscr{M}e^{\omega s^\alpha}\qquad \textit{for}\quad 0\leq s<\infty.$$

Proof. First we will show that $\|\mathscr{S}(s)\| \leq \mathscr{M}$ for $0 \leq s^{\alpha} \leq b$ for some $b > 0$.
Suppose not, there exist a sequence $s_n \geq 0$ such that $s_n \to 0$ but $\|\mathscr{S}(s_n)\| \geq n$.
From the contrapositive of the uniform bounded theorem, there exists $y \in Y$ such that $\|\mathscr{S}(s_n)y\|$ is unbounded which is a contradiction ($\mathscr{S}(s_n)y \to y$ from definition of c_0-α-semigroup) so,

$$\|\mathscr{S}(s)\| \leq \mathscr{M}$$

$\forall \quad 0 \leq s^{\alpha} \leq b$. Now, if $s^{\alpha} > b$, then $s^{\alpha} = nb + a$, where $a < b$.

$$\begin{aligned}\|\mathscr{S}(s)\| = \|\mathscr{S}\left((s^{\alpha})^{\frac{1}{\alpha}}\right)\| = \|\mathscr{S}\left((nb+a)^{\frac{1}{\alpha}}\right)\| &= \|\mathscr{S}\left((nb)^{\frac{1}{\alpha}}\right)\|\|\mathscr{S}\left(a^{\frac{1}{\alpha}}\right)\| \\ &= \|\mathscr{S}^n\left(b^{\frac{1}{\alpha}}\right)\|\|\mathscr{S}\left(a^{\frac{1}{\alpha}}\right)\| \\ &\leq \mathscr{M}^n\mathscr{M}.\end{aligned}$$

If we let $\omega = \frac{\log \mathscr{M}}{b}$, then $\omega b = \log \mathscr{M}$ and

$$\|\mathscr{S}(s)\| \leq \mathscr{M}\mathscr{M}^n \leq \mathscr{M}\mathscr{M}^{\frac{s^{\alpha}}{b}} = \mathscr{M}((e^{\omega b})^{\frac{s^{\alpha}}{b}}) = \mathscr{M}e^{\omega s^{\alpha}}$$

Therefore, $\|\mathscr{S}(s)\| \leq \mathscr{M}e^{\omega s^{\alpha}}$.

Corollary 1. *If $\{\mathscr{S}(s)\}_{s\geq 0}$ is a c_0-α-semigroup, then for all $y \in Y$, $s \to \mathscr{S}(s)y$ is continuous on $[0,\infty)$.*

Proof. Let $s, h \geq 0$. The continuity of $s \to \mathscr{S}(s)y$ follows from

$$\begin{aligned}\|\mathscr{S}(s+h)y - \mathscr{S}(s)y\| &= \|\mathscr{S}\left((s^{\alpha} - s^{\alpha} + (s+h)^{\alpha})^{\frac{1}{\alpha}}\right)y - \mathscr{S}(s)y\| \\ &= \|\mathscr{S}\left((s^{\alpha} + (s+h)^{\alpha} - s^{\alpha})^{\frac{1}{\alpha}}\right)y - \mathscr{S}(s)y\| \\ &= \|\mathscr{S}\left((s^{\alpha})^{\frac{1}{\alpha}}\right)\mathscr{S}\left(((s+h)^{\alpha} - s^{\alpha})^{\frac{1}{\alpha}}\right)y - \mathscr{S}(s)y\| \\ &\leq \|\mathscr{S}(s)\|\|\mathscr{S}\left(((s+h)^{\alpha} - s^{\alpha})^{\frac{1}{\alpha}}\right)y - y\| \\ &\leq \mathscr{M}e^{\omega s^{\alpha}}\|\mathscr{S}\left(((s+h)^{\alpha} - s^{\alpha})^{\frac{1}{\alpha}}\right)y - y\| \to 0 \quad \text{as} \quad h \to 0^+.\end{aligned}$$

Therefore, $\mathscr{S}(s)$ is continuous from right.
Similarly one can show that $\|\mathscr{S}(s-h)y - \mathscr{S}(s)y\| \to 0 \quad \text{as} \quad h \to 0^+$.

Proposition 1. *Let $\{\mathscr{S}(s)\}_{s\geq 0}$ be a c_0-α-semigroup of linear operators on Y. If $\mathscr{S}(s_0)$ is compact for some $s_0 > 0$, then $\mathscr{S}(s)$ is compact for all $s \in (s_0,\infty)$.*

Proof. Since $\mathscr{S}(s_0)$ is compact for $s_0 > 0$, the identity

$$\mathscr{S}(s) = \mathscr{S}\left(s^{\alpha} - s_0^{\alpha} + s_0^{\alpha})^{\frac{1}{\alpha}}\right) = \mathscr{S}\left((s^{\alpha} - s_0^{\alpha})^{\frac{1}{\alpha}}\right)\mathscr{S}(s_0)$$

implies that $\mathscr{S}(s)$ is compact for all $s \in (s_0,\infty)$. This ends the proof.

Theorem 2. *Let $\{\mathscr{S}(s)\}_{s\geq 0}$ be a c_0-α-semigroup of linear operators on Y. If $\mathscr{S}(s)$ is compact for all $s \in (s_0,\infty)$ for some $s_0 > 0$, then the map $s \to \mathscr{S}(s)$ is continuous from the right in the uniform operator topology on (s_0,∞).*

Proof. Let $\|\mathscr{S}(s)\| \leq \mathscr{K}$ for $0 \leq s \leq 1$. Then for $s > s_0$, the set $U_s = \{\mathscr{S}(s)y : \|y\| \leq 1\}$ is relatively compact. For $\varepsilon > 0$, the collection of sets $\mathscr{W} = \{\mathscr{B}\left(\mathscr{S}(s)y, \frac{\varepsilon}{2(\mathscr{K}+1)}\right) : \|y\| \leq 1\}$ forms an open cover for $\overline{U_s}$. Since $\overline{U_s}$ is compact, then their exist $y_1, y_2, \ldots, y_{\mathscr{N}}$ in Y such that $\bigcup_{i=1}^{\mathscr{N}} \mathscr{B}\left(\mathscr{S}(s)y_i, \frac{\varepsilon}{3(\mathscr{K}+1)}\right)$ covers U_s. This means that for each $y \in Y$; $\|y\| \leq 1$ there exists r (r depends on y), $1 \leq r \leq \mathscr{N}$ such that:

$$\|\mathscr{S}(s)y - \mathscr{S}(s)y_r\| \leq \frac{\varepsilon}{3(\mathscr{K}+1)}.$$

But $\mathscr{S}(s)$ is strongly continuous. So there exists h_0 , $0 \le h_0 \le 1$ such that:

$$\left\|\mathscr{S}(s^\alpha + h^\alpha)^{\frac{1}{\alpha}} y_r - \mathscr{S}(s) y_r\right\| \le \frac{\varepsilon}{3},$$

for $0 \le h \le h_0$ and all j, $1 \le r \le \mathscr{N}$.

Now, for $0 \le h \le h_0$, and $y \in Y$; $\|y\| \le 1$, set $L = \left\|\mathscr{S}(s^\alpha + h^\alpha)^{\frac{1}{\alpha}} y - \mathscr{S}(s) y\right\|$. Then:

$$\begin{aligned}
L &= \left\| \begin{array}{c} \mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y - \mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y_r \\ + \mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y_r - \mathscr{S}(s) y_r + \mathscr{S}(s) y_r - \mathscr{S}(s) y \end{array} \right\| \\
&\le \left\|\mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y - \mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y_r\right\| \\
&\quad + \left\|\mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y_r - \mathscr{S}(s) y_r\right\| + \|\mathscr{S}(s) y_r - \mathscr{S}(s) y\| \\
&\le \|\mathscr{S}(s)\mathscr{S}(h) y - \mathscr{S}(s)\mathscr{S}(h) y_r\| \\
&\quad + \left\|\mathscr{S}\left((s^\alpha + h^\alpha)^{\frac{1}{\alpha}}\right) y_r - \mathscr{S}(s) y_r\right\| + \|\mathscr{S}(s) y_r - \mathscr{S}(s) y\| \\
&\le \|\mathscr{S}(h)\| \, \|\mathscr{S}(s) y - \mathscr{S}(s) y_r\| \\
&\quad + \left\|\mathscr{S}(s^\alpha + h^\alpha)^{\frac{1}{\alpha}} y_r - \mathscr{S}(s) y_r\right\| + \|\mathscr{S}(s) y_r - \mathscr{S}(s) y\| \\
&\le \mathscr{K} \frac{\varepsilon}{3(\mathscr{K}+1)} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3(\mathscr{K}+1)} = \varepsilon.
\end{aligned}$$

Thus $\mathscr{S}(s)$ is continuous in the uniform operator topology.

Theorem 3. *Let $\{\mathscr{S}(s)\}_{s \ge 0}$ be a c_0-α-semigroup on Y with infinitesimal generator $\mathscr{A}$. If $\mathscr{S}(s)$ is compact for $s \in (0, \infty)$, then the resolvent operator $\mathscr{R}(\mu : \mathscr{A})$ is compact for all $\mu \in \rho(\mathscr{A})$.*

Proof. By Theorem 1 there exists $\mathscr{M} > 0$ such that $\|\mathscr{S}(s)\| \le \mathscr{M} e^{\omega \frac{s^\alpha}{\alpha}}$. Suppose that $\mathscr{S}(s)$ is compact for $s > 0$. Using Theorem 2 it follows that $\mathscr{S}(s)$ is continuous in the uniform operator topology. Therefore for $\operatorname{Re}\mu > \omega$, define the operator

$$\mathscr{R}(\mu : \mathscr{A}) y = \int_0^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}(s) y \, d_\alpha s, .$$

and the integral exists in the uniform operator topology. For $b > 0$, Define:$\mathscr{R}_b(\mu) y = \int_b^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}(s) y \, d_\alpha s$,

$$\begin{aligned}
\mathscr{R}_b(\mu) y &= \int_b^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}(s) y \, d_\alpha s \\
&= \int_b^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}((s^\alpha - t^\alpha + t^\alpha)^{\frac{1}{\alpha}}) y \, d_\alpha s \\
&= \int_b^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}\left((s^\alpha - t^\alpha)^{\frac{1}{\alpha}}\right) \mathscr{S}(t) y \, d_\alpha s \\
&= \mathscr{S}(t) \int_b^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}\left((s^\alpha - t^\alpha)^{\frac{1}{\alpha}}\right) y \, d_\alpha s.
\end{aligned}$$

Since $\mathscr{S}(t)$ is compact and the operator C, $Cy = \int\limits_b^\infty e^{-\mu\frac{s^\alpha}{\alpha}}\mathscr{S}\left((s^\alpha - t^\alpha)^{\frac{1}{\alpha}}\right) y\, d_\alpha s$ is bounded in $\mathscr{L}(Y)$ for $\operatorname{Re}\mu > \omega$ and $b > 0$, the operators $\mathscr{R}_b(\mu)$ is compact for all $b > 0$, $\mu \in \rho(A), \operatorname{Re}\mu > \omega \geq 0$. Further:

$$
\begin{aligned}
\|\mathscr{R}_b(\mu) - \mathscr{R}(\mu : \mathscr{A})\| &= \left\| \int\limits_b^\infty e^{-\mu\frac{s^\alpha}{\alpha}}\mathscr{S}(s)\, d_\alpha s - \int\limits_0^\infty e^{-\mu\frac{s^\alpha}{\alpha}}\mathscr{S}(s)\, d_\alpha s \right\| \\
&\leq \left\| \int\limits_0^b e^{-\mu\frac{s^\alpha}{\alpha}}\mathscr{S}(s)\, d_\alpha s \right\| \\
&\leq \mathscr{M}\int\limits_0^b e^{(\omega-\mu)\frac{s^\alpha}{\alpha}}\, d_\alpha s.
\end{aligned}
$$

Since $\lim\limits_{b\to 0^+}\mathscr{M}\int\limits_0^b e^{(\omega-\mu)\frac{s^\alpha}{\alpha}}\, d_\alpha s = 0$, $\mathscr{R}_b(\mu)$ is compact for each $b > 0$, it follows that $\mathscr{R}(\mu : \mathscr{A})$ is compact for all $\mu \in \rho(A)$, $\operatorname{Re}\mu > \omega \geq 0$. Using the resolvent identity, [16]

$$
\mathscr{R}(\mu : \mathscr{A}) = \mathscr{R}(\lambda : \mathscr{A}) + (\lambda - \mu)\mathscr{R}(\lambda : \mathscr{A})\mathscr{R}(\mu : \mathscr{A}),
$$

it follows that $\mathscr{R}(\mu : \mathscr{A})$ is compact for any $\mu \in \rho(A)$.

Theorem 4. *Let $\{\mathscr{S}(s)\}_{s\geq 0}$ be a c_0-α-semigroup on Y with infinitesimal generator $\mathscr{A}$. If $\mathscr{S}(s)$ is uniformly continuous on $(0,\infty)$ and the resolvent operator $\mathscr{R}(\mu : \mathscr{A})$ is compact for all $\mu \in \rho(A)$, then $\mathscr{S}(s)$ is compact for all $s > 0$.*

Proof. Since $\mathscr{R}(\mu : \mathscr{A})$ is compact for $\mu \in \rho(A)$ and $\mathscr{S}(s) \in \mathscr{L}(Y)$ for $s > 0$, it follows that $\mu\mathscr{R}(\mu : \mathscr{A})\mathscr{S}(s)$ is compact. Now for $\mu \in \rho(A)$, $\operatorname{Re}\mu > \omega$, we have $\mathscr{R}(\mu : \mathscr{A})y = \int\limits_0^\infty e^{-\mu\frac{t^\alpha}{\alpha}}\mathscr{S}(t)y\, d_\alpha t$. Let $J = \|\mu\mathscr{R}(\mu : \mathscr{A})\mathscr{S}(s) - \mathscr{S}(s)\|$. Then

$$
\begin{aligned}
J &= \left\| \mu \int_0^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \mathscr{S}(t)\mathscr{S}(s)\, d_\alpha t - \mu \int_0^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \mathscr{S}(s)\, d_\alpha t \right\| \\
&= \left\| \mu \int_0^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \mathscr{S}\left((s^\alpha + t^\alpha)^{\frac{1}{\alpha}} \right) d_\alpha t - \mu \int_0^\infty e^{-\mu \frac{s^\alpha}{\alpha}} \mathscr{S}(s)\, d_\alpha t \right\| \\
&= \left\| \mu \int_0^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \left(\mathscr{S}\left((s^\alpha + t^\alpha)^{\frac{1}{\alpha}} \right) - \mathscr{S}(s) \right) d_\alpha t \right\| \\
&\leq \mu \int_0^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \left\| \mathscr{S}\left((s^\alpha + t^\alpha)^{\frac{1}{\alpha}} \right) - \mathscr{S}(s) \right\| d_\alpha t \\
&= \mu \int_0^b e^{-\mu \frac{t^\alpha}{\alpha}} \left\| \mathscr{S}\left((s^\alpha + t^\alpha)^{\frac{1}{\alpha}} \right) - \mathscr{S}(s) \right\| d_\alpha t \\
&\quad + \mu \int_b^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \left\| \mathscr{S}\left((s^\alpha + t^\alpha)^{\frac{1}{\alpha}} \right) - \mathscr{S}(s) \right\| d_\alpha t \\
&\leq \mu \sup_{0 \leq t \leq b} \left\| \mathscr{S}(s^\alpha + t^\alpha)^{\frac{1}{\alpha}} - \mathscr{S}(s) \right\| \int_0^b e^{-\mu \frac{t^\alpha}{\alpha}} d_\alpha t \\
&\quad + \mu \int_b^\infty e^{-\mu \frac{t^\alpha}{\alpha}} \mathscr{M}\left(e^{\omega \frac{(s^\alpha + t^\alpha)}{\alpha}} + e^{\omega \frac{s^\alpha}{\alpha}}\right) d_\alpha t \\
&= \sup_{0 \leq t \leq b} \left\| \mathscr{S}(s^\alpha + t^\alpha)^{\frac{1}{\alpha}} - \mathscr{S}(s) \right\| + \mathscr{M} \mu e^{\omega s^\alpha} \left(\frac{e^{(\omega - \mu)\frac{b^\alpha}{\alpha}}}{\mu - \omega} - \frac{e^{-\mu \frac{b^\alpha}{\alpha}}}{\mu} \right) \\
&\leq \varepsilon + \mathscr{M} \mu e^{\omega s^\alpha} \left(\frac{e^{(\omega - \mu)\frac{b^\alpha}{\alpha}}}{\mu - \omega} - \frac{e^{-\mu \frac{b^\alpha}{\alpha}}}{\mu} \right)
\end{aligned}
$$

which implies $\limsup_{\mu \to \infty} \|\mu \mathscr{R}(\mu : \mathscr{A})\mathscr{S}(s) - \mathscr{S}(s)\| \leq \varepsilon$ for every $b > 0$. Since b is arbitrary we get, $\|\mu \mathscr{R}(\mu : \mathscr{A})\mathscr{S}(s) - \mathscr{S}(s)\| = 0$. Thus $\mathscr{S}(s)$ is compact.

Theorem 5. *Let $\{\mathscr{S}(s)\}_{s \geq 0}$ be a a differentiable c_0-α-semigroup on Y with infinitesimal generator $\mathscr{A}$. Then $\mathscr{S}(s)$ is compact for all $s \in (0, \infty)$ if the following conditions are satisfied:*

1- $\mathscr{R}(\mu_0 : \mathscr{A})$is compact for some $\mu_0 \in \rho(A)$.

2- $\mathscr{S}(s)$ is uniformly continuous.

Proof. Let $\mu_0 = 0 \in \rho(A)$. Define $\mathscr{G}(u)y = \int_0^u \mathscr{S}(s)y\, d_\alpha s = \int_0^u \frac{\mathscr{S}(s)y}{s^{1-\alpha}} ds$. Then $\mathscr{G}(u)y \in \mathscr{D}(\mathscr{A})$ and

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t + \varepsilon t^{1-\alpha}} \frac{\mathscr{S}(s)y}{s^{1-\alpha}} ds &= \lim_{\varepsilon \to 0} \frac{\int_0^{t + \varepsilon t^{1-\alpha}} \frac{\mathscr{S}(s)y}{s^{1-\alpha}} ds - \int_0^t \frac{\mathscr{S}(s)y}{s^{1-\alpha}} ds}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\mathscr{G}(t + \varepsilon t^{1-\alpha})y - \mathscr{G}(t)y}{\varepsilon} \\
&= \mathscr{G}^\alpha(t)y \\
&= \mathscr{S}(t)y.
\end{aligned}
$$

Now:

$$\frac{\mathscr{S}(h+\varepsilon h^{1-\alpha})-\mathscr{S}(h)}{\varepsilon}\int_a^t \mathscr{S}(s)y\,d_\alpha s=\frac{1}{\varepsilon}\left(\begin{array}{c}\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\\ -\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\end{array}\right)$$

$$=\left(\begin{array}{c}\frac{1}{\varepsilon}\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\\ +\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\\ +\int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\\ -\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\end{array}\right)$$

$$=\frac{1}{\varepsilon}\left(\int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u-\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}\mathscr{S}(u)y\,d_\alpha u\right)$$

Since

$$\begin{aligned}\mathscr{G}^\alpha\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y&=v^{1-\alpha}\alpha v^{\alpha-1}\frac{1}{\alpha}(v^\alpha+h^\alpha)^{\frac{1}{\alpha}-1}\mathscr{G}'\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y\\&=(v^\alpha+h^\alpha)^{\frac{1}{\alpha}-1}\mathscr{G}'\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y\\&=\mathscr{G}^\alpha\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y=\mathscr{S}\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y.\end{aligned}$$

As $\varepsilon\to 0$, it follows that

$$\begin{aligned}&\lim_{\in\to 0}\frac{\mathscr{S}(h+\varepsilon h^{1-\alpha})-\mathscr{S}(h)}{\varepsilon}\int_a^t\mathscr{S}(s)y\,d_\alpha s\\&=\mathscr{G}^\alpha\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y\Big|_{v=t}-\mathscr{G}^\alpha\left((v^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y\Big|_{v=a}.\\&=\mathscr{S}\left((t^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y-\mathscr{S}\left((a^\alpha+h^\alpha)^{\frac{1}{\alpha}}\right)y,\end{aligned}$$

as $h\to 0$, we have

$$\mathscr{A}\int_a^t\mathscr{S}(s)y\,d_\alpha s=\mathscr{S}(t)y-\mathscr{S}(a)y.$$

As $a\to 0$, we get

$$\mathscr{A}\mathscr{G}(t)y=\mathscr{A}\int_0^t\mathscr{S}(s)y\,d_\alpha s=\mathscr{S}(t)y-y.$$

Now, since $\mathscr{S}(t)$ is uniformly continuous, then $\mathscr{G}^\alpha(t)$ exists and

$$\mathscr{G}^\alpha(t)=\lim_{\varepsilon\to 0}\frac{\mathscr{G}(t+\varepsilon t^{1-\alpha})y-\mathscr{G}(t)y}{\varepsilon}.$$

Taking $\varepsilon = \frac{1}{n}$, then

$$\begin{aligned}
\mathscr{G}^{\alpha}(t) &= \lim_{h\to 0} \frac{\mathscr{G}(t+ht^{1-\alpha})y - \mathscr{G}(t)y}{h} \\
&= \lim_{n\to\infty} n\left(\mathscr{G}(t+\frac{1}{n}t^{1-\alpha}) - \mathscr{G}(t)\right) \\
&= \lim_{n\to\infty} n\left(\mathscr{R}(0:\mathscr{A})(I-\mathscr{S}(t+\frac{1}{n}t^{1-\alpha})) - \mathscr{R}(0:\mathscr{A})(I-\mathscr{S}(t))\right) \\
&= \lim_{n\to\infty} n\mathscr{R}(0:\mathscr{A})\left(\mathscr{S}(t) - \mathscr{S}(t+\frac{1}{n}t^{1-\alpha})\right).
\end{aligned}$$

Define: $D_n(t) = n\mathscr{R}(0:\mathscr{A})\left(\mathscr{S}(t) - \mathscr{S}(t+\frac{1}{n}t^{1-\alpha})\right)$. Since $\mathscr{R}(0:\mathscr{A})$ is compact, it follows that $D_n(t)$ is compact for all $t > 0$ and all $n \in \mathbb{N}$. Thus $\mathscr{G}^{\alpha}(t) = \mathscr{S}(t)$ is compact $\mathscr{G}^{\alpha}(t) = \mathscr{S}(t)$, for all $t > 0$.

For $\mu_0 > 0$, define $\mathscr{S}_1(t) = e^{-\mu_0 \frac{t^{\alpha}}{\alpha}}\mathscr{S}(t)$. Then if $\mathscr{A}$ is the generator of $\mathscr{S}(t)$, then $\mathscr{A}-\mu_0$ is the generator of $e^{-\mu_0 t^{\alpha}}\mathscr{S}(t)$. So if $\mu_0 \in \rho(\mathscr{A}-\mu_0)$, then $0 \in \rho(\mathscr{A})$. This completes the proof.

3 Conclusion

The theory semigroups of linear operators and their infinitesimal generators plays an important role in the applications in partial differential equations, Markov processes, and ergodic theory, In this paper if Y is a Banach space and $\{\mathscr{S}(s)\}_{s\geq 0}$ is a one paprameter conformable semigroup of bounded linear operators be a one parameter conformable semigroup of bounded linear operators on Y, we give an answer to the question: "As $S(s)$ is a bounded linear operator under what conditions a strongly continuous conformable α-semigroup is compact". Further we show that under certain conditions the resolvent operator $R(\mu : A)$ of the generator of $\{\mathscr{S}(s)\}_{s\geq 0}$ can be compact.

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