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Dynamical Analysis and Simulation of a General Two-Strain SEIR Epidemic Model with Caputo Fractional Derivative and Vaccination

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Abstract: In this study, the dynamical behavior of a two-strain SEIR epidemic model with fractional order of differentiation and having general non-linear incidence rates. The mathematical representation of the epidemic model is given and the constant solution is evaluated according to the reproduction number of the two strains. The boundedness and uniqueness of the solution are studied. The stability of the model has been investigated by examining the stability of each constant solution of the system. Constructing appropriate Lyapunov functions helps to investigate the global stability of the system's constant solutions. A new numerical technique based on approximating the Caputo fractional order derivative by difference schemes of a high-order approximation of the L_2 type. This scheme is called "The Non-uniform L_2 Fractional differentiation numerical scheme (NU L_2 FDNS)" which is used to verify the analytically proven results and also clarify the effect of system transactions on the control of the disease. Especially the vaccination rate controls the disease very well.

Keywords: Stability analysis, Lyapunov functions, SEIR model, Caputo fractional derivative.

1 Introduction

Sometimes these days, infectious diseases surprise us with great challenges, despite progress in the field of treatment and prevention. An example of this is what happened in the Corona pandemic, which the world was unable to confront for many months. Knowing how the infection is transmitted, the duration of infection and the rate of contact between individuals, and the dynamics of the spread of infection are essential to control the spread of this infection. Therefore, many researchers were interested in studying mathematical models expressing the spread of diseases in general and infectious ones in particular, [1,2,3,4]. The COVID-19 pandemic causes a frightening health and economic shock that affected all countries and affected their health, economic, and political conditions. Therefore, researching the causes of disease and finding appropriate treatments, vaccinations, and ways to reduce its spread were preoccupying the entire world. Dias and Ratnayaka [5] studied the disease transmission, identified symptoms of the disease, and diagnosed the injured. The effect of memory on the dynamics of epidemiology using Caputo's concept is considered by many researchers. For example, Saedian et al. [6], interested in studying the memory effect on the behaviour of the epidemic models using Caputo's concept. Also, Hikal et al [7] presented a fractional-order derivative of the COVID-19 model with a delay in implementing the quarantine strategy.

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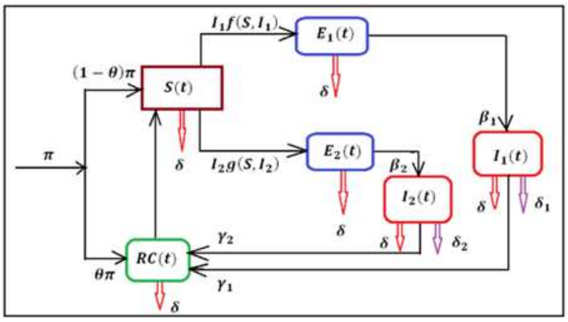


Fig. 1: Flow diagram of the General Two-Strain SEIR model.

With the occurrence of mutations in many pathogens of some diseases such as influenza [8], tuberculosis [9], and HIV/AIDS [10,11,12]. This phenomenon may lead to the existence of two or more strains of pathogens of a disease. Therefore, the mathematical expression of diseases caused by different strains must express the multiplicity of these strains. Although these models that express the multiplicity of disease-causing strains are more complex, their study is useful in understanding the conditions that allow coexistence between all strains. Various works on the multi-strain SEIR model are presented. Amine [13], and Meskaf et al [14] used Lyapunov functions to study the stability of a two-strain epidemic model with non-monotone incidence rates. And Bentaleb and Amine [15] studied a two-strain SEIR epidemic model considering both bilinear and non-monotone incidence rates. Also Khyar and Allali [16] constructed suitable Lyapunov functions to prove the global stability of the equilibrium points of the two-strain SEIR epidemic model with general incidence rates.

In this work, the dynamic of time-fractional, nonlinear general incidence rates epidemic model with two-strain SEIR is considered. Our model has the dynamic relation between the system variables illustrated in figure 1 and given by:

$$\begin{aligned}
D_t^{\alpha} S(t) &= (1-\theta)\pi - I_1 f(S,I_1) - I_2 g(S,I_2) - \delta S + \theta_1 RC,\\
D_t^{\alpha}\, E_1(t) &= I_1 f(S,I_1) - (\beta_1+\delta)\; E_1,\\
D_t^{\alpha}\, E_2(t) &= I_2 g(S,I_2) - (\beta_2+\delta)\; E_2,\\
D_t^{\alpha}\, I_1(t) &= \beta_1 E_1 - (\gamma_1+\delta_1+\delta)\; I_1,\\
D_t^{\alpha}\, I_2(t) &= \beta_2\, E_2 - (\gamma_2+\delta_2+\delta)\; I_2,\\
D_t^{\alpha} RC(t) &= \gamma_1 I_1 + \;\gamma_2\, I_2 + \theta\pi - (\theta_1+\delta) RC, \quad 0<\alpha\le 1,
\end{aligned} \tag{1}$$

where D_t^{α} is the fractional operator in the Caputo sense. Figure 1 gives the relations between the six different categories of the system that are represented in equations in (1). The population is divided into the following, the number of susceptible individuals class is S(t), the number of latent individuals of two strains is $E_1(t)$ and $E_2(t)$, the number of infected individuals classes $I_1(t)$ and $I_2(t)$ and the number of recovered individuals is $RC(t)$. The system parameters all are positive. The parameter θ is the ratio of vaccinated individuals, π is the recruitment rate, δ is the normal rate of mortality, θ_1 is the rate from recovered to susceptible, β_1 and β_2 are the latency rates, γ_1, and γ_2 are the transfer rates from infected classes to recovered class, δ_1 and δ_2 are death rates due to the disease, for strain 1 and strain 2 respectively. For the initial conditions:

$$S(0)\ge 0, E_1(0)\ge 0, E_2(0)\ge 0, I_1(0)\ge 0, I_2(0)\ge 0 \textit{ and } RC(0)\ge 0, \tag{2}$$

and $X=[S(t),E_1(t),E_2(t),I_1(t),I_2(t),RC(t)]\in R^{+}$.

The general incidence rate for strain 1 is $f(S,I_1)$ and for strain 2 is $g(S,I_2)$. Assume that these general incidence rates for the two strains satisfy the following conditions:

$$f(0,I_1)=g(0,I_2)=0, \text{ for } I_i\ge 0, i\in 1,2, \tag{3}$$

$$\frac{\partial f(S,I_1)}{\partial S}>0, \frac{\partial g(S,I_2)}{\partial S}>0, S>0 \text{ and } I_i\ge 0, i\in 1,2, \tag{4}$$

$$\frac{\partial f(S,I_1)}{\partial I_1}\le 0, \frac{\partial g(S,I_2)}{\partial I_2}\le 0, S>0 \text{ and } I_i\ge 0, i\in 1,2. \tag{5}$$

Assume the fractional order differential equations defined by the system

$$D_t^{\alpha} X(t) = f(X), \quad \alpha\in(0,1], \;\; X\in R^{n}. \tag{6}$$

Definition 1.[17, 18, 19] *The Caputo fractional derivative of a function $u(t)$ of order α is given by*

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\eta)^{-\alpha} u^{(1)}(t) d\eta , \qquad 0 < \alpha \leq 1. \tag{7}$$

The stability of behavior of the constant solutions that satisfy $f(x^*) = 0$ of the system (6) can be generalized to the system itself. Where the constant solution x^* is locally asymptotically stable whenever each eigenvalue of the Jacobian matrix of system (6) calculated at x^* satisfies the inequality [20]

$$|arg\lambda| > \frac{\alpha\pi}{2}. \tag{8}$$

The Lyapunov direct method may be considered the most efficient tool for studying the stability and boundedness of solutions of nonlinear ordinary differential equations. The theory of this method is based on the use of positive definite functions that are non-increasing along the solutions of differential equations under consideration. However, we will recall some definitions and theorems which were given by Wilson [21] in the following: Assuming that the vector function $f(X, t)$ in the R. H. S. of Eq. (6) is continuous and lipschitizian on a region $\rho \times [t_0, \infty)$ in $R^n \times R$ and the origin is an isolated critical point of Eq. (6) in ρ. Let $G \subset \rho$ be a neighborhood of the origin.

Definition 2. *A real-valued function $W : R^n \to R$ is said to be positive (negative) definite on G if $W(x) > 0\,(W(x) < 0)$ for all $x \neq 0$ in G and $W(0) = 0$. A function $L(X,t)$ defined on a cylinder $G \times [t_1, \infty)$ is called positive (negative) definite if $L(0,t) = 0$ for $t \geq t_1$ and there exists a positive (a negative) definite function W on G such that $W(x) \leq L(X,t)$ $(W(x) \geq L(X,t)$ for all (x,t) in $G \times [t_1, \infty)$.*

Definition 3. *A continuous real-valued function $L(X,t)$ is called a Lyapunov function for Eq. (6) at the origin (when the order of the derivative is unity) if:*

1. there is a cylinder $G \times [t_0, \infty)$, on which $L(X,t)$ is positive definite.
2. when $X(t)$ is a solution of (6) with $X(t_0)$ in G, then $L(X(t), t)$ is non-increasing in t for $t \geq t_1 \geq t_0$.

Definition 4. *A positive definite function $L(X,t)$ which is defined on a cylinder $G \times [t_0, \infty)$, is called decrescent if there exists a positive definite function U in G such that:*

$$L(X,t) \leq U(X) \;\forall\; (X,t) \text{ in } G \times [t_0, \infty).$$

Theorem 1. *If the differential equation (6) has a decrescent Lyapunov function $L(X,t)$ at the origin with DV (a derivative of V along the solution of Eq.(6)) negative definite, then the origin is uniformly asymptotically stable.*

Also, the following lemma will be considered in our analysis [16].

Lemma 1.[16] *Let $x_1, x_2, \ldots, x_n$ be n positive numbers. Then their arithmetic mean is greater than or equal to their geometric mean.*

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1\, x_2\, \ldots x_n}. \tag{9}$$

2 Constant Solutions of the Two Strains Epidemic Model and Boundedness of Solution.

2.1 *Constant solutions of the two strains epidemic model.*

Assume that the considerable domain for solutions of system (1) is defined by $\Gamma = (R^+)^6$ and the constant solutions can given by

$$D_t^{\alpha} S = D_t^{\alpha} E_1 = D_t^{\alpha} E_2 = D_t^{\alpha} I_1 = D_t^{\alpha} I_2 = D_t^{\alpha} RC = 0. \tag{10}$$

For simplicity, let

$$a = \beta_1 + \delta, \quad b = \beta_2 + \delta, \quad c = \gamma_1 + \delta_1 + \delta, \quad d = \gamma_2 + \delta_2 + \delta \quad \text{and} \quad e = \theta_1 + \delta, \tag{11}$$

and solving the six nonlinear equations in Eq. (10), at the constant solutions we have:

$$E_1 = \frac{c}{\beta_1} I_1, \qquad E_2 = \frac{d}{\beta_2} I_2, \qquad RC = \frac{1}{e}\left(\theta\pi + \gamma_1 I_1 + \gamma_2 I_2\right),$$
$$S = \frac{1}{\delta}\left[\pi\left(1-\theta+\frac{\theta\ \theta_1}{e}\right) - I_1\left(\frac{a\,c}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right) - I_2\left(\frac{b\,d}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right)\right]. \tag{12}$$

Hence we have the following:

$$I_1 = 0 \ \ or \ \ f(S, I_1) = \frac{ac}{\beta_1} \quad \text{and} \quad I_2 = 0 \ \ or \ g(S, I_2) = \frac{bd}{\beta_2}. \tag{13}$$

Then we get four constant solutions, they are as shown in the following:

$$P_0(S_0,\ 0,\ 0,\ 0,\ 0,\ RC_0), \quad S_0 = \frac{\pi}{\delta}\left(1-\theta+\frac{\theta\ \theta_1}{e}\right), \quad RC_0 = \frac{\theta}{e}\pi, \tag{14}$$

$$P_1(S_1, E_{11}, E_{21}, I_{11}, I_{21}, C_1), \quad S_1 = \left[S_0 - \frac{1}{\delta}\left(\frac{ac}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right) I_{11}\right], \ E_{11} = \frac{c}{\beta_1} I_{11}$$
$$f(S_1, I_{11}) = \frac{ac}{\beta_1}, \quad RC_1 = \frac{1}{e}\left[\gamma_1\ I_{11} + \theta\ \pi\right], \tag{15}$$

$$P_2(S_2, E_{12}, E_{22}, I_{12}, I_{22}, C_2), \quad S_2 = \left[S_0 - \frac{1}{\delta}\left(\frac{bd}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right) I_{22}\right], \ E_{22} = \frac{d}{\beta_2} I_{22},$$
$$g(S_2, I_{22}) = \frac{bd}{\beta_2}, \quad E_{12} = I_{12} = 0, \quad RC_2 = \frac{1}{e}\left[\gamma_2\ I_{22} + \theta\ \pi\right], \tag{16}$$

$$P^*(S^*, E_1^*, E_2^*, I_1^*, I_2^*, RC^*), S^* = \left[S_0 - \frac{1}{\delta}\left(\frac{ac}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right) I_1^* - \frac{1}{\delta}\left(\frac{bd}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right) I_2^*\right],$$
$$E_1^* = \frac{c}{\beta_1} I_1^*, \quad E_2^* = \frac{d}{\beta_2} I_2^*, \quad f(S^*, I_1^*) = \frac{ac}{\beta_1}, \quad g(S^*, I_2^*) = \frac{cd}{\beta_2}, \quad RC^* = \frac{1}{e}\left[\gamma_1\ I_1^* + \gamma_2\ I_2^* + \theta\ \pi\right]. \tag{17}$$

2.2 *The reproduction number R_0*

Following [20], we can get the reproduction number R_0 for the Eqns (1) by rewriting it as:

$$D_t^{\alpha}\ \phi(t) = \psi(\phi) - \eta(\phi), \quad \phi = (E_1, E_2, I_1, I_2, RC, S)^T, \tag{18}$$

and

$$\psi(\phi) = \begin{pmatrix} I_1 f(S, I_1) \\ I_2 g(S, I_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \eta(\phi) = \begin{pmatrix} a\ E_1 \\ b\ E_2 \\ cI_1 - \beta_1 E_1 \\ dI_2 - \beta_2 E_2 \\ -\gamma_1 I_1 - \gamma_2\ I_2 - \theta\pi + (\theta_1+\delta) RC \\ I_1 f(S, I_1) + I_2 g(S, I_2) - (1-\theta)\pi + \delta\ S - \theta_1 RC \end{pmatrix}, \tag{19}$$

$$J_\psi = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \ J_\eta = \begin{pmatrix} V & 0 \\ F_1 & F_2 \end{pmatrix} \ \text{where} F = \begin{pmatrix} 0\ 0 & f(S_0,\ 0) & 0 \\ 0\ 0 & 0 & g(S_0,\ 0) \\ 0\ 0 & 0 & 0 \\ 0\ 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$V = \begin{pmatrix} a & 0 & 0\ 0 \\ 0 & b & 0\ 0 \\ -\beta_1 & 0 & c\ 0 \\ 0 & -\beta_2 & 0\ d \end{pmatrix}, V^{-1} = \frac{1}{abcd}\begin{pmatrix} acd & 0 & 0 & 0 \\ 0 & acd & 0 & 0 \\ bd\beta_1 & 0 & abd & 0 \\ 0 & ac\beta_2 & 0 & abc \end{pmatrix}, \tag{21}$$

$$FV^{-1} = \frac{1}{abcd}\begin{pmatrix} \frac{\beta_1 f(S_0,\,0)}{ac} & 0 & \frac{f(S_0,\,0)}{c} & 0 \\ 0 & \frac{\beta_2\, g(S_0,\,0)}{bd} & 0 & \frac{g(S_0,\,0)}{d} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

Hence, the reproduction number has the form:

$$R_0 = \max\ R_0^1, R_0^2\ , \quad R_0^1 = \beta_1 \frac{f\,(S_0,0)}{ac}, \quad R_0^2 = \beta_2 \frac{g\,(S_0,0)}{bd}. \tag{23}$$

According to the representation of the reproduction number R_0 we can summarize the conditions for the existence of the constant solution of system (1) as:

Theorem 2. *The model (1) has the free-disease constant solution $P_0(\,S_0,\ 0,\ 0,\ 0,\ 0,\ RC_0)$ defined by (14) only when $R_0 \leq 1$. When $R_0 > 1$, the constant solutions of the system are P_0 in addition to one of the following constant solutions:*

(i) The endemic solution of strain 1, $P_1\,(\,S_1, E_{11},\ E_{21},\ I_{11},\ I_{21},\ C_1)$ defined by (15) exists if $R_0 > 1 > R_0^2$,
(ii) The endemic solution of strain 2, $P_2\,(S_2, E_{12},\ E_{22},\ I_{12},\ I_{22}, C_2)$, defined by (16) exists if $R_0 > 1 > R_0^1$,
(ii) The endemic solution of the two strains, $P^*\,(S^*, E_1^*, E_2^*, I_1^*, I_2^*, RC^*)$ defined by (17) exists if $R_0^1 > 1$ and, $R_0^2 > 1$.

Proof. Solving equation (10) and using the relations in (13), The first constant solution is when $I_1 = I_2 = 0$, the free disease P_0 defined in (14).

(i) The second solution is when $I_{21} = E_{21} = 0$ and $f\,(S_1,\ I_{11}) = \frac{ac}{\beta_1}$. And $I_{11} = \frac{\delta}{\pi}\ \frac{1}{\left(\frac{ac}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right)}\ (S_0 -\ S_1)$, it is clear that a positive value for I_{11} is only when $S_1 \in [0, S_0]$.
Let $H_1\,(S)$ be a function defined for $S \in [0,\infty]$ and $I_1 = \frac{\delta}{\pi}\ \frac{1}{\left(\frac{ac}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right)}\ (S_0 - S)$

$$H_1\,(S) = f\,(S,\ \ I_1) - \frac{ac}{\beta_1}, \tag{24}$$

$$\frac{\partial\ H_1(S)}{\partial\ S} = \frac{\partial\ f\,(S,\ \ I_1)}{\partial\ S} + \frac{\partial\ f\,(S,\ \ I_1)}{\partial\ I_1}\ (-\ \frac{\delta}{\pi}\ \frac{1}{\left(\frac{ac}{\beta_1} - \frac{\theta_1\ \gamma_1}{e}\right)}\ )\ , \tag{25}$$

From conditions (4) and (5), we have

$$\frac{\partial\ H_1(S)}{\partial\ S} \geq 0, \tag{26}$$

and we have:

$$H_1\,(0) = f\,(0,\ \ I_1) - \frac{ac}{\beta_1} = -\frac{ac}{\beta_1} < 0,\ \ H_1\,(S_1) = f\,(S_1,\ \ I_{11}) - \frac{ac}{\beta_1} = 0, \tag{27}$$

and $H_1\,(S_0) = f\,(S_0,0) - \frac{ac}{\beta_1} = \frac{ac}{\beta_1}\left(R_0^1 - 1\right)$. So from (26) and (27), $H_1\,(S_0) > 0$ which is true if $R_0^1 > 1$.

(ii) The third solution is when $I_{12} = E_{12} = 0$ and $g\,(S_2,\ I_{21}) = \frac{b\,d}{\beta_2}$. And $I_{22} = \frac{\delta}{\pi}\ \frac{1}{\left(\frac{bd}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right)}\ (S_0 -\ S_2)$, it is clear that a positive value for I_{22} is only when $S_2 \in [0, S_0]$.
Let $H_2\,(S)$ be a function defined for $S \in [0,\infty]$ and $I_2 = \frac{\delta}{\pi}\ \frac{1}{\left(\frac{bd}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right)}\ (S_0 - S)$,

$$H_2\,(S) = g\,(S,\ \ I_2) - \frac{bd}{\beta_2} \tag{28}$$

$$\frac{\partial\ H_2(S)}{\partial\ S} = \frac{\partial\ g\,(S,\ \ I_2)}{\partial\ S} + \frac{\partial\ g\,(S,\ \ I_2)}{\partial\ I_2}\ (-\ \frac{\delta}{\pi}\ \frac{1}{\left(\frac{bd}{\beta_2} - \frac{\theta_1\ \gamma_2}{e}\right)}\ ). \tag{29}$$

From conditions (4) and (5), we have

$$\frac{\partial\ H_2(S)}{\partial\ S} \geq 0, \tag{30}$$

and we have

$$H_2(0) = g(0,\ I_2) - \frac{bd}{\beta_2} = -\frac{bd}{\beta_2} < 0,\ \ H_2(S_2) = g(S_2,\ I_{22}) - \frac{bd}{\beta_2} = 0, \tag{31}$$

and $H_2(S_0) = g(S_0, 0) - \frac{bd}{\beta_2} = \frac{bd}{\beta_2}\left(R_0^2 - 1\right)$. So from (30) and (31), $H_2(S_0) > 0$ which is true if $R_0^2 > 1$.

(iii) The fourth solution is when $f(S^*, I_1^*) = \frac{ac}{\beta_1}$ and $g(S^*, I_2^*) = \frac{b\,d}{\beta_2}$, $S^* = \left[S_0 - \frac{1}{\delta}\left(\frac{ac}{\beta_1} - \frac{\theta_1\,\gamma_1}{e}\right)I_1^* - \frac{1}{\delta}\left(\frac{bd}{\beta_2} - \frac{\theta_1\,\gamma_2}{e}\right)I_2^*\right]$, and $S^* \in [0, S_0]$. Similarly defined $H^*(S)$ be a function defined for $S \in [0, \infty]$ and $S = \left[S_0 - \frac{1}{\delta}\left(\frac{ac}{\beta_1} - \frac{\theta_1\,\gamma_1}{e}\right)I_1 - \frac{1}{\delta}\left(\frac{bd}{\beta_2} - \frac{\theta_1\,\gamma_2}{e}\right)I_2\right]$.

$$\frac{\partial H^*(S)}{\partial\, S} = \frac{\partial f(S, I_1)}{\partial\, S} + \frac{\partial f(S, I_1)}{\partial\, I_1}\left(-\frac{\delta}{\pi}\frac{1}{\left(\frac{ac}{\beta_1} - \frac{\theta_1\gamma_1}{e}\right)}\right) + \frac{\partial g(S, I_2)}{\partial S} + \frac{\partial g(S, I_2)}{\partial I_2}\left(-\frac{\delta}{\pi}\frac{1}{\left(\frac{bd}{\beta_2} - \frac{\theta_1\gamma_2}{e}\right)}\right). \tag{32}$$

From conditions (4) and (5), we have

$$\frac{\partial\, H^*(S)}{\partial\, S} \geq 0. \tag{33}$$

And we have

$$H^*(0) = f(0, I_1) - \frac{ac}{\beta_1} + g(0, I_2) - \frac{bd}{\beta_2} = -\left(\frac{ac}{\beta_1} + \frac{bd}{\beta_2}\right) < 0,\ H^*(S^*) = f(S^*, I_1^*) - \frac{ac}{\beta_1} + g(S^*, I_2^*) - \frac{bd}{\beta_2} = 0, \tag{34}$$

$H^*(S_0) = f(S_0, 0) - \frac{ac}{\beta_1} + g(S_0, 0) - \frac{bd}{\beta_2} = \frac{ac}{\beta_1}\left(R_0^2 - 1\right) + \frac{bd}{\beta_2}\left(R_0^2 - 1\right) > 0$. And from (33) and (34) $H^*(S_0) > 0$ when $R_0^1 > 1$, and $R_0^2 > 1$.

2.3 *Limited solutions to the pandemic system*

Consider the total population $N(t)$, which has the form

$$N(t) = S(t) + E_1(t) + E_2(t) + I_1(t) + I_2(t) + RC(t), \tag{35}$$

and the initial value for the total population is:

$$N^* = S(0) + E_1(0) + E_2(0) + I_1(0) + I_2(0) + RC(0). \tag{36}$$

The fractional-order derivative of the function $N(t)$ is defined by:

$$D_t^{\alpha}N(t) = D_t^{\alpha}S(t) + D_t^{\alpha}E_1(t) + D_t^{\alpha}E_2(t) + D_t^{\alpha}I_1(t) + D_t^{\alpha}I_2(t) + D_t^{\alpha}RC(t). \tag{37}$$

Using equation Eq. (1), we have:

$$D_t^{\alpha}N(t) + \delta N(t) = \pi - (\delta_1 I_1(t) + \delta_2 I_2(t)), \tag{38}$$

and consequently since the term $(\delta_1 I_1(t) + \delta_2 I_2(t)\ )$ is always positive, then we can write:

$$D_t^{\alpha}N(t) + \delta\, N(t) \leq \pi. \tag{39}$$

Applying Laplace transform on (39), and using the initial condition (36), we have:

$$N(s) \leq \frac{\pi}{s}\frac{1}{s^{\alpha} + \delta} + \frac{N^*}{s^{\alpha} + \delta}, \tag{40}$$

using inverse Laplace transform on Eq.(40), one have

$$N(t) \leq N^*\, t^{\alpha-1}\, E_{\alpha,\alpha}(-\delta\, t^{\alpha}) + \pi\int_0^t (t-\tau)^{\alpha-1}\, E_{\alpha,\alpha}\left(-\delta\,(t-\tau)^{\alpha}\right)\, d\tau. \tag{41}$$

Applying the definition and properties of the Mittag-Leffler function, the function $N(t)$ satisfies the inequality:

$$N(t) \leq N^*\, t^{\alpha-1}\, E_{\alpha,\alpha}(-\delta\, t^{\alpha}) + \pi t^{\alpha}\, E_{\alpha,\alpha-1}(-\delta\, t^{\alpha}). \tag{42}$$

Hence, $N(t)$ is bounded for $t \geq 0$. And we can conclude the above result in the following theorem:

Theorem 3. *Any solution of system (1) is uniformly bounded whenever it has bounded initial conditions.*

3 Stability analysis of the two-strain epidemic model

In this section, the behavior of the constant solutions of system (1) is studied.

Stability of the free disease constant solution P_0:

Let $f(S,I_1)$ satisfies the following condition:

$$\left(\frac{RC}{RC_0}-1\right)\left(1-\frac{f(S_0,0)}{f(S,0)}\right)\leq 0. \tag{43}$$

Theorem 4. *The free equilibrium point P_0 of the system (1) is globally asymptotically stable when $R_0 \leq 1$.*

*Proof.*Consider that we have a function $L_0(X)$ where$X=(S,E_1,E_2,I_1,I_2,RC)$

$$L_0(X)=S-S_0-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du+E_1+E_2+\frac{a}{\beta_1}I_1+\frac{b}{\beta_2}I_2. \tag{44}$$

It is sufficient to prove that $L_0(X)$ is a positive definite function if $h(S)=S-S_0-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du$ is a positive definite. From conditions (3), (4), $f(S,\ I_1)$ is a positive increasing function with respect to S. It is clear that $h(S_0)=0$ and for $S>S_0$, we have $\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du<\int_{S_0}^{S}du=S-S_0$ then, $-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du>-(S-S_0)$ and

$$h(S)=S-S_0-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du>0,\ \forall\ \ S>S_0, \tag{45}$$

also, for $S<S_0$, $-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du=\int_{S}^{S_0}\frac{f(S_0,0)}{f(u,0)}\,du>S_0-S$ and hence,

$$h(S)=S-S_0-\int_{S_0}^{S}\frac{f(S_0,0)}{f(u,0)}\,du>0,\ \forall\ S<S_0, \tag{46}$$

from (45) and (46), $h(S)$ and consequently $L_0(X)$ are positive definite functions and by definition, $L_0(X)$ is a Lyapunov function. The time derivative of $L_0(X)$ is:

$$L_0^{\bullet}(S,\ E_1,E_2,I_1,I_2,\ RC)=S^{\bullet}-\frac{f(S_0,0)}{f(S,0)}\,S^{\bullet}+E_1^{\bullet}+E_2^{\bullet}+\frac{a}{\beta_1}I_1^{\bullet}+\frac{b}{\beta_2}\,I_2^{\bullet}. \tag{47}$$

Substitute from Eq. (1) in Eq. (47), then using Eq. (14) and the equation

$$(1-\theta)\pi=\delta S_0-\theta_1 RC_0.$$

We have

$$L_0^{\bullet}(S,\ E_1,E_2,I_1,I_2,\ C)=\delta S_0\left(1-\frac{S}{S_0}\right)\left(1-\frac{f(S_0,0)}{f(S,0)}\right)+\theta C_0\left(\frac{RC}{RC_0}-1\right)\left(1-\frac{f(S_0,0)}{f(S,0)}\right)$$
$$+\frac{ac}{\beta_1}\,I_1\,\left(R_0^1\,\frac{f(S,I_1)}{f(S,0)}-1\right)+\frac{bd}{\beta_2}\,I_2\,\left(R_0^2\,\frac{f(S_0,0)}{f(S,0)}\,\frac{g(S,I_2)}{g(S_0,0)}-1\right). \tag{48}$$

From condition (4), we have: $\frac{f(S,I_1)}{f(S,0)}\leq 1$

$$L_0^{\bullet}(S,E_1,E_2,I_1,I_2,C)\leq\delta S_0\left(1-\frac{S}{S_0}\right)\left(1-\frac{f(S_0,0)}{f(S,0)}\right)+\theta C_0\left(\frac{RC}{RC_0}-1\right)\left(1-\frac{f(S_0,0)}{f(S,0)}\right)+\frac{ac}{\beta_1}\,I_1\,\left(R_0^1\,-1\right)+$$
$$+\frac{bd}{\beta_2}\,I_2\,\left(R_0^2\,\frac{f(S_0,0)}{f(S,0)}\,\frac{g(S,I_2)}{g(S_0,0)}-1\right). \tag{49}$$

We have from condition (43), that the second term in (49) is not positive. Also, the third term is always not positive where $R_0^1\leq 1$. Now we have two cases that need to be considered:

Case 1 For $S \leq S_0$ and using condition (4), $\frac{f(S_0,0)}{f(S,0)} > 1$, the first term in the inequality (49) is not positive. In addition to $\frac{g(S,I_2)}{g(S_0,0)} \leq 1$, so $R_0^2 \frac{f(S_0,0)}{f(S,0)} \frac{g(S,I_2)}{g(S_0,0)} - 1 \leq R_0^2 \frac{f(S_0,0)}{f(S,0)} - 1$, and since $R_0^2 \leq \frac{f(S,0)}{f(S_0,0)} \leq 1$, then $R_0^2 \frac{f(S_0,0)}{f(S,0)} \leq 1$ and now the $R_0^2 \frac{f(S_0,0)}{f(S,0)} - 1 \leq 0$.

Case 2 For $S > S_0$ and by condition (4), $\frac{f(S_0,0)}{f(S,0)} \leq 1$, the first two terms are negative and the last term satisfies that $R_0^2 \frac{f(S_0,0)}{f(S,0)} \frac{g(S,I_2)}{g(S_0,0)} - 1 \leq R_0^2 \frac{g(S,I_2)}{g(S_0,0)} - 1$ and since $R_0^2 \leq \frac{g(S_0,0)}{g(S,I_2)} \leq 1$. So $R_0^2 \frac{g(S,I_2)}{g(S_0,0)} - 1 \leq 0$.

So we conclude that when $R_0 \leq 1 \left(i.e. R_0^2 \leq 1 \; and \; R_0^2 \leq 1\right)$.

$$L_0^{\bullet}(S,E_1,E_2,I_1,I_2,RC) \leq 0. \tag{50}$$

Stability of the endemic equilibrium point for strain 1 P_1:

Let we assume that

$$\left(\frac{f(S,I_{11})}{f(S,I_1)} - \frac{I_1}{I_{11}}\right)\left(1 - \frac{f(S,I_1)}{f(S,I_{11})}\right) \leq 0 and \left(\frac{RC}{RC_1} - 1\right)\left(1 - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right) \leq 0. \tag{51}$$

Theorem 5. *The equilibrium point P_1 is asymptotically stable when $R_0^2 \leq 1 < R_0^1$.*

Proof. Consider a function $L_1(x)$ where:

$$L_1(x) = S - S_1 - \int_{S_1}^{S} \frac{f(S_1,I_{11})}{f(u,I_{11})} du + E_{11}\left[\frac{E_1}{E_{11}} - \ln\frac{E_1}{E_{11}} - 1\right] + E_2 + \frac{a}{\beta_1} I_{11}\left[\frac{I_1}{I_{11}} - \ln\frac{I_1}{I_{11}} - 1\right] + \frac{b}{\beta_2} I_2. \tag{52}$$

It is sufficient to prove that $L_1(X)$ is a positive definite function if $h_1(S) = S - S_1 - \int_{S_1}^{S} \frac{f(S_1,I_{11})}{f(u,I_{11})} du$ and $h_{11}(x) = x - \ln x - 1$ are positive definite functions. In a similar way as in theorem 3.1, $h_1(x)$ can be proved to be a positive definite function. Now $h_{11}(1) = 0$, and it has a minimum value at x=1 where the second derivative of $h_{11}(x)$ is always positive and this minimum value is zero. Then $h_{11}(x) > 0, \; \forall x \neq 1$. And $L_1(X)$ is a Lyapunov function.

The time derivative of the Lyapunov function $L_1(x)$ is:

$$L_1^{\bullet}(x) = S^{\bullet} - \frac{f(S_1,I_{11})}{f(S,I_{11})} S^{\bullet} + E_{11}\left[\frac{E_1^{\bullet}}{E_{11}} - \frac{E_1^{\bullet}}{E_1}\right] + E_2^{\bullet} + \frac{a}{\beta_1} I_{11}\left[\frac{I_1^{\bullet}}{I_{11}} - \frac{I_1^{\bullet}}{I_1}\right] + \frac{b}{\beta_2} I_2^{\bullet}. \tag{53}$$

Substitute from equation (1) in (53) and then use (15) and the equation

$$(1-\theta)\pi = I_{11} f(S_1,I_{11}) + \delta S_1 - \theta_1 RC_1.$$

Then, we have

$$L_1^{\bullet}(x) = \delta S_1\left(1 - \frac{S}{S_1}\right)\left(1 - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right) + \theta_1 RC_1\left(\frac{RC}{RC_1} - 1\right)\left(1 - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right) + \frac{bd}{\beta_2} I_2\left[R_0^2 \frac{f(S_1,I_{11})}{f(S,I_{11})} \frac{g(S,I_2)}{g(S_0,0)} - 1\right]$$
$$+ aE_{11}\left(\frac{f(S,I_{11})}{f(S,I_1)} - \frac{I_1}{I_{11}}\right)\left(1 - \frac{f(S,I_1)}{f(S,I_{11})}\right). \tag{54}$$

Since from the conditions (4), and (5) we have $\frac{g(S,I_2)}{g(S_0,0)} \leq 1$

$$L_1^{\bullet}(x) \leq \delta S_1\left(1 - \frac{S}{S_1}\right)\left(1 - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right) + \theta_1 RC_1\left(\frac{RC}{RC_1} - 1\right)\left(1 - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right) + \frac{bd}{\beta_2} I_2\left[R_0^2 \frac{f(S_1,I_{11})}{f(S,I_{11})} - 1\right]$$
$$+ aE_{11}\left(\frac{f(S,I_{11})}{f(S,I_1)} - \frac{I_1}{I_{11}}\right)\left(1 - \frac{f(S,I_1)}{f(S,I_{11})}\right) + aE_{11}\left(4 - \frac{f(S,I_{11})}{f(S,I_1)} - \frac{I_1 f(S,I_1)}{a E_1} - \frac{E_1}{E_{11}} \frac{I_{11}}{I_1} - \frac{f(S_1,I_{11})}{f(S,I_{11})}\right). \tag{55}$$

By assumption (50), the second and fourth terms in the inequality (55) are not positive. And using lemma 1.1, the fifth term also satisfies that:

$$\left(4 - \frac{f(S_1,I_{11})}{f(S,I_{11})} - \frac{I_1 f(S,I_1)}{a E_1} - \frac{E_1}{E_{11}} \frac{I_{11}}{I_1} - \frac{f(S,I_{11})}{f(S,I_1)}\right) \leq 0.$$

Now two cases are considered:
When $S \leq S_1$, applying condition (4), $\frac{f(S_1,I_{11})}{f(S,I_{11})} > 1$ we have the first and third terms in the inequality (55) are not positive. In addition, by condition (4), $R_0^2 \leq \frac{f(S,I_{11})}{f(S_1,I_{11})} \leq 1$, hence $R_0^2 \leq \frac{f(S,I_{11})}{f(S_1,I_{11})}$, then $R_0^2\,\frac{f(S_1,I_{11})}{f(S,I_{11})} \leq 1$ and now the $R_0^2\,\frac{f(S_1,I_{11})}{f(S,I_{11})} - 1 \leq 0$. When $S > S_1$ and by condition (4), $\frac{f(S_1,I_{11})}{f(S,I_{11})} < 1$ and so the first and third terms in the inequality (55) are not positive. So we conclude that when $R_0^2 \leq 1 < R_0^1$, $L_1^{\bullet}(S,\ E_1,E_2,I_1,I_2,\ RC) \leq 0$.

Stability of the endemic equilibrium point for strain 2 P_2:

Let we assume that

$$\left(\frac{g(S,I_{22})}{g(S,I_2)} - \frac{I_2}{I_{22}}\right)\left(1 - \frac{g(S,I_2)}{g(S,I_{22})}\right) \leq 0 \; and \left(\frac{RC}{RC_2} - 1\right)\left(1 - \frac{g(S_2,I_{22})}{g(S,I_{22})}\right) \leq 0. \tag{56}$$

Theorem 6. *The equilibrium point P_2 is asymptotically stable when $R_0^1 \leq 1 < R_0^2$.*

Proof. Consider a function $L_2(x)$ where:

$$L_2(x) = S - S_2 - \int_{S_2}^{S} \frac{g(S_2,I_{22})}{g(u,I_{22})} du + E_{22}\left[\frac{E_2}{E_{22}} - \ln\frac{E_2}{E_{22}} - 1\right] + E_1 + \frac{a}{\beta_1} I_1 + \frac{b}{\beta_2} I_{22}\left[\frac{I_2}{I_{22}} - \ln\frac{I_2}{I_{22}} - 1\right]. \tag{57}$$

In the same way that used in the proof of theorem 3.2, it can be proved that $L_2(X)$ is a Lyapunov function. And the time derivative of the Lyapunov function $L_2(x)$ is:

$$L_2^{\bullet}(x) = S^{\bullet} - \frac{g(S_2,I_{22})}{g(S,I_{22})} S^{\bullet} + E_{22}\left[\frac{E_2^{\bullet}}{E_{22}} - \frac{E_2^{\bullet}}{E_2}\right] + E_1^{\bullet} + \frac{a}{\beta_1} I_1^{\bullet} + \frac{b}{\beta_2} I_{22}\left[\frac{I_2^{\bullet}}{I_{22}} - \frac{I_2^{\bullet}}{I_2}\right]. \tag{58}$$

Substitute from equation (1) in (58) and use (16) and the equation

$$(1-\theta)\pi = \frac{bd}{\beta_2} I_{22} + \delta S_2 - \theta_1 RC_2, \tag{59}$$

$$\begin{aligned} L_2^{\bullet}(x) = {} & \delta S_2\left(1 - \frac{S}{S_2}\right)\left(1 - \frac{g(S_2,I_{22})}{g(S,I_{22})}\right) + \theta_1\, RC_2\left(\frac{RC}{RC_2} - 1\right)\left(1 - \frac{g(S_2,I_{22})}{g(S,I_{22})}\right) + \frac{ac}{\beta_1} I_1\left[R_0^1 \frac{g(S_2,I_{22})}{g(S,I_{22})}\frac{f(S,I_1)}{f(S_0,0)} - 1\right] \\ & + bE_{22}\left(\frac{g(S,I_{22})}{g(S,I_2)} - \frac{I_2}{I_{22}}\right)\left(1 - \frac{g(S,I_2)}{g(S,I_{22})}\right) + bE_{22}\left(4 - \frac{g(S,I_2)}{g(S,I_{22})} - \frac{I_2\, g(S_2,I_2)}{b\, E_2} - \frac{E_2}{E_{22}}\frac{I_{22}}{I_2} - \frac{g(S_2,I_{22})}{f(S,I_{22})}\right). \end{aligned} \tag{60}$$

From the conditions (4), (5), the following inequality $\frac{f(S,I_1)}{f(S_0,0)} le 1$ is satisfied, so we have:

$$\begin{aligned} L_2^{\bullet}(x) \leq {} & \delta S_2\left(1 - \frac{S}{S_2}\right)\left(1 - \frac{g(S_2,I_{22})}{g(S,I_{22})}\right) + \theta_1 RC_2\left(\frac{RC}{RC_2} - 1\right)\left(1 - \frac{g(S_2,I_{22})}{g(S,I_{22})}\right) + \frac{ac}{\beta_1} I_1\left[R_0^1 \frac{g(S_2,I_{22})}{g(S,I_{22})} - 1\right] \\ & + bE_{22}\left(\frac{g(S,I_{22})}{g(S,I_2)} - \frac{I_2}{I_{22}}\right)\left(1 - \frac{g(S,I_2)}{g(S,I_{22})}\right) + bE_{22}\left(4 - \frac{g(S,I_2)}{g(S,I_{22})} - \frac{I_2\, g(S_2,I_2)}{b\, E_2} - \frac{E_2}{E_{22}}\frac{I_{22}}{I_2} - \frac{g(S_2,I_{22})}{f(S,I_{22})}\right). \end{aligned} \tag{61}$$

The fourth term in the inequality (61) satisfies that:

$$bE_{22}\left(\frac{g(S,I_{22})}{g(S,I_2)} - \frac{I_2}{I_{22}}\right)\left(1 - \frac{g(S,I_2)}{g(S,I_{22})}\right) \leq 0,$$

by the assumption (55) in addition, the second term in (61) is not positive. And indicating the relation between the geometric and arithmetic means we get

$$\left(4 - \frac{g(S,I_2)}{g(S,I_{22})} - \frac{I_2\, g(S_2,I_2)}{b\, E_2} - \frac{E_2}{E_{22}}\frac{I_{22}}{I_2} - \frac{g(S_2,I_{22})}{f(S,I_{22})}\right) \leq 0.$$

Now two cases are considered:
When $S \le S_2$, applying condition (4), $\frac{g(S_2,I_{22})}{g(S,I_{22})} > 1$ and so the first term in the inequality (61) is not positive. In addition, by condition (4), $R_0^1 \le \frac{g(S,I_{22})}{g(S_2,I_{22})} \le 1$, so since $R_0^1 \le \frac{g(S,I_{22})}{g(S_2,I_{22})}$, then $R_0^1 \frac{g(S_2,I_{22})}{g(S,I_{22})} \le 1$ and now the $R_0^1 \frac{g(S_2,I_{22})}{g(S,I_{22})} - 1 \le 0$.
When $S > S_2$ and by condition (4), $\frac{g(S_2,I_{22})}{g(S,I_{22})} < 1$ and so the first and third terms in the inequality (61) are not positive. So we conclude that, $L_2^{\bullet}(x) \le 0$ when $R_0^1 \le 1 < R_0^2$.

Stability of the endemic equilibrium point P^*

Assume that the functions $f(S,I_1)$ and $g(S,I_2)$ at the constant solution P^* satisfy that:

$$\left(\frac{g(S^*,I_2^*)}{g(S,I_2)}\frac{f(S,I_1^*)}{f(S^*,I_1^*)} - \frac{I_2}{I_2^*}\right)\left(1 - \frac{g(S,I_2)}{g(S^*,I_2^*)}\frac{f(S^*,I_1^*)}{f(S,I_1^*)}\right) \le 0 \quad \text{and} \quad \left(\frac{RC}{RC^*} - 1\right)\left(1 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)}\right) \le 0. \tag{62}$$

Theorem 7. *The equilibrium point P^* is asymptotically stable when $R_0^1 > 1 < R_0^2$.*

Proof. Consider a function $L^*(x)$ where:

$$L^*(x) = S - S^* - \int_{S^*}^{S} \frac{f(S^*,I_1^*)}{f(u,I_1^*)}\,du + E_1^*\left[\frac{E_1}{E_1^*} - \ln\frac{E_1}{E_1^*} - 1\right] + E_2^*\left[\frac{E_2}{E_2^*} - \ln\frac{E_2}{E_2^*} - 1\right] + \frac{a}{\beta_1}I_1^*\left[\frac{I_1}{I_1^*} - \ln\frac{I_1}{I_1^*} - 1\right]$$
$$+ \frac{b}{\beta_2}I_2^*\left[\frac{I_2}{I_2^*} - \ln\frac{I_2}{I_2^*} - 1\right]. \tag{63}$$

In the same way that used in theorem 3.2, it can be proved that $L^*(x)$ is a Lyapunov function. And the time derivative of the Lyapunov function $L^*(x)$ is:

$$L^{*\bullet}(x) = S^{\bullet} - \frac{f(S^*,I_1^*)}{f(S,I_1^*)}S^{\bullet} + E_1^{\bullet}\left[1 - \frac{E_1^*}{E_1}\right] + E_2^{\bullet}\left[1 - \frac{E_2^*}{E_2}\right] + \frac{a}{\beta_1}I_1^{\bullet}\left[1 - \frac{I_1^*}{I_1}\right] + \frac{b}{\beta_2}I_2^{\bullet}\left[1 - \frac{I_2^*}{I_2}\right]. \tag{64}$$

Substitute equation (1) in equation (64) and using (17), we have

$$L^{*\bullet}(x) = \delta S^*\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)}\right) + \theta_1 RC^*\left(\frac{RC}{RC^*} - 1\right)\left(1 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)}\right)$$
$$+ aE_1^*\left(\frac{f(S,I_1^*)}{f(S,I_1)} - \frac{I_1}{I_1^*}\right)\left(1 - \frac{f(S,I_1)}{f(S,I_1^*)}\right) + bE_2^*\left(\frac{g(S^*,I_2^*)}{g(S,I_2)}\frac{f(S,I_1^*)}{f(S^*,I_1^*)} - \frac{I_2}{I_2^*}\right)\left(1 - \frac{g(S,I_2)}{g(S^*,I_2^*)}\frac{f(S^*,I_1^*)}{f(S,I_1^*)}\right)$$
$$+ aE_1^*\left(4 - \frac{a\,E_1^*}{f(S,I_1^*)\,I_1^*} - \frac{I_1\,f(S,I_1)}{a\,E_1} - \frac{E_1}{E_1^*}\frac{I_1^*}{I_1} - \frac{f(S,I_1^*)}{f(S,I_1)}\right)$$
$$+ bE_2^*\left(4 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)} - \frac{I_2\,g(S,I_2)}{b\,E_2} - \frac{E_2}{E_2^*}\frac{I_2^*}{I_2} - \frac{f(S,I_1^*)}{f(S^*,I_1^*)}\frac{b\,E_2^*}{I_2^*\,g(S,I_2)}\right). \tag{65}$$

By condition (62) it guarantees that the second and the fourth term in the inequality (65) are not positive. And indicating the relation between the geometric and arithmetic means we get

$$4 - \frac{aE_1^*}{f(S,I_1^*)\,I_1^*} - \frac{I_1 f(S,I_1)}{a\,E_1} - \frac{E_1}{E_1^*}\frac{I_1^*}{I_1} - \frac{f(S,I_1^*)}{f(S,I_1)} \le 0 \text{ and } 4 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)} - \frac{I_2\,g(S,I_2)}{bE_2} - \frac{E_2}{E_2^*}\frac{I_2^*}{I_2} - \frac{f(S,I_1^*)}{f(S^*,I_1^*)}\frac{bE_2^*}{I_2^* g(S,I_2)} \le 0.$$

From condition (4), we have:

$$\frac{f(S^*,I_1^*)}{f(S,I_1^*)} \le 1 \text{ if } S \ge S^* \quad \text{and} \quad \frac{f(S^*,I_1^*)}{f(S,I_1^*)} \ge 1 \text{ if } S \le S^*,$$

then

$$1 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)} \ge 0 \;\; if\ S \ge S^* \quad \text{and} \quad 1 - \frac{f(S^*,I_1^*)}{f(S,I_1^*)} \le 0 \;\; \text{if } S \le S^*.$$

Then the first term of the right-hand side of (65) is not positive. Also by condition (51), we have:

$$\left(\frac{f(S,I_1^*)}{f(S,I_1)} - \frac{I_1}{I_1^*}\right)\left(1 - \frac{f(S,I_1)}{f(S,I_1^*)}\right) \leq 0,$$

which is related to the third term of (65).

From the previous relations, we conclude that $L^{*\bullet}(x) \leq 0$ and the constant solution P^* is asymptotically stable when $R_0^1 > 1$ and $R_0^2 > 1$.

4 The Non-uniform L_2 Fractional differentiation numerical scheme

This section discusses a new scheme to get an approximate solution of the fractional order differential equation defined in Caputo sense that is represented in the following equation:

$$D^{\alpha}\phi(t) = \chi(\phi), \qquad \phi(0) = \phi_0, \tag{66}$$

In the beginning, the uniqueness of the solution of system (66) is studied. Assume that $\chi(\phi)$ satisfies the Lipschitz condition as

$$\|\chi(\phi_1) - \chi(\phi_2)\| < \ell\|\phi_1 - \phi_2\|,\ \ell > 0. \tag{67}$$

Hence, the solution of system (66) is given by

$$\phi = \phi_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi(\phi)ds = \Psi(\phi). \tag{68}$$

Then, we have

$$\begin{aligned}
\|\Psi(\phi_1) - \Psi(\phi_2)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\chi(\phi_1(s)) - \chi(\phi_2(s))\|ds \\
&\leq \frac{\ell}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{s\in[0,T]}|\phi_1(s) - \phi_2(s)| \\
&\leq \frac{\ell\|\phi_1 - \phi_2\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds
\end{aligned}$$

$$\|\Psi(\phi_1) - \Psi(\phi_2)\| \leq \frac{\ell T^{\alpha}}{\Gamma(\alpha+1)}\|\phi_1 - \phi_2\|. \tag{69}$$

Then we get $\|\Psi(\phi_1) - \Psi(\phi_2)\| \leq \|\phi_1 - \phi_2\|$ as $\frac{\ell T^{\alpha}}{\Gamma(\alpha+1)} \leq 1$, this suggests that our model has a unique solution. The above results are summarized in the following theorem.

Theorem 8. *The system given by (66) has a unique solution under that* $\frac{\ell T^{\alpha}}{\Gamma(\alpha+1)} \leq 1$ *and condition (67) is satisfied.*

To get the Non-uniform L_2 Fractional differentiation numerical scheme (NU L_2 FDNS) of the proposed model, we need to develop a new different representation of the Caputo fractional derivative of a function $\phi(t) \in C^3[0,T]$, with order $\alpha, 0 < \alpha < 1$, with approximation of order $o(\tau)^{3-\alpha}$. Let we consider that the total time of simulation is T and it is divided into N nonuniform grid as

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T, t_j = T\left(\frac{j}{N}\right)^r,\ \ r > 0\ and\, \tau_j = t_j - t_{j-1},\ \ j \in \{1,2,\ldots,N\}. \tag{70}$$

$$D_{t_{j+1}}^{\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+1}} \frac{\phi'(\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi \tag{71}$$

$$D_{t_{j+1}}^{\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_2} \frac{\phi'(\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi + \frac{1}{\Gamma(1-\alpha)}\sum_{k=2}^{j}\int_{t_k}^{t_{k+1}} \frac{\phi'(\xi)}{(t_{j+1}-\xi)^{\alpha}}d\xi. \tag{72}$$

The function $\phi(t)$ will be interpolated by the quadratic interpolant $\Omega_{2,\,s}\ \phi(t)$ through the subinterval $[t_k,\ t_{k+1}], (1 \le k \le j)$. Then using three points $(t_{k-1},\ \phi(t_{k-1}))$, $(t_k,\ \phi(t_k))$ and $(t_{k+1},\ \phi(t_{k+1}))$, the interpolant function $\Omega_{2,\,s}\ \phi(t)$ can be represented as

$$\Omega_{2,\,k}\phi\,(t) = \frac{\phi(t_{k-1})(t-t_k)(t-t_{k+1})}{(t_{k-1}-t_k)(t_{k-1}-t_{k+1})} + \frac{\phi(t_k)(t-t_{k-1})(t-t_{k+1})}{(t_k-t_{k-1})(t_k-t_{k+1})} + \frac{\phi(t_{k+1})(t-t_{k-1})(t-t_k)}{(t_{k+1}-t_{k-1})(t_{k+1}-t_k)}, \tag{73}$$

$$t_k - t_{k-1} = \tau_k, \quad t_{k+1} - t_{k-1} = \tau_k + \tau_{k+1}, \quad \text{and} \quad t_{k+1} - t_k = \tau_{k+1}.$$

Differentiate Eq. (73) with respect to the time, we have

$$\Omega'_{2,\,k}\phi\,(t) = \frac{\phi(t_{k-1})(2t-(t_k+t_{k+1}))}{\tau_k(\tau_k+\tau_{k+1})} - \frac{\phi(t_k)(2t-(t_{k-1}+t_{k+1}))}{\tau_k\ \tau_{k+1}} + \frac{\phi(t_{k+1})(2t-(t_{k-1}+t_k))}{\tau_{k+1}(\tau_k+\tau_{k+1})}. \tag{74}$$

Insert equation (74) in equation (71), and simplifying the obtained integrations, we get the following results:

a. $\int_0^{t_2} \frac{\xi}{(t_{j+1}-\xi)^{\alpha}} d\xi = \frac{1}{(1-\alpha)(2-\alpha)} \left[{t_{j+1}}^{2-\alpha} - (t_{j+1}-t_2)^{2-\alpha} \right] - \frac{1}{(1-\alpha)} t_2\ (t_{j+1}-t_2)^{1-\alpha}$.

b. $\int_0^{t_2} \frac{1}{(t_{j+1}-\xi)^{\alpha}} d\eta = \frac{1}{(1-\alpha)} \left[{t_{j+1}}^{1-\alpha} - (t_{j+1}-t_2)^{1-\alpha} \right]$.

c. $\int_{t_k}^{t_{k+1}} \frac{\xi}{(t_{j+1}-\xi)^{\alpha}} d\xi = \frac{1}{(1-\alpha)} \left[(t_k\ (t_{j+1}-t_k)^{1-\alpha} - t_{k+1}\ (t_{j+1}-t_{k+1})^{1-\alpha} \right]$
$+ \frac{1}{(1-\alpha)(2-\alpha)} \left[(t_{j+1}-t_k)^{2-\alpha} - (t_{j+1}-t_{k+1})^{2-\alpha} \right]$.

d. $\int_{t_k}^{t_{k+1}} \frac{1}{(t_{j+1}-\xi)^{\alpha}} d\xi = \frac{1}{(1-\alpha)} \left[(t_{j+1}-t_k)^{1-\alpha} - (t_{j+1}-t_{k+1})^{1-\alpha} \right]$.

Theorem 9. *The Non-uniform L_2 Fractional differentiation numerical scheme (NU L_2 FDNS) of the Caputo derivative defined in Eq. (71) is given by the following relation:*

$$D^{\alpha}_{t_{j+1}}\phi\,(t) \cong \mathbf{D}^{\alpha}_{t_{j+1}}\phi\,(t) = \frac{1}{\Gamma\,(2-\alpha)} \sum_{k=1}^{j} [c_1(k)\ \phi(tk-1) - c_2(k)\ \phi(t_k) + c_3(k)\ \phi(t_{k+1})], 1 \le j \le N-1, \tag{75}$$

where, for $1 \le k \le j$,

$$c_1\,(k) = \tau_{k+1}\ [a_k - b_k\,(t_k + t_{k+1})], \tag{76}$$

$$c_2\,(k) = (\tau_k + \tau_{k+1})\ [a_k - b_k\,(t_{k-1} + t_{k+1})], \tag{77}$$

$$c_3\,(k) = \tau_k\ [a_k - b_k\,(t_{k-1} + t_k)], \tag{78}$$

and:

$$\begin{aligned}
a_1 &= \left(\frac{2}{\tau_1^*}\right) \left[\frac{1}{(2-\alpha)} \left({t_{j+1}}^{2-\alpha} - (t_{j+1}-t_2)^{2-\alpha} \right) - t_2\ (t_{j+1}-t_2)^{1-\alpha} \right], \\
b_1 &= \left(\frac{1}{\tau_1^*}\right) [\ {t_{j+1}}^{1-\alpha} - (t_{j+1}-t_2)^{1-\alpha}], \\
a_k &= \left(\frac{2}{\tau_s^*}\right) \left[\frac{1}{(2-\alpha)} \left((t_{j+1}-t_k)^{2-\alpha} - (t_{j+1}-t_{k+1})^{2-\alpha} \right) + t_s\ (t_{j+1}-t_k)^{1-\alpha} - t_{s+1}\ (t_{j+1}-t_{k+1})^{1-\alpha} \right], \\
b_k &= \left(\frac{1}{\tau_s^*}\right) [(t_{j+1}-t_k)^{1-\alpha} - (t_{j+1}-t_{k+1})^{1-\alpha}] \quad \textit{and} \quad \tau_k^* = \tau_k\ \tau_{k+1}(\tau_k+\tau_{k+1}).
\end{aligned} \tag{79}$$

Lemma 2. *Let $\varepsilon_k = D^{\alpha}_{t_{j+1}}\phi\,(t) - \mathbf{D}^{\alpha}_{t_{j+1}}\phi\,(t)$, $k = 1, 2, \ldots, N-1, 0 < \alpha < 1$ and $\phi\,(t) \in C^3\,[0, t_{k+1}]$, then $|\varepsilon_k| = O\left(\tau^{3-\alpha}\right)$.*

Proof. let

$$\begin{aligned}
\varepsilon_k &= D^{\alpha}_{t_{j+1}}\phi(t) - \mathbf{D}^{\alpha}_{t_{j+1}}\phi(t) \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^{t_2}\frac{(\phi(t)-\Omega_{2,\,1}\phi(t))'}{(t_{j+1}-\xi)^{\alpha}}d\xi + \sum_{k=2}^{j}\int_{t_k}^{t_{k+1}}\frac{(\phi(t)-\Omega_{2,\,k}\phi(t))'}{(t_{j+1}-\xi)^{\alpha}}d\xi\right] \\
&= \frac{-\alpha}{\Gamma(1-\alpha)}\left[\int_0^{t_2}\frac{\phi(t)-\Omega_{2,\,1}\phi(t)}{(t_{j+1}-\xi)^{\alpha+1}}d\xi + \sum_{k=2}^{j}\int_{t_k}^{t_{k+1}}\frac{\phi(t)-\Omega_{2,\,k}\phi(t)}{(t_{j+1}-\xi)^{\alpha+1}}d\xi\right] \\
&= \frac{-\alpha}{6\Gamma(1-\alpha)}\left[\int_0^{t_2}\phi'''(\xi_1)\,\xi(\xi-t_1)(\xi-t_2)(t_{j+1}-\xi)^{-\alpha-1}d\xi\right. \\
&\qquad\left. + \sum_{k=2}^{j}\int_{t_k}^{t_{k+1}}\phi'''(\xi_2)(\xi-t_{k-1})(\xi-t_k)(\xi-t_{k+1})(t_{j+1}-\xi)^{-\alpha-1}d\xi\right]
\end{aligned}$$

$$\begin{aligned}
|\varepsilon_k| \le{}& \frac{\alpha}{6\Gamma(1-\alpha)}\left|\int_0^{t_2}\phi'''(\xi_1)\,\xi(\xi-t_1)(\xi-t_2)(t_{j+1}-\xi)^{-\alpha-1}d\xi\right| \\
&+ \frac{\alpha}{6\Gamma(1-\alpha)}\left|\sum_{k=2}^{j}\int_{t_k}^{t_{k+1}}\phi'''(\xi_2)(\xi-t_{k-1})(\xi-t_k)(\xi-t_{k+1})(t_{j+1}-\xi)^{-\alpha-1}d\xi\right|.
\end{aligned}$$

Following the same procedure in [22], we have

$$|\varepsilon_k| \le \kappa\tau^{3-\alpha},$$

where $\kappa = \left(2^{1-\alpha}\alpha + \frac{\sqrt{3}}{9} + \frac{\alpha}{1-\alpha}\right)\frac{|\phi'''(\xi)|}{3\Gamma(1-\alpha)}$.

In order to get the high-order approximation for our system, we apply the NU L_2 FDNS scheme of the system given by Eq.(1) as:

$$\begin{aligned}
\mathbf{D}^{\alpha}_{t_{j+1}}S(t_{j+1}) &= (1-\theta)\pi - I_1 f(S(t_{j+1}),I_1(t_{j+1})) - I_2(t_{j+1})g(S(t_{j+1}),I_2(t_{j+1})) - \delta\, S(t_{j+1}) + \theta_1 RC(t_{j+1}), \\
\mathbf{D}^{\alpha}_{t_{j+1}}E_1(t_{j+1}) &= I_1(t_{j+1})f(S(t_{j+1}),I_1(t_{j+1})) - (\beta_1+\delta)E_1(t_{j+1}), \\
\mathbf{D}^{\alpha}_{t_{j+1}}E_2(t_{j+1}) &= I_2(t_{j+1})g(S(t_{j+1}),I_2(t_{j+1})) - (\beta_2+\delta)E_2(t_{j+1}), \\
\mathbf{D}^{\alpha}_{t_{j+1}}I_1(t_{j+1}) &= \beta_1 E_1(t_{j+1}) - (\gamma_1+\delta_1+\delta)\, I_1(t_{j+1}), \\
\mathbf{D}^{\alpha}_{t_{j+1}}I_2(t_{j+1}) &= \beta_2\, E_2(t_{j+1}) - (\gamma_2+\delta_2+\delta)\, I_2(t_{j+1}), \\
\mathbf{D}^{\alpha}_{t_{j+1}}RC(t_{j+1}) &= \gamma_1 I_1(t_{j+1}) + \gamma_2\, I_2(t_{j+1}) + \theta\pi - (\theta_1+\delta)RC(t_{j+1}),
\end{aligned}$$

5 Results and simulation

Case 1: Consider the bilinear incidence rates for the two strains given by [23,24]:

$$f(S,I_1) = A\,S(t),\quad g(S,I_2) = B\,S(t). \tag{80}$$

Example 1. In case 1, when the system parameters are $\theta = 0.1, \pi = 1, \delta = 0.11, \theta_1 = 0.3,\ \beta_1 = 0.5, \beta_2 = 0.16, \gamma_1 = 0.165, \gamma_2 = 0.0175, \delta_1 = \delta_2 = 0.1\, A = 0.03,\ B = 0.02$. Let the initial conditions be $S(0) = 0.15, E_1(0) = E_2(0) = I_1(0) = I_2(0) = RC(0) = 0.1$, And considering different values of the order of differentiation $\alpha = 0.99,\ 0.84,\ 0.69$ and 0.54. The reproduction number for the two strains are: $R_0^1 = 0.54,\ R_0^2 = 0.43$ and hence, $R_0 = 0.54$ is less than one. So the free epidemic point $P_0(S_0,0,0,0,0,RC_0),\ S_0 = \frac{\pi}{\delta}\left(1-\frac{\theta\delta}{\theta_1+\delta}\right), RC_0 = \frac{\theta}{\theta_1+\delta}\pi$ is the only critical point for the system and by theorem 3.1 P_0 is asymptotically stable. Applying the NU L_2 FDNS for solving the fractional order Caputo differential equation displayed in equations (74)-(79), the effect of differentiation order α, on the behavior of the system (1) is shown in figure 2 (a, b, c, d, e, f). Where for α decease far from unity the time interval for disappearing the disease increases. While by changing the values of, $A = 0.8,\ B = 0.5$, hence $R_0^1 = 14.3,\ R_0^2 = 10.6, R_0 = 14.3$. The free disease constant solution is unstable and by theorem 3.4, the epidemic disease constant solution P^* asymptotically stable. The approximate solution of system (1) for different values of α is displayed in figure 3 (a, b, c, d, e, f).

Example 2. Considering case 1, and the set of parameters are $\alpha = 0.95$, $\pi = 1$, $\delta = 0.11$, $\theta_1 = 0.1$, $\beta_1 = 0.5$, $\beta_2 = 0.16$, $\gamma_1 = 0.165$, $\gamma_2 = 0.0175$, $\delta_1 = \delta_2 = 0.1$, $A = 0.8$, $B = 0.5$. Let the initial conditions be $S(0) = 0.5, E_1(0) = E_2(0) = I_1(0) = I_2(0) = RC(0) = 0.1$, And considering different values of the ratio of vaccinated individuals, the reproduction number for the two strains are: The epidemic solution of system (1) P^* is asymptotically stable by theorem 3.4. the effect

Table 1: The reproduction number for the two strains.

θ	0	0.3	0.6	0.9
R_0^1	15.9	11.1	6.4	1.6
$R_0^2 = R_0$	16.9	11.8	6.8	1.7

of decreasing the portion of vaccinated individuals leads to increasing $E_1(t)E_2(t)$, $I_1(t)$, $I_2(t)$ and RC(t) decreasing

Case 2:
Let the incidence rates be as in [25,26]:

$$f(S,I_1) = \frac{\eta\ S(t)}{1+\eta_1 I_1^2}\ ,g(S,I_2) = \frac{\xi\ S(t)}{1+\xi_1 I_2^2}. \tag{81}$$

*Example 3.*If the parameters of the epidemic model (1) with incidence rates defined in (81) have the following values: $\alpha = 0.95$, $0 < \theta \le 1$, $\pi = 1$, $\delta = 0.11$, $\theta_1 = 0.9$, $\beta_1 = 0.5$, $\beta_2 = 0.6$, $\gamma_1 = 0.65$, $\gamma_2 = 0.75$, $\delta_1 = 0.1$, $\delta_2 = 0.1$. And the parameters of the incidence rates for the two strains are $\eta = 0.5$, $\eta_1 = 2.5$, $\xi = 0.6$ and $\xi_1 = 3$. Let the initial conditions be $S(0) = 0.15, E_1(0) = E_20 = I10 = I20 = RC(0) = 0.01$, And considering different values of the ratio of vaccinated individuals $\theta = 0.7, 0.5, 0.3$ and 0.1. The corresponding values of the reproduction numbers are illustrated in the following table: Then the values of R_0 are greater than one in the four cases displayed. Hence, the epidemic solution of

Table 2: The reproduction number for the two strains.

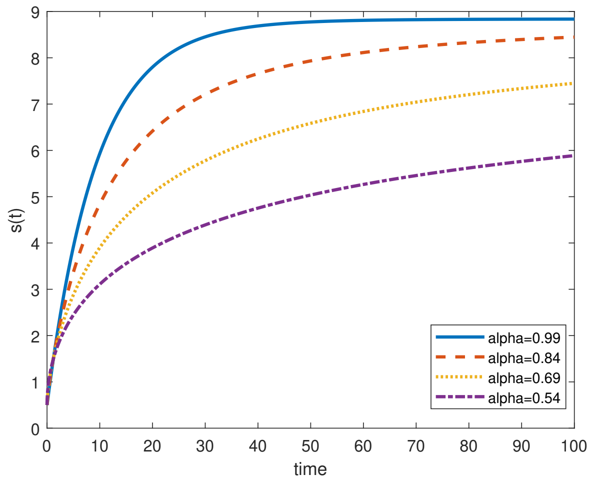
θ	0.7	0.5	0.3	0.1
R_0^1	1.2997	2.17	3.33	3.9
$R_0^2 = R_0$	1.44	2.4	3.36	4.32

system (1) P^* is asymptotically stable by theorem 3.4. Applying our numerical technique to get the approximate solution of system (1) and the effect of increasing the ratio of the vaccinated individual is clear in decreasing the number of latent individuals of two strains $E_1(t)$ and $E_2(t)$, the number of infected individuals $I_1(t)$ and $I_2(t)$ and help in increasing the number of recovered individuals $RC(t)$ the above results are given in figure 5.

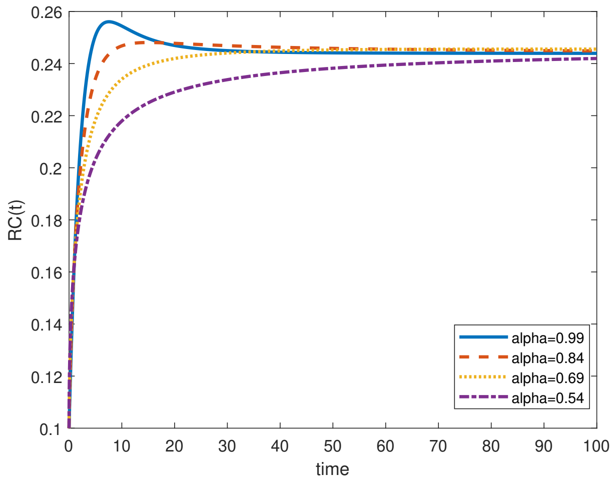
Case 3
In this case two Beddington-DeAngelis incidence functions [27] that has the forms:

$$f(S,I_1) = \frac{A\ S(t)}{1+\omega_1 S+\omega_2\ I_1} and g(S,I_2) = \frac{B\ S(t)}{1+\omega_3 S+\omega_4\ I_2}. \tag{82}$$

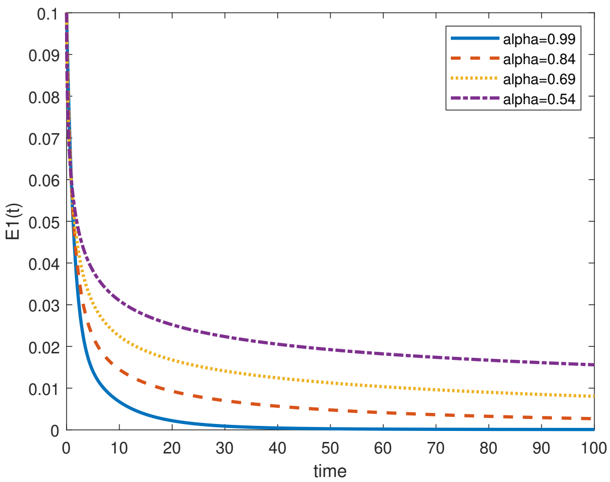
Example 4. System (1) they satisfy condition (3) - (5) with the parameters $\theta = 0.8$, $\pi = 1$, $\delta = 0.11$, $\theta_1 = 0.3$, $\beta_1 = 0.5$, $\beta_2 = 0.6$, $\gamma_1 = 0.65$, $\gamma_2 = 0.75$, $\delta_1 = 0.1$, $\delta_2 = 0.1$. And the parameters of the incidence rates for the two strains are$A = 1.8$, $B = 2.8$, $\omega_1 = 0.4$, $\omega_2 = 0.6$, $\omega_3 = 0.5$, $\omega_4 = 0.8$. The reproduction number $R_0 = 4$ and it is greater than 1, and according to theorem 4 the epidemic equilibrium point is asymptotically stable, see figure 6. While for $\alpha = 0.7$, studying the effect of vaccination is displayed in figure 7.



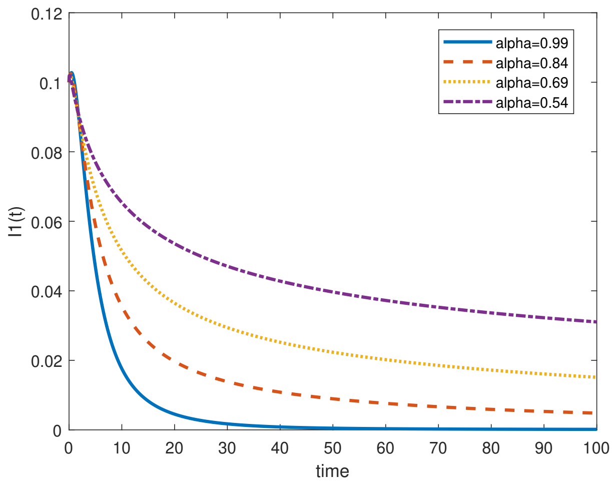
Time response of $S(t)$



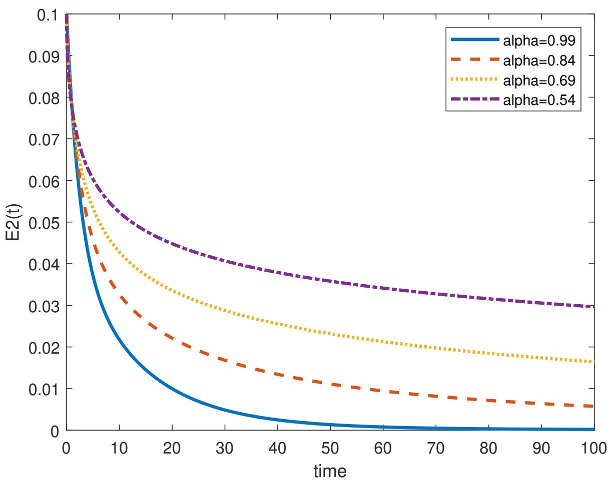
Time response of $RC(t)$



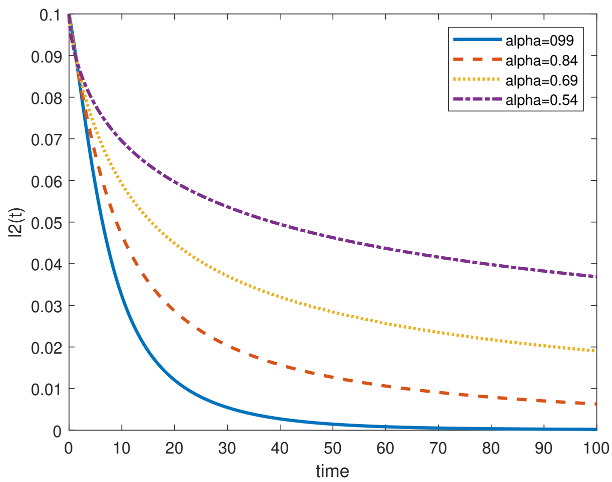
Time response of $E_1(t)$



Time response of $I_1(t)$



Time response of $E_2(t)$



Time response of $I_2(t)$

Fig. 2: Time response of $S(t), RC(t), E_1(t), I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively for example 1 when $A = 0.03, B = 0.02$.

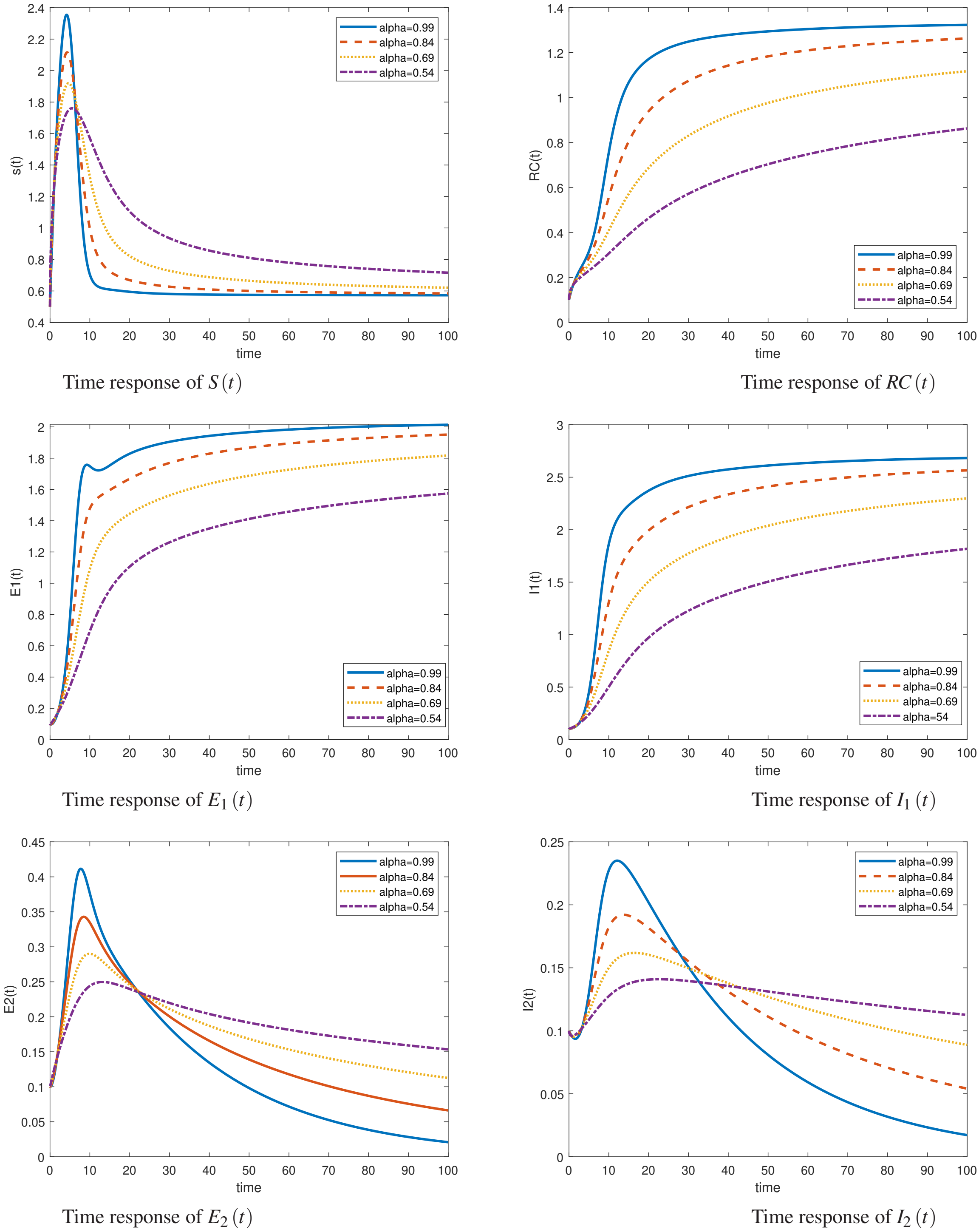
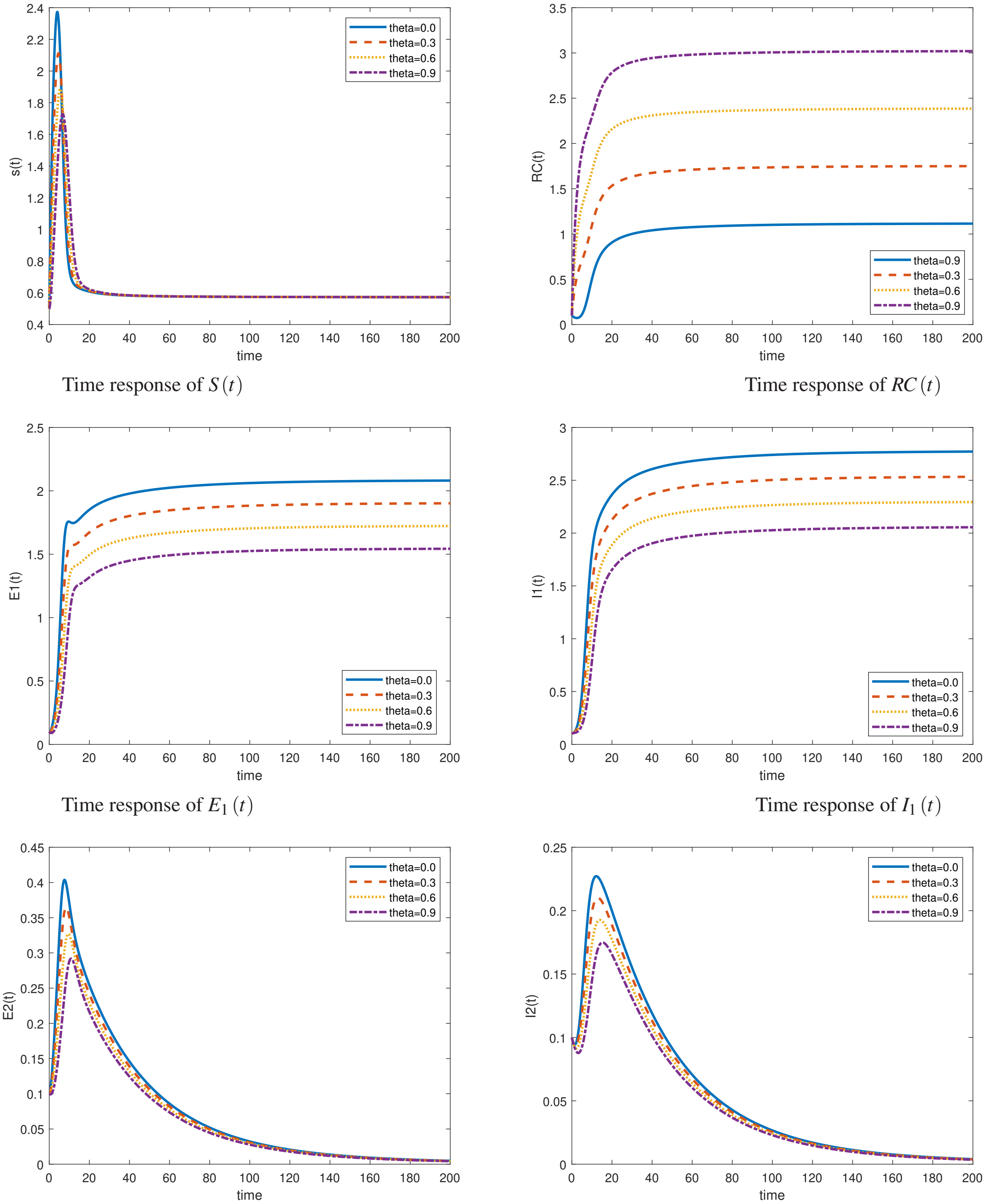


Fig. 3: Time response of $S(t), RC(t), E_1(t), I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively for example 1 when $A = 0.8, B = 0.5$.

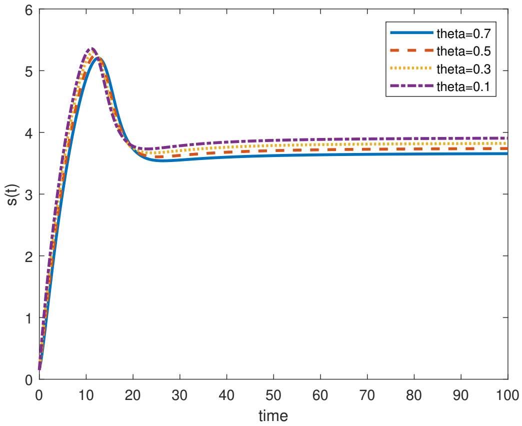


Time response of $S(t)$ Time response of $RC(t)$

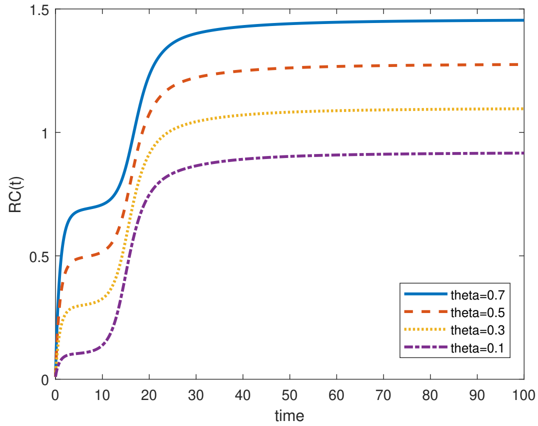
Time response of $E_1(t)$ Time response of $I_1(t)$

Time response of $E_2(t)$ Time response of $I_2(t)$

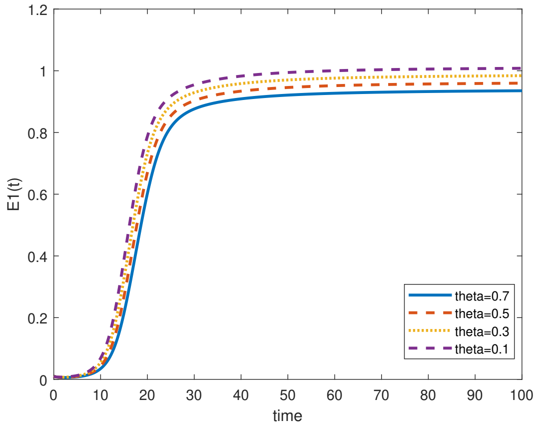
Fig. 4: Time response of $S(t), RC(t), E_1(t), I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively for example 2 case 1.



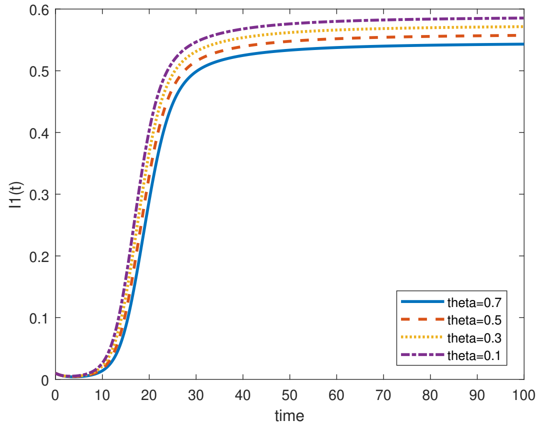
Time response of $S(t)$



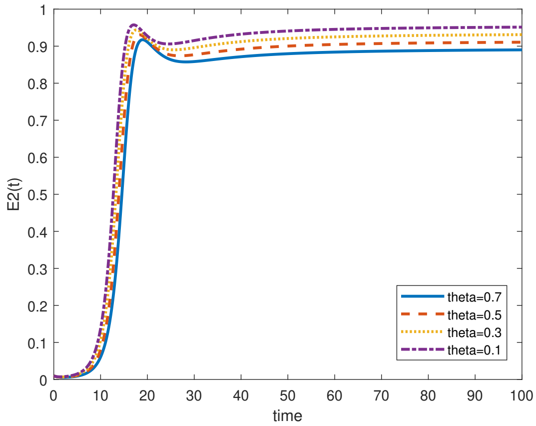
Time response of $RC(t)$



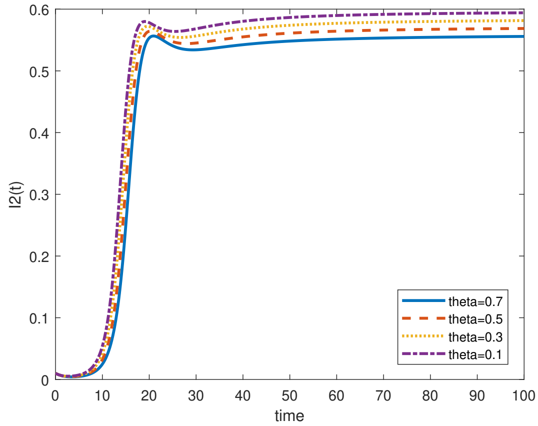
Time response of $E_1(t)$



Time response of $I_1(t)$

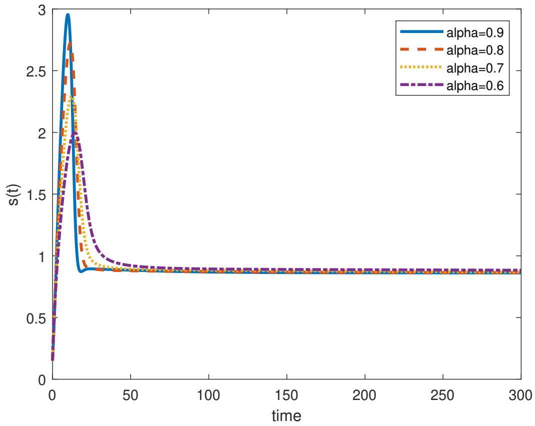


Time response of $E_2(t)$

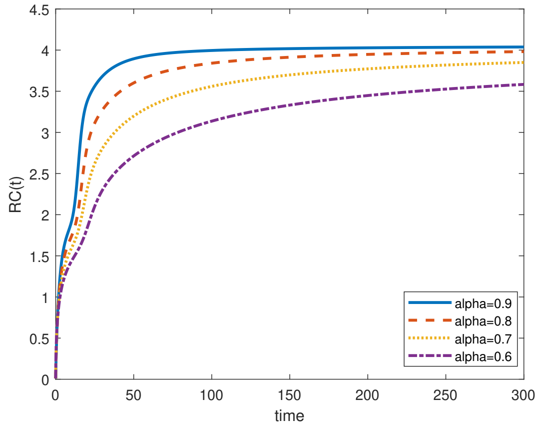


Time response of $I_2(t)$

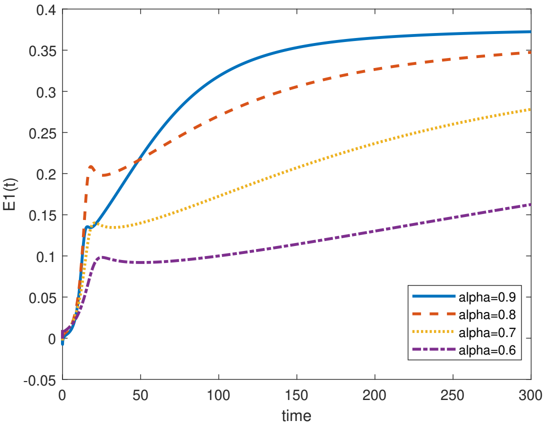
Fig. 5: Time response of $S(t)$, $RC(t)$, $E_1(t)$, $I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively for example 3 for different values of portion of vaccinated individuals θ.



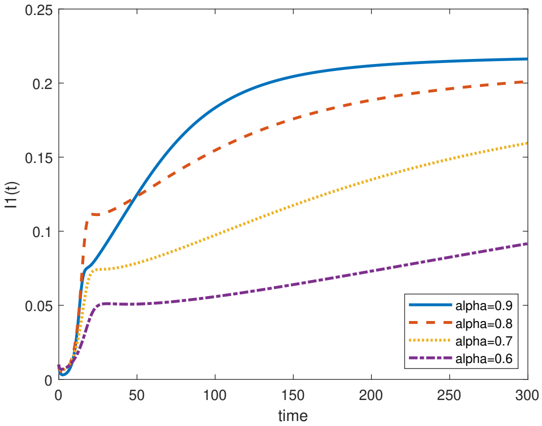
Time response of $S(t)$



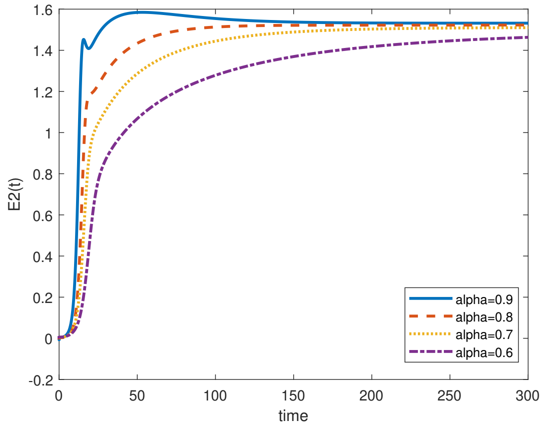
Time response of $RC(t)$



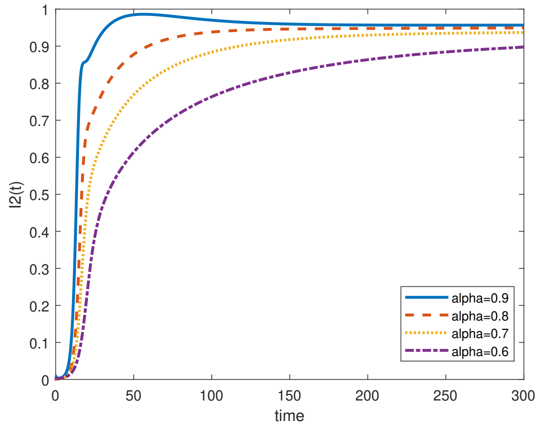
Time response of $E_1(t)$



Time response of $I_1(t)$

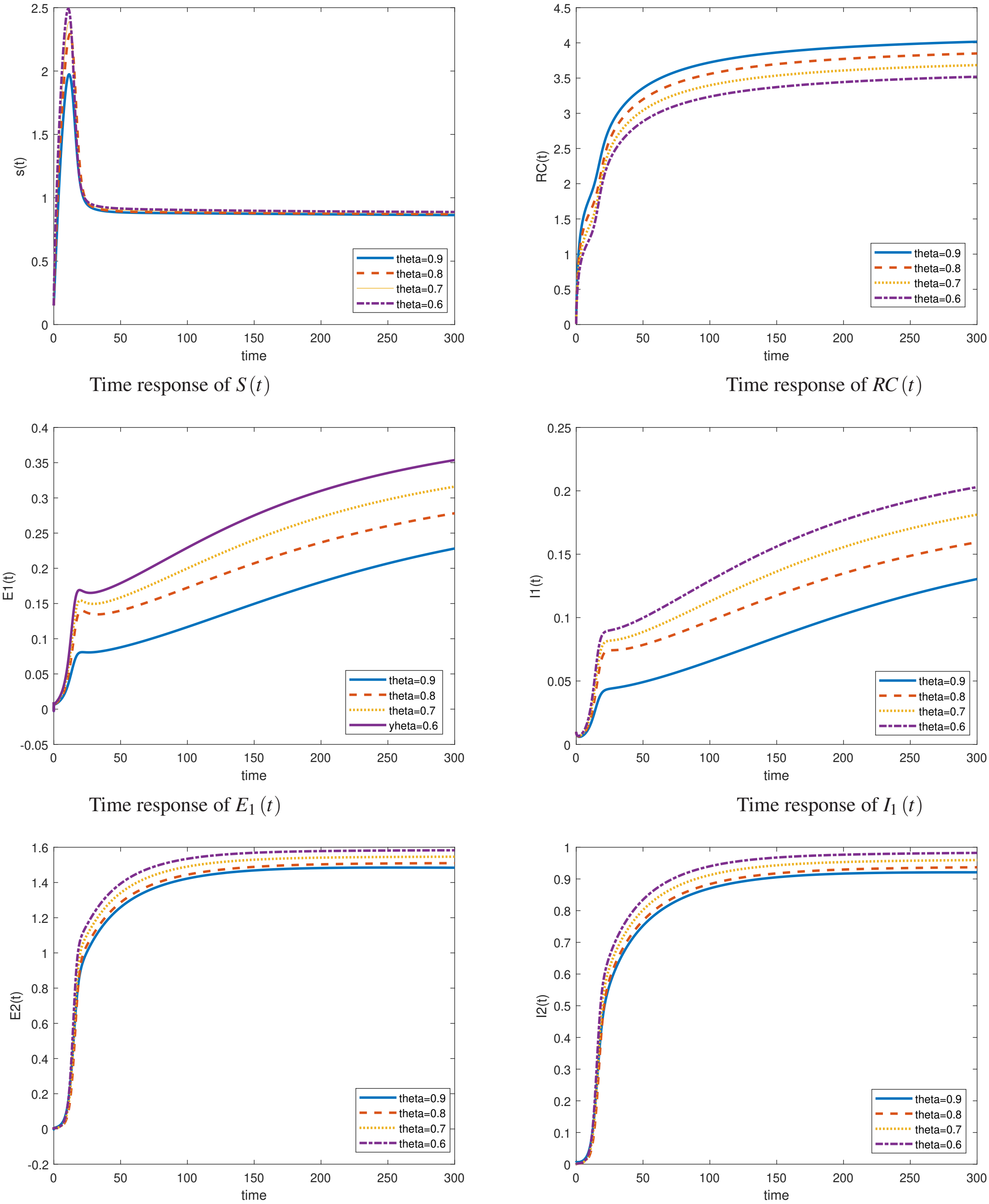


Time response of $E_2(t)$



Time response of $I_2(t)$

Fig. 6: Time response of $S(t), RC(t), E_1(t), I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively with different values of α.



Time response of $S(t)$

Time response of $RC(t)$

Time response of $E_1(t)$

Time response of $I_1(t)$

Time response of $E_2(t)$

Time response of $I_2(t)$

Fig. 7: Time response of $S(t)$, $RC(t)$, $E_1(t)$, $I_1(t)$, $E_2(t)$ and $I_2(t)$ respetively with different values of θ.

6 Conclusion

This work considers a fractional two-strain SEIR epidemic model with Caputo fractional derivative, $0 < \alpha \leq 1$, and having general non-linear incidence rates. Vaccination is considered in our model and a part of the recovered individuals is included in the susceptible individuals class. The mathematical model is represented and the system reproduction number is evaluated. The system constant solutions are given according to the nature of the incidence rates functions $f(S,I_1)$, $g(S,I_2)$ and the values of the reproduction number for each strain. There is only the free disease constant solution P_0 when $R_0 \leq 0$. If the reproduction number of the first strain only, $R_0^1 > 1$, there will be the strain 1 epidemic constant solution P_1 in addition to P_0. and when $R_0^2 > 1$, there will be the strain 2 epidemic constant solution in addition to P_2 and if $R_0^1 > 1$ and $R_0^2 > 1$ there is the system endemic constant solution P^*. The boundedness and uniqueness of the solution are studied. Suitable Lyapunov functions are constructed to prove the global stability of all constant solutions of the system. A new numerical technique based on approximating the Caputo fractional order derivative by difference schemes of a heightened order of approximation of the L_2 type. This scheme is called "The Non-uniform L_2 Fractional differentiation numerical scheme (NU L_2 FDNS)". This scheme is used to verify the analytic results of this work. We apply our analytic and numerical results to the Covid-19 pandemic model and the effect of the order of differentiation α on the behavior of the system is studied. The percentage of vaccinated individuals, θ helps control the disease. Where by increasing θ the infection individual decreases.

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