Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/190319

A New Generalized Local Derivative of Two Parameters

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Received: 12 Dec. 2024, Revised: 2 Feb. 2025, Accepted: 21 Mar. 2025
Published online: 1 May 2025

Abstract: We introduce a novel generalized derivative, the biparametric derivative, which constitutes an extension of the deformable derivative introduced by Ahuja Priyanka et al. (2017). This generalization is achieved when the secondary parameter, denoted by ψ, assumes the value of unity. Fundamental properties of the biparametric derivative are rigorously examined, and generalized forms of Rolle's theorem and the mean value theorem are derived within this new framework. The biparametric integral, intrinsically associated with the biparametric derivative, is defined, and a version of the fundamental theorem of calculus adapted to this setting is established. Finally, we address and solve certain biparametric fractional differential equations as illustrative applications of the proposed operator.

Keywords: biparametric, generalized derivative, Rolle's theorem.

1 Introduction

When the definition of a mathematical concept is varied, it is possible to obtain new results, as is the case with the fractional derivative. Several definitions of fractional derivatives have been introduced since the time when L'Hôpital asked Leibniz for an explanation about $\frac{d^{1/2}n}{dm^{1/2}}$ in a letter [7] in 1695; including those defined in the nineteenth century using the integral, such as the Riemann-Liouville and Caputo derivatives:

Definition 1.*Riemann – Liouville fractional derivative*
For $\varphi \in [n-1,n)$, the φ-derivative of g is defined by

$$D_a^{\varphi}(g)(t) = \frac{1}{\Gamma(n-\varphi)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{g(x)}{(t-x)^{\varphi-n+1}}\,\mathrm{d}x$$

Definition 2.*Caputo fractional derivative*
For $\varphi \in [n-1,n)$, the φ-derivative of g is defined by

$$D_a^{\varphi}(g)(t) = \frac{1}{\Gamma(n-\varphi)}\int_a^t \frac{g^{(n)}(x)}{(t-x)^{\varphi-n+1}}\,\mathrm{d}x$$

These derivatives can be found in [5,9], but they have some drawbacks.

1. The Riemann-Liouville derivative does not satisfy $D_a^{\varphi}(1) = 0$
2. The two fractional derivatives do not satisfy the rule for the derivative of the product of two functions

$$D_a^{\varphi}(f\cdot g) = fD_a^{\varphi}(g) + gD_a^{\varphi}(f)$$

A new definition of fractional derivative was recently introduced by R. Khalil [6] in 2014. This derivative, unlike the Riemann-Liouville and Caputo derivatives, is defined as a local fractional operator and is called the conformable fractional derivative. Related work on the conformable derivatives, non conformable derivatives and fractionals derivatives can be found in [2,3,4,8,11,12,17,13,14,15,16]

Definition 3.*Conformable Fractional Derivative*
Given a function $g:[0,\infty)\longrightarrow \mathbb{R}$. Then the conformable fractional derivative of g of order φ is defined by

$$T_{\varphi}(g)(t) = \lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{1-\varphi}) - g(t)}{\varepsilon}$$

for all $t>0$ and $\varphi\in(0,1)$.

This definition does not include zero or negative numbers, in the domain of the functions. In 2017, F. Zulfeqarr in [18] introduced a new derivative that overcomes the drawback of the conformable fractional derivative, called the deformable derivative.

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Definition 4. ***Deformable Derivative***
Given a function $\varphi \in [0,1]$, the deformable Derivative de $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$D^{\varphi} g(t) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon(1-\varphi))g(t+\varepsilon\varphi)-g(t)}{\varepsilon}$$

always whenever the limit exists.

In this article, a generalization of the deformable derivative is presented, called the V-derivative or biparametric derivative, as it depends on two parameters.

Definition 5. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}, \varphi \geq 0, \psi > 0$, the V-derivative (Biparametric) is defined as*

$$V^{\varphi,\psi}(g(t)) := \lim_{h \to 0} \frac{(\psi + h(\psi-\varphi))\, g\left(t+h\dfrac{\varphi}{\psi}\right) - \psi g(t)}{\psi \cdot h} \tag{1}$$

equivalently

$$V^{\varphi,\psi}(g(t)) := \lim_{h \to 0} \frac{\left(1+h\left(1-\dfrac{\varphi}{\psi}\right)\right) g\left(t+h\dfrac{\varphi}{\psi}\right) - g(t)}{h} \tag{2}$$

always whenever the limit exists.

Remark. For some values of φ and ψ the following results are obtained:

1.6.1. If $\psi = 1$, we obtain

$$\begin{aligned} V^{\varphi,1}(g(t)) &= \lim_{h \to 0} \frac{(1+h(1-\varphi))g(t+h\varphi)-g(t)}{h} \\ &= D^{\varphi} g(t) \end{aligned}$$

1.6.2. If $\varphi = \psi$ and g is differentiable, then

$$\begin{aligned} V^{\varphi,\varphi}(g(t)) &= \lim_{h \to 0} \frac{\psi(g(t+h)) - \psi g(t)}{\psi h} \\ &= \lim_{h \to 0} \frac{(g(t+h)) - g(t)}{h} = g'(t) \end{aligned}$$

1.6.3. If $\varphi = 0$ and $\psi = 1$, we obtain

$$\begin{aligned} V^{0,1}(g(t)) &= \lim_{h \to 0} \frac{(1+h)g(t+0) - 1g(t)}{1 \cdot h} \\ &= \lim_{h \to 0} \frac{hg(t)}{h} = g(t) \end{aligned}$$

1.6.4. If $\varphi = 1$ and $\psi > 0$, then

$$V^{1,\psi}(g(t)) = \lim_{h \to 0} \frac{(\psi + h(\psi-1))\, g\left(t+\dfrac{1}{\psi}\right) - \psi g(t)}{\psi \cdot h}$$

Remark. It is worth noting that the V-derivative of order (ϕ,ψ) is different from the V-derivative of order (ψ,ϕ).

2 Preliminary Results

In this section, the relationship between the biparametric derivative, the function and its ordinary derivative is exposed. The first theorem states that differentiability implies V-differentiability of order (φ,ψ). In this section and the following ones, we assume that $\varphi \geq 0$ and $\psi > 0$.

Theorem 1. *A function g differentiable at a point $t \in (\mu,\nu)$ is always V-differentiable of order (φ,ψ) at t, moreover,*

$$V^{\varphi,\psi}(g(t)) = \frac{\varphi}{\psi} g'(t) + \frac{(\psi-\varphi)}{\psi} g(t) \tag{3}$$

Proof.

$$\begin{aligned} V^{\varphi,\psi}(g(t)) &= \lim_{\varepsilon \to 0} \frac{(\psi + \varepsilon(\psi-\varphi))\, g\left(t+\varepsilon\dfrac{\varphi}{\psi}\right) - \psi g(t)}{\psi \cdot \varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\psi g\left(t+\varepsilon\dfrac{\varphi}{\psi}\right) - \psi g(t)}{\psi \cdot \varepsilon} + \\ &\quad \frac{\varepsilon(\psi-\varphi)\, g\left(t+\varepsilon\dfrac{\varphi}{\psi}\right)}{\psi \cdot \varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{g\left(t+\varepsilon\dfrac{\varphi}{\psi}\right) - g(t)}{\varepsilon} + \\ &\quad \frac{\psi-\varphi}{\psi}\, g\left(t+\varepsilon\frac{\varphi}{\psi}\right) \\ &= \frac{\varphi}{\psi} g'(t) + \frac{(\psi-\varphi)}{\psi} g(t) \end{aligned}$$

The second result of this section gives answers the question of whether V-differentiability of order (φ,ψ) implies continuity, the answer is affirmative, but before proving the result, we must demonstrate the following proposition, for which we recall the definition of a locally bounded function.

A function is locally bounded at $t \in (\mu,\nu)$ if there exist positive constants N y γ, such that

$$|g(t+\varepsilon)| \leq N, \quad \text{if} |\varepsilon| < \gamma$$

γ is taken to be sufficiently small such that $t+\varepsilon \in (\mu,\nu)$.

Proposition 1. *Suppose g is V-differentiable of order (φ,ψ) at $t \in (\mu,\nu)$, then g is locally bounded at t.*

Proof. If g is V-differentiable of order (φ,ψ) at $t \in (\mu,\nu)$, then there exists $\gamma > 0$ such that

$$\left| \frac{(\psi + \varepsilon(\psi-\varphi))\, g\left(t+\varepsilon\dfrac{\varphi}{\psi}\right) - g(t)}{\psi \cdot \varepsilon} - V^{\varphi,\psi}(g(t)) \right| < \rho = 1$$

Whenever $|\varepsilon| < \gamma$, where ρ is as small as desired.

Thus, for $|\varepsilon| < \gamma$, it is found that

$$\left|(\psi+\varepsilon(\psi-\varphi))\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)-\psi\cdot\varepsilon V^{\varphi,\psi}(g(t))\right|<|\psi\cdot\varepsilon|$$

Then,

$$\left|(\psi+\varepsilon(\psi-\varphi))\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right|\leq |\psi\cdot\varepsilon||V^{\varphi,\psi}(g(t))+1|+|g(t)|$$

and it is found that

$$\left|g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right|\leq\frac{|\psi\cdot\varepsilon||V^{\varphi,\psi}(g(t))+1|}{|\psi+\varepsilon(\psi-\varphi)|}+\frac{|g(t)|}{|\psi+\varepsilon(\psi-\varphi)|},\qquad |\varepsilon|<\gamma.$$

The last inequality proves that g is locally bounded at t.

The next theorem states that V-differentiability of order (φ,ψ) implies continuity.

Theorem 2. *Let g be an V-differentiable of order (φ,ψ) function at $t\in(\mu,\nu)$ for some $\varphi\in[0.1]$ and $\psi>0$, then g is continuous at t.*

Proof. It can be proven, solely that

$$\lim_{\varepsilon\to 0}\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)\right)=0$$

Now,

$$\lim_{\varepsilon\to 0}\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)\right)$$

$$=\lim_{\varepsilon\to 0}\frac{\psi\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)\right)}{\psi}$$

$$=\lim_{\varepsilon\to 0}\frac{\left[(\psi+\varepsilon(\psi-\varphi))\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\right.}{1}$$

$$\frac{\left.\varepsilon(\psi-\varphi)g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\psi g(t)\right]\varepsilon}{\psi\cdot\varepsilon}$$

$$=\lim_{\varepsilon\to 0}\left\{\frac{\left[(\psi+\varepsilon(\psi-\varphi))\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\psi g(t)\right]\varepsilon}{\psi\cdot\varepsilon}-\frac{\left[\varepsilon(\psi-\varphi)\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right]\varepsilon}{\psi\cdot\varepsilon}\right\}$$

$$=\frac{1}{\psi}\lim_{\varepsilon\to 0}\frac{\left[(\psi+\varepsilon(\psi-\varphi))\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\psi g(t)\right]\varepsilon}{\varepsilon}-$$

$$\lim_{\varepsilon\to 0}\varepsilon\frac{\psi-\varphi}{\psi}g\left(t+\varepsilon\frac{\varphi}{\psi}\right)$$

$$=\frac{1}{\psi}V^{\varphi,\psi}(g(t))\cdot\lim_{\varepsilon\to 0}\varepsilon-\frac{\psi-\varphi}{\psi}\lim_{\varepsilon\to 0}\varepsilon\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)$$

$$=-\frac{\psi-\varphi}{\psi}\lim_{\varepsilon\to 0}\varepsilon\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)=0$$

The last equality is derived from Proposition 1. Thus g is continuous at $t\in(\mu,\nu)$.

Corollary 1. *A function g that is V-differentiable of order (φ,ψ) at $t\in(\mu,\nu)$ is also differentiable.*

Proof. To prove the existence of the derivative, we will use the following definition

$$g'(t)=\frac{\psi}{\varphi}\lim_{\varepsilon\to 0}\frac{g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)}{\varepsilon}$$

$$=\frac{1}{\varphi}\lim_{\varepsilon\to 0}\frac{\psi\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)\right)}{\varepsilon}$$

$$=\frac{1}{\varphi}\lim_{\varepsilon\to 0}\frac{\psi\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right)-\psi g(t)}{\varepsilon}$$

$$=\frac{1}{\varphi}\lim_{\varepsilon\to 0}\frac{\left[\psi g\left(t+\varepsilon\frac{\varphi}{\psi}\right)+\varepsilon(\psi-\varphi)\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\right.}{1}$$

$$\frac{\left.\varepsilon(\psi-\varphi)\,g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\psi g(t)\right]}{\varepsilon}$$

$$= \frac{1}{\varphi} \lim_{\varepsilon \to 0} \frac{\left[(\psi + \varepsilon(\psi - \varphi)) g\left(t + \varepsilon \frac{\varphi}{\psi}\right) - \psi g(t) \right]}{\varepsilon} - \lim_{\varepsilon \to 0} \frac{\psi - \varphi}{\varphi} g\left(t + \varepsilon \frac{\varphi}{\psi}\right)$$

Since g is V-differentiable of order (φ, ψ) and by Theorem 2, it follows that

$$g'(t) = \frac{\psi}{\varphi} V^{\varphi,\psi}(g(t)) - \frac{\psi - \varphi}{\varphi} g(t)$$

*Remark.*It is found that from

$$g'(t) = \frac{\psi}{\varphi} V^{\varphi,\psi}(g(t)) - \frac{\psi - \varphi}{\varphi} g(t),$$

we can obtain

$$V^{\varphi,\psi} g(t) = \frac{\varphi}{\psi} g'(t) + \frac{\psi - \varphi}{\psi} g(t).$$

Theorem 3.*Let g defined on (μ, ν). It is found that g is V-differentiable of order (φ, ψ) if and only if g is differentiable.*

*Remark.*Although it is true that the function g is V-differentiable of order (φ, ψ) and g is differentiable, the values of the two derivatives are different.

3 Properties of the V-Derivative

This section is dedicated to establishing some fundamental properties of the V-derivative.

Theorem 4.*Let a and b be constants, and f and g be V-differentiable of order (φ, ψ) functions. The operator $V^{\varphi,\psi}(g(t))$ satisfies the following properties:*

1. $V^{\varphi,\psi}(af + bg)(t) = aV^{\varphi,\psi} f(t) + bV^{\varphi,\psi} g(t)$

2. $V^{\varphi,\psi}(f \cdot g)(t) = \frac{\varphi}{\psi} \left[f'(t)g(t) + f(t)g'(t) \right] + \frac{\psi - \varphi}{\psi} f(t)g(t).$

3. $V^{\varphi,\psi}\left(\frac{f}{g}\right)(t) = \frac{\varphi}{\psi} \frac{[f'(t)g(t) + f(t)g'(t)]}{[g(t)]^2} + \frac{\psi - \varphi}{\psi} \frac{f(t)}{g(t)}, \quad g(t) \neq 0.$

4. $V^{\varphi,\psi}(k) = \frac{\psi - \varphi}{\psi} k, \quad k$ **constant**.

5. $V^{\varphi_1,\psi_1} V^{\varphi_2,\psi_2} = V^{\varphi_2,\psi_2} V^{\varphi_1,\psi_1}$

*Proof.*1. It is evident, thanks to the definition.

2. $V^{\varphi,\psi}(f \cdot g)(t) =$

$$\lim_{h \to 0} \frac{(\psi + h(\psi - \varphi)) f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right) - \psi f(t)g(t)}{\psi \cdot h}$$

$$= \lim_{h \to 0} \frac{\psi f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right) + h(\psi - \varphi) f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right) - \psi f(t)g(t)}{\psi \cdot h}$$

$$= \lim_{h \to 0} \frac{f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right) - f(t)g\left(t + h\frac{\varphi}{\psi}\right)}{h} + \frac{f(t)g\left(t + h\frac{\varphi}{\psi}\right) - f(t)g(t)}{h} + \frac{h(\psi - \varphi) f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right)}{\psi \cdot h}$$

$$= \lim_{h \to 0} \frac{\left[f\left(t + h\frac{\varphi}{\psi}\right) - f(t) \right] g\left(t + h\frac{\varphi}{\psi}\right)}{h} + \lim_{h \to 0} \frac{f(t) \left[g\left(t + h\frac{\varphi}{\psi}\right) - g(t) \right]}{h} + \lim_{h \to 0} \frac{\psi - \varphi}{\psi} f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right)$$

If $h\frac{\varphi}{\psi} = \varepsilon$, then $h = \varepsilon\frac{\psi}{\varphi}$, if $h \to 0$ then $\varepsilon \to 0$ and it is found that

$$V^{\varphi,\psi}(f \cdot g)(t) = \frac{\varphi}{\psi} \lim_{\varepsilon \to 0} \frac{[f(t + \varepsilon) - f(t)]}{\varepsilon} g(t + \varepsilon) + \lim_{\varepsilon \to 0} \frac{\varphi}{\psi} f(t) \frac{g(t + \varepsilon) - g(t)}{\varepsilon} + \lim_{h \to 0} \frac{\psi - \varphi}{\psi} f\left(t + h\frac{\varphi}{\psi}\right) g\left(t + h\frac{\varphi}{\psi}\right)$$

$$= \frac{\varphi}{\psi} f'(t)g(t) + \frac{\varphi}{\psi} f(t)g'(t) + \frac{\psi - \varphi}{\psi} f(t)g(t)$$

$$= \frac{\varphi}{\psi} [f(t) \cdot g(t)]' + \frac{\psi - \varphi}{\psi} f(t)g(t)$$

3. Applying (3) from Theorem 1, it is found

$$V^{\varphi,\psi}\left(\frac{f}{g}\right)(t)=\frac{\varphi}{\psi}\left(\frac{f}{g}\right)'(t)+\frac{\psi-\varphi}{\psi}\left(\frac{f}{g}\right)(t)$$
$$=\frac{\varphi}{\psi}\frac{[f'(t)g(t)-g'(t)f(t)]}{[g(t)]^2}+$$
$$\frac{\psi-\varphi}{\psi}\left(\frac{f}{g}\right)(t)$$

4.

$$V^{\varphi,\psi}(k)=\frac{\varphi}{\psi}(k)'+\frac{\psi-\varphi}{\psi}k$$
$$=\frac{\varphi}{\psi}(0)+\frac{\psi-\varphi}{\psi}k=\frac{\psi-\varphi}{\psi}k$$

5. If (3) from Theorem 1 is applied once again, we obtain the following

$$V^{\varphi_1,\psi_1}V^{\varphi_2,\psi_2}(g(t))=V^{\varphi_1,\psi_1}\left(\frac{\varphi_2}{\psi_2}g'(t)+\frac{\psi_2-\varphi_2}{\psi_2}g(t)\right)$$
$$=\frac{\varphi_1}{\psi_1}\left(\frac{\varphi_2}{\psi_2}g''(t)+\frac{\psi_2-\varphi_2}{\psi_2}g'(t)\right)+$$
$$\frac{\psi_1-\varphi_1}{\psi_1}\left(\frac{\varphi_2}{\psi_2}g'(t)+\frac{\psi_2-\varphi_2}{\psi_2}g(t)\right)$$
$$=\frac{\varphi_2}{\psi_2}\frac{\varphi_1}{\psi_1}g''(t)+\frac{\varphi_2}{\psi_2}\left(\frac{\psi_1-\varphi_1}{\psi_1}\right)g'(t)+\frac{\psi_2-\varphi_2}{\psi_2}\frac{\varphi_1}{\psi_1}g'(t)+$$
$$\left(\frac{\psi_2-\varphi_2}{\psi_2}\right)\left(\frac{\psi_1-\varphi_1}{\psi_1}\right)g(t)$$
$$=\frac{\varphi_2}{\psi_2}\left[\frac{\varphi_1}{\psi_1}g''(t)+\frac{\psi_1-\varphi_1}{\psi_1}g'(t)\right]+$$
$$\frac{\psi_2-\varphi_2}{\psi_2}\left[\frac{\varphi_1}{\psi_1}g'(t)+\frac{\psi_1-\varphi_1}{\psi_1}g(t)\right]$$
$$=\frac{\varphi_2}{\psi_2}\left[\frac{\varphi_1}{\psi_1}g'(t)+\frac{\psi_1-\varphi_1}{\psi_1}g(t)\right]'+$$
$$\frac{\psi_2-\varphi_2}{\psi_2}\left[\frac{\varphi_1}{\psi_1}g'(t)+\frac{\psi_1-\varphi_1}{\psi_1}g(t)\right]$$
$$=V^{\varphi_2,\psi_2}\left[\frac{\varphi_1}{\psi_1}g'(t)+\frac{\psi_1-\varphi_1}{\psi_1}g(t)\right]$$
$$=V^{\varphi_2,\psi_2}V^{\varphi_1,\psi_1}(g(t))$$

Parts *2* y *3* violate the Leibniz rule for derivatives, which is why $V^{\varphi,\psi}$ is considered a fractional derivative. Interested readers can refer to [10].

Proposition 2. *The following V-derivatives can be obtained:*

1. $V^{\varphi,\psi}(t^n)=n\frac{\varphi}{\psi}t^{n-1}+\frac{\psi-\varphi}{\psi}t^n, \quad n\in\mathbb{R}$
2. $V^{\varphi,\psi}(e^t)=e^t$
3. $V^{\varphi,\psi}(\sin(t))=\frac{\varphi}{\psi}\cos(t)+\frac{\psi-\varphi}{\psi}\sin(t)$
4. $V^{\varphi,\psi}(\ln(t))=\frac{\varphi}{\psi}\cdot\frac{1}{t}+\frac{\psi-\varphi}{\psi}\ln(t) \quad t>0$

Proof. The four V-derivatives can be obtained using (3) from theorem 1.

4 Theorems of the V-Derivative

Next, we will demonstrate some classical results from calculus for the V-derivative, including the Mean Value Theorem, the Rolle's Theorem, and the Chain Rule, as well as present some examples of their application.

Theorem 5. ***Rolle's Theorem for the V-Derivative***
Let $g:[\mu,\nu]\longrightarrow\mathbb{R}$ *a function that satisfies:*

1. *g is continuous in* $[\mu,\nu]$
2. *g is V-differentiable of order* (φ,ψ) *in* (μ,ν)
3. $g(\mu)=g(\nu)$

Then, there exists $c\in(\mu,\nu)$ *such that* $V^{\varphi,\psi}(g(c))=\frac{\psi-\varphi}{\psi}g(c)$.

Proof. Since g is V-differentiable of order (φ,ψ), then g is differentiable and therefore satisfies the hypotheses of Rolle's Theorem for ordinary derivatives, so there exists $c\in(\mu,\nu)$ such that $g'(c)=0$.
Then, from

$$V^{\varphi,\psi}(g(t))=\frac{\varphi}{\psi}g'(t)+\frac{\psi-\varphi}{\psi}g(t)$$

it is found that,

$$V^{\varphi,\psi}(g(c))=\frac{\varphi}{\psi}g'(c)+\frac{\psi-\varphi}{\psi}g(c)=\frac{\varphi}{\psi}(0)+\frac{\psi-\varphi}{\psi}g(c)$$

Therefore,

$$V^{\varphi,\psi}(g(c))=\frac{\psi-\varphi}{\psi}g(c).$$

Example 1. Given the function $g(t)=t-t^3$, $\varphi=\frac{1}{9}$, and $\psi=\frac{1}{3}$, the function $g(t)$ satisfies the conditions of the Rolle's Theorem for V-derivatives, since g is continuous on $[0,1]$ and V-differentiable of order (φ,ψ) on $(0,1)$. It is also true that $g(0)=g(1)$.

Therefore, we can apply the Rolle's Theorem for V-derivatives. Then, we can find a $c\in(0,1)$ such that

$$V^{\frac{1}{9},\frac{1}{3}}(g(c))=\frac{\frac{1}{3}-\frac{1}{9}}{\frac{1}{3}}g(c)$$

When calculating the V-derivative of order (φ,ψ) of g, we have

$$V^{\frac{1}{9},\frac{1}{3}}(t-t^3)=\frac{1}{3}+\frac{2}{3}t-t^2-\frac{2}{3}t^3$$

which, when evaluated at c, gives us

$$V^{\frac{1}{9},\frac{1}{3}}(c-c^3)=\frac{1}{3}+\frac{2}{3}c-c^2-\frac{2}{3}c^3$$

and we equate this with

$$V^{\frac{1}{9},\frac{1}{3}}(c-c^3)=\frac{\frac{1}{3}-\frac{1}{9}}{\frac{1}{3}}(c-c^3)=\frac{2}{3}(c-c^3)$$

which gives

$$\frac{1}{3}+\frac{2}{3}c-c^2-\frac{2}{3}c^3=\frac{2}{3}(c-c^3)$$

implying

$$\frac{1}{3}-c^2=0$$

So $c=\sqrt{\frac{1}{3}}$.

Theorem 6.***Mean Value Theorem for the V-Derivative***
Let $g:[\mu,\nu]\longrightarrow\mathbb{R}$ be a function that satisfies:

1.*g is continuous in $[\mu,\nu]$*
2.*g is V-differentiable of order (φ,ψ) in (μ,ν)*

Then, there exist $c\in(\mu,\nu)$ such that

$$V^{\varphi,\psi}(g(c))=\frac{\varphi}{\psi}\frac{g(\nu)-g(\mu)}{\nu-\mu}+\frac{\psi-\varphi}{\psi}g(c).$$

*Proof.*Let's consider the auxiliary function

$$h(t)=g(t)-g(\mu)-\frac{g(\nu)-g(\mu)}{\nu-\mu}t$$

it is found that h satisfies the hypotheses of the Rolle's Theorem, so there exists $c\in(\mu,\nu)$ such that

$$V^{\varphi,\psi}(h(c))=\frac{\psi-\varphi}{\psi}h(c)$$

Additionally, it is found that,

$$\begin{aligned}V^{\varphi,\psi}(h(t))&=\frac{\varphi}{\psi}h'(t)+\frac{\psi-\varphi}{\psi}h(t)\\&=\frac{\varphi}{\psi}\left[g'(t)-\frac{g(\nu)-g(\mu)}{\nu-\mu}\right]+\frac{\psi-\varphi}{\psi}h(t)\end{aligned}$$

Then, it is found that

$$\frac{\psi-\varphi}{\psi}h(c)=\frac{\varphi}{\psi}\left[g'(c)-\frac{g(\nu)-g(\mu)}{\nu-\mu}h(c)\right]+\frac{\psi-\varphi}{\psi}h(c)$$

By simplifying, we obtain,

$$\frac{\varphi}{\psi}\left[g'(c)-\frac{g(\nu)-g(\mu)}{\nu-\mu}\right]=0$$

which implies

$$\frac{\varphi}{\psi}g'(c)=\frac{\varphi}{\psi}\frac{g(\nu)-g(\mu)}{\nu-\mu} \tag{4}$$

From $V^{\varphi,\psi}(g(c))=\frac{\varphi}{\psi}g'(c)+\frac{\psi-\varphi}{\psi}g(c)$ it follows that

$$\frac{\varphi}{\psi}g'(c)=V^{\varphi,\psi}(g(c))-\frac{\psi-\varphi}{\psi}g(c)$$

Thus, from (4) it is found that

$$\frac{\varphi}{\psi}\frac{g(\nu)-g(\mu)}{\nu-\mu}=V^{\varphi,\psi}(g(c))-\frac{\psi-\varphi}{\psi}g(c)$$

then,

$$V^{\varphi,\psi}(g(c))=\frac{\varphi}{\psi}\frac{g(\nu)-g(\mu)}{\nu-\mu}+\frac{\psi-\varphi}{\psi}g(c).$$

*Remark.*If in the Mean Value Theorem for the V-Derivative, it is found that φ is equal to ψ, the Mean Value Theorem for ordinary derivatives is obtained.

*Example 2.*For the function and the V-derivative of order (φ,ψ) of the Rolle's Theorem example $g(t)=t-t^3$, with $\varphi=\frac{1}{9}$ and $\psi=\frac{1}{3}$, it is observed that the function g satisfies the conditions of the Mean Value Theorem for V-derivatives, since g is continuous on $[0,2]$ and V-differentiable of order (φ,ψ) on $(0,2)$.

Therefore, the Mean Value Theorem for V-derivatives can be applied. Then, we can find a $c\in(0,2)$ such that

$$V^{\frac{1}{9},\frac{1}{3}}(g(c))=\frac{\frac{1}{9}}{\frac{1}{3}}\frac{g(2)-g(0)}{2-0}+\frac{\frac{1}{3}-\frac{1}{9}}{\frac{1}{3}}g(c)$$

When calculating the V-derivative of order (φ,ψ) of g, we have

$$V^{\frac{1}{9},\frac{1}{3}}(t-t^3)=\frac{1}{3}+\frac{2}{3}t-t^2-\frac{2}{3}t^3$$

which, when evaluated at c, gives us

$$V^{\frac{1}{9},\frac{1}{3}}(c-c^3)=\frac{1}{3}+\frac{2}{3}c-c^2-\frac{2}{3}c^3$$

and we equate this with

$$\begin{aligned}V^{\frac{1}{9},\frac{1}{3}}(c-c^3)&=\frac{\frac{1}{9}}{\frac{1}{3}}\cdot\frac{g(2)-g(0)}{2-0}+\frac{\frac{1}{3}-\frac{1}{9}}{\frac{1}{3}}(c-c^3)\\&=\frac{1}{3}\left(-\frac{6}{2}\right)+\frac{2}{3}(c-c^3)\end{aligned}$$

which gives

$$\frac{1}{3}+\frac{2}{3}c-c^2-\frac{2}{3}c^3=-1+\frac{2}{3}(c-c^3)$$

implying

$$\frac{4}{3}-c^2=0$$

So $c=\frac{2}{\sqrt{3}}$.

Theorem 7.*Chain Rule for V-Derivatives*
If g is an V-differentiable of order (φ,ψ) function and f is differentiable at $g(t)$, then $f\circ g$ is V-differentiable at t of order (φ,ψ) and

$$V^{\varphi,\psi}(f\circ g)(t)=\frac{\varphi}{\psi}f'(g(t))g'(t)+\frac{\psi-\varphi}{\psi}f(g(t))$$

*Proof.*By the definition of the V-derivative of $f\circ g$, it is found

$$V^{\varphi,\psi}(f\circ g)(t)$$

$$=\lim_{\varepsilon\to 0}\frac{(\psi+\varepsilon(\psi-\varphi))\,(f\circ g)\left(t+\varepsilon\frac{\varphi}{\psi}\right)-\psi(f\circ g)(t)}{\psi\cdot\varepsilon}$$

$$=\lim_{\varepsilon\to 0}\frac{(\psi+\varepsilon(\psi-\varphi))f\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right)-\psi f(g(t))}{\psi\cdot\varepsilon}$$

If $h=g(t+\varepsilon\frac{\varphi}{\psi})-g(t)$ and $y=g(t)$, it is found that $g(t+\varepsilon\frac{\varphi}{\psi})=y+h$.

By Theorem 2, if g is V-differentiable at t of order (φ,ψ), then g is continuous at t, which ensures that $g(t+\varepsilon\frac{\varphi}{\psi})\to g(t)$ when $\varepsilon\to 0$, thus $h\to 0$.

Therefore, it can be written

$$\begin{aligned}
&V^{\varphi,\psi}(f\circ g)(t)\\
&=\lim_{\varepsilon\to 0}\frac{(\psi+\varepsilon(\psi-\varphi))f(y+h)-\psi f(y)}{\psi\cdot\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{f(y+h)-f(y)}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{\varepsilon(\psi-\varphi)f(y+h)}{\psi\cdot\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{f(y+h)-f(y)}{h}\cdot\frac{h}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{\psi-\varphi}{\psi}f\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right)\\
&=\lim_{h\to 0}\frac{f(y+h)-f(y)}{h}\cdot\lim_{\varepsilon\to 0}\frac{g\left(t+\varepsilon\frac{\varphi}{\psi}\right)-g(t)}{\varepsilon}+\\
&\lim_{\varepsilon\to 0}\frac{\psi-\varphi}{\psi}f\left(g\left(t+\varepsilon\frac{\varphi}{\psi}\right)\right)\\
&=f'(y)\cdot\frac{\varphi}{\psi}g'(t)+\frac{\psi-\varphi}{\psi}f(g(t))\\
&=\frac{\varphi}{\psi}f'(g(t))g'(t)+\frac{\psi-\varphi}{\psi}f(g(t))
\end{aligned}$$

The following are some examples that illustrate the use of the Chain Rule Theorem for the V-derivative.

*Example 3.*If $h(t)=\cos^2(t)=(f\circ g)(t)$, where $f(t)=t^2$ y $g(t)=\cos(t)$, then using the Chain Rule Theorem for the V-derivative, it is found

$$\begin{aligned}
V^{\varphi,\psi}(h(t))&=\frac{\varphi}{\psi}f'(g(t))g'(t)+\frac{\psi-\varphi}{\psi}f(g(t))\\
&=\frac{\varphi}{\psi}f'(\cos(t))(\cos(t))'+\frac{\psi-\varphi}{\psi}\cos^2(t)\\
&=\frac{\varphi}{\psi}2\cos(t)\cdot\sin(t)+\frac{\psi-\varphi}{\psi}\cos^2(t)\\
&=\frac{\varphi}{\psi}\sin(2t)+\frac{\psi-\varphi}{\psi}\cos^2(t)
\end{aligned}$$

*Example 4.*If $h(t)=\cos(t^3)=(f\circ g)(t)$, where $f(t)=\cos(t)$ y $g(t)=t^3$, then using the Chain Rule Theorem for the V-derivative., it is found

$$\begin{aligned}
V^{\varphi,\psi}(h(t))&=\frac{\varphi}{\psi}f'(t^3)(t^3)'+\frac{\psi-\varphi}{\psi}\cos(t^3)\\
&=\frac{\varphi}{\psi}\sin(t^3)(3t^2)+\frac{\psi-\varphi}{\psi}\cos(t^3)\\
&=3\frac{\varphi}{\psi}t^2\sin(t^3)+\frac{\psi-\varphi}{\psi}\cos(t^3)
\end{aligned}$$

5 V-Integral(Biparametric Integral)

In this section, the inverse operator to the V-derivative is defined, along with some properties of the V-integral.

Definition 6.*Let g be a continuous function defined on $[\mu,\nu]$. The V-integral of order (φ,ψ) (Biparametric integral of order (φ,ψ)), denoted by $I_\mu^{\varphi,\psi}(g)$, is defined by the expression;*

$$I_\mu^{\varphi,\psi}(g)(t)=\frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\int_\mu^t e^{\frac{\psi-\varphi}{\varphi}y}g(y)\,\mathrm{d}y.$$

*Remark.*If $\psi=\varphi$, the Biparametric integral of order (φ,ψ) coincides with the Riemann integral.

Theorem 8.*Let $c,\ d,\ \varphi_1,\ \psi_1,\ \varphi_2,\ \psi_2\in\mathbb{R}$ and g , h be continuous functions, then:*

1.$I_\mu^{\varphi,\psi}(cg+dh)=cI_\mu^{\varphi,\psi}g+dI_\mu^{\varphi,\psi}h$ (linearity)

2.$I_\mu^{\varphi_1,\psi_1}I_\mu^{\varphi_2,\psi_2}=I_\mu^{\varphi_2,\psi_2}I_\mu^{\varphi_1,\psi_1}$ (commutativity)

Proof. 1.Linearity follows from the definition.
2.For commutativity, the following procedure is followed.

$$I_\mu^{\varphi_1,\psi_1}I_\mu^{\varphi_2,\psi_2}g(t)=I_\mu^{\varphi_1,\psi_1}\left(\frac{\psi_2}{\varphi_2}e^{-\frac{\psi_2-\varphi_2}{\varphi_2}t}\int_\mu^t e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi}g(\pi)\,\mathrm{d}\pi\right)$$

$$= \frac{\psi_1}{\varphi_1} e^{-\frac{\psi_1-\varphi_1}{\varphi_1}t} \int_\mu^t e^{\frac{\psi_1-\varphi_1}{\varphi_1}y} \left(\frac{\psi_2}{\varphi_2} e^{-\frac{\psi_2-\varphi_2}{\varphi_2}y} \int_\mu^t e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi} g(\pi)\,\mathrm{d}\pi \right) \mathrm{d}y$$

$$= \frac{\psi_1}{\varphi_1}\frac{\psi_2}{\varphi_2} e^{-\frac{\psi_1-\varphi_1}{\varphi_1}t} \int_\mu^t \int_\mu^y e^{\left(\frac{\psi_1-\varphi_1}{\varphi_1}-\frac{\psi_2-\varphi_2}{\varphi_2}\right)y} e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi} g(\pi)\,\mathrm{d}\pi\,\mathrm{d}y$$

$$= \frac{\psi_1}{\varphi_1}\frac{\psi_2}{\varphi_2} e^{-\frac{\psi_1-\varphi_1}{\varphi_1}t} \int_\mu^t \int_\mu^y e^{\left(\frac{\psi_1-\varphi_1}{\varphi_1}-\frac{\psi_2-\varphi_2}{\varphi_2}\right)y} e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi} g(\pi)\,\mathrm{d}y\,\mathrm{d}\pi$$

$$= \frac{\psi_1}{\varphi_1}\frac{\psi_2}{\varphi_2} e^{-\frac{\psi_1-\varphi_1}{\varphi_1}t} \int_\mu^t e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi} g(\pi) \left(\int_\pi^t e^{\left(\frac{\psi_1-\varphi_1}{\varphi_1}-\frac{\psi_2-\varphi_2}{\varphi_2}\right)y} \mathrm{d}y \right) \mathrm{d}\pi$$

$$= \frac{\psi_1\psi_2}{(\psi_1-\varphi_1)\varphi_2-(\psi_2-\varphi_2)\varphi_1} \left(e^{-\frac{\psi_2-\varphi_2}{\varphi_2}t} \int_\mu^t e^{\frac{\psi_2-\varphi_2}{\varphi_2}\pi} g(\pi)\,\mathrm{d}\pi - e^{-\frac{\psi_1-\varphi_1}{\varphi_1}t} \int_\mu^t e^{\frac{\psi_1-\varphi_1}{\varphi_1}\pi} g(\pi)\,\mathrm{d}\pi \right)$$

$$= \frac{\psi_1\psi_2}{(\psi_1-\varphi_1)\varphi_2-(\psi_2-\varphi_2)\varphi_1} \left(\frac{\varphi_2}{\psi_2} I_\mu^{\varphi_2,\psi_2} g(t) - \frac{\varphi_1}{\psi_1} I_\mu^{\varphi_1,\psi_1} g(t) \right)$$

When $\varphi_1,\ \psi_1$ is exchanged for $\varphi_2,\ \psi_2$ you have to

$$I_\mu^{\varphi_2,\psi_2} I_\mu^{\varphi_1,\psi_1} g(t) = \frac{\psi_1\psi_2}{(\psi_2-\varphi_2)\varphi_1-(\psi_1-\varphi_1)\varphi_2} \left(\frac{\varphi_1}{\psi_1} I_\mu^{\varphi_1,\psi_1} g(t) - \frac{\varphi_2}{\psi_2} I_\mu^{\varphi_2,\psi_2} g(t) \right) = I_\mu^{\varphi_1,\psi_1} I_\mu^{\varphi_2,\psi_2} g(t)$$

This completes the proof.

The following theorem is an adaptation of the fundamental theorem of calculus for the V-integral of order (φ,ψ), and establishes that it is the inverse operation to V-differentiation of order (φ,ψ).

Theorem 9. *Let g be a continuous function defined on $[\mu,\nu]$. Then $I_\mu^{\varphi,\psi}g$ is V-differentiable of order (φ,ψ) in (μ,ν), and it holds that*

$$V^{\varphi,\psi}(I_\mu^{\varphi,\psi} g(y)) = g(y)$$

Additionally, if h is a continuous function and h is an V-antiderivative of order (φ,ψ) of g in (μ,ν), that is, $h = V^{\varphi,\psi}g$, then it holds that

$$I_\mu^{\varphi,\psi}(V^{\varphi,\psi} g(t)) = g(t) - g(\mu) e^{\frac{\psi-\varphi}{\varphi}(\mu-t)} \tag{5}$$

Proof. The first part of the theorem's proof is obtained from the fact that g is continuous, which implies that g is differentiable, and thanks to Theorem 3, it follows that g is V-differentiable of order (φ,ψ). If we let $h = I_\mu^{\varphi,\psi}$, then it follows that

$$V^{\varphi,\psi}(I_\mu^{\varphi,\psi} g(t)) = V^{\varphi,\psi}(h(t)) = \frac{\varphi}{\psi} h'(t) + \frac{\psi-\varphi}{\psi} h(t)$$

Since a particular solution of the differential equation

$$g = \frac{\varphi}{\psi} h'(t) + \frac{\psi-\varphi}{\psi} h(t),$$

is given by

$$g(t) = \frac{\psi}{\varphi} e^{-\frac{\psi-\varphi}{\varphi}t} \int_\mu^t e^{\frac{\psi-\varphi}{\varphi}y} g(y)\,\mathrm{d}y$$

the first part of the theorem is proven.
For the second part, if

$$h(t) = V^{\varphi,\psi}(g(t)) = \frac{\varphi}{\psi} g'(t) + \frac{\psi-\varphi}{\psi} g(t),$$

we have

$$I_\mu^{\varphi,\psi}(h(t)) = I_\mu^{\varphi,\psi}\left(\frac{\varphi}{\psi} g'(t) + \frac{\psi-\varphi}{\psi} g(t) \right)$$

$$= \frac{\psi}{\varphi} \int_\mu^t \frac{\varphi}{\psi} e^{\frac{\psi-\varphi}{\varphi}(y-t)} g'(y)\,\mathrm{d}y + \frac{\psi-\varphi}{\psi} I_\mu^{\varphi,\psi} g(t)$$

$$= e^{-\frac{\psi-\varphi}{\varphi}t} \int_\mu^t e^{\frac{\psi-\varphi}{\varphi}y} g'(y)\,\mathrm{d}(y) + \frac{\psi-\varphi}{\psi} I_\mu^{\varphi,\psi} g(t)$$

$$= e^{-\frac{\psi-\varphi}{\varphi}t} \left[g(y) e^{\frac{\psi-\varphi}{\varphi}y} \Big|_\mu^t - \int_\mu^t \frac{\psi-\varphi}{\varphi} e^{\frac{\psi-\varphi}{\varphi}y} g(y)\,\mathrm{d}(y) \right] + \frac{\psi-\varphi}{\varphi} I_\mu^{\varphi,\psi} g(t)$$

$$= g(t) - g(\mu) e^{\frac{\psi-\varphi}{\varphi}(\mu-t)} - \frac{\psi-\varphi}{\varphi} \cdot \frac{\psi}{\varphi} e^{-\frac{\psi-\varphi}{\varphi}t} \int_\mu^t e^{\frac{\psi-\varphi}{\varphi}y} g(y)\,\mathrm{d}(y) + \frac{\psi-\varphi}{\varphi} I_\mu^{\varphi,\psi} g(t)$$

$$= g(t) - g(\mu) e^{\frac{\psi-\varphi}{\varphi}(\mu-t)} - \frac{\psi-\varphi}{\varphi} I_\mu^{\varphi,\psi} g(t) + \frac{\psi-\varphi}{\varphi} I_\mu^{\varphi,\psi} g(t)$$

$$= g(t) - g(\mu) e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}$$

We apply the V-integral of order (φ,ψ) to some functions, resulting in the following proposition.

Proposition 3. *The following results hold*

1. $I_{\mu}^{\varphi,\psi}(\sin(t))$
$$= \frac{\varphi\psi}{(\psi-\varphi)^2+\varphi^2}\left[\left(\frac{\psi-\varphi}{\varphi}\sin(t)-\cos(t)\right)+\left(\cos(\mu)-\frac{\psi-\varphi}{\varphi}\sin(\mu)\right)e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}\right]$$
2. $I_{\mu}^{\varphi,\psi}(e^t) = e^t - e^{\mu}e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}$
3. $I_{\mu}^{\varphi,\psi}(\lambda) = \frac{\psi\lambda}{(\psi-\varphi)}\left[1-e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}\right]$

Proof. The proof of the four equalities is obtained from the definition of the V-integral of order (φ,ψ).
For *1)*, we have that

$$I_{\mu}^{\varphi,\psi}(\sin(t)) = \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\int_{\mu}^{t} e^{\frac{\psi-\varphi}{\varphi}y}\sin(y)\,\mathrm{d}y$$

To solve the V-integral of order (φ,ψ) of $\sin(t)$, we integrate by parts.

$$\int_{\mu}^{t} e^{\frac{\psi-\varphi}{\varphi}y}\sin(y)\,\mathrm{d}y$$
$$= \left(-\cos(y)e^{\frac{\psi-\varphi}{\varphi}y}\right)\Big|_{\mu}^{t} + \frac{\psi-\varphi}{\varphi}\int_{\mu}^{t}\cos(y)e^{\frac{\psi-\varphi}{\varphi}y}\mathrm{d}y$$

$$= -\cos(t)e^{\frac{\psi-\varphi}{\varphi}t} + \cos(\mu)e^{\frac{\psi-\varphi}{\varphi}\mu} + \frac{\psi-\varphi}{\varphi}\left[\left(\sin(y)e^{\frac{\psi-\varphi}{\varphi}y}\right)\Big|_{\mu}^{t} - \frac{\psi-\varphi}{\varphi}\int_{\mu}^{t}\sin(y)e^{\frac{\psi-\varphi}{\varphi}y}\mathrm{d}y\right]$$

$$= -\cos(t)e^{\frac{\psi-\varphi}{\varphi}t} + \cos(\mu)e^{\frac{\psi-\varphi}{\varphi}\mu} + \frac{\psi-\varphi}{\varphi}\sin(t)e^{\frac{\psi-\varphi}{\varphi}t} - \frac{\psi-\varphi}{\varphi}\sin(\mu)e^{\frac{\psi-\varphi}{\varphi}\mu} - \frac{(\psi-\varphi)^2}{\varphi^2}\int_{\mu}^{t}\sin(y)e^{\frac{\psi-\varphi}{\varphi}y}\mathrm{d}y$$

Then,

$$\int_{\mu}^{t} e^{\frac{\psi-\varphi}{\varphi}y}\sin(y)\,\mathrm{d}y = \frac{\varphi^2}{(\psi-\varphi)^2+\varphi^2} \times\left[\left(-\cos(t)+\frac{\psi-\varphi}{\varphi}\sin(t)\right)e^{\frac{\psi-\varphi}{\varphi}t}+\left(\cos(\mu)-\frac{\psi-\varphi}{\varphi}\sin(\mu)\right)e^{\frac{\psi-\varphi}{\varphi}\mu}\right]$$

Therefore,

$$I_{\mu}^{\varphi,\psi}(\sin(t)) = \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\left[\frac{\varphi^2}{(\psi-\varphi)^2+\varphi^2}\left(\left(-\cos(t)+\frac{\psi-\varphi}{\varphi}\sin(t)\right)e^{\frac{\psi-\varphi}{\varphi}t}+\left(\cos(\mu)-\frac{\psi-\varphi}{\varphi}\sin(\mu)\right)e^{\frac{\psi-\varphi}{\varphi}\mu}\right)\right]$$

$$= \frac{\varphi\psi}{(\psi-\varphi)^2+\varphi^2}\left[\frac{\psi-\varphi}{\varphi}\sin(t)-\cos(t)+\left(\cos(\mu)-\frac{\psi-\varphi}{\varphi}\sin(\mu)\right)e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}\right]$$

For *2)* we have that

$$\begin{aligned}
I_{\mu}^{\varphi,\psi}(e^t) &= \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\int_{\mu}^{t} e^{\frac{\psi-\varphi}{\varphi}y}e^y\,\mathrm{d}y \\
&= \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\left(\frac{\varphi}{\psi}e^{\frac{\psi-\varphi}{\varphi}y}e^y\right)\Big|_{\mu}^{t} \\
&= \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\left(\frac{\varphi}{\psi}\left(e^{\frac{\psi-\varphi}{\varphi}t}e^t - e^{\frac{\psi-\varphi}{\varphi}\mu}e^{\mu}\right)\right) \\
&= e^t - e^{\mu}e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}
\end{aligned}$$

For *3)*, whenever $\varphi \neq \psi$, we have that

$$\begin{aligned}
I_{\mu}^{\varphi,\psi}(\lambda) &= \frac{\psi}{\varphi}e^{-\frac{\psi-\varphi}{\varphi}t}\int_{\mu}^{t} e^{\frac{\psi-\varphi}{\varphi}y}\lambda\,\mathrm{d}y \\
&= \frac{\psi}{\varphi}\lambda e^{-\frac{\psi-\varphi}{\varphi}t}\left[\frac{\varphi}{\psi-\varphi}e^{\frac{\psi-\varphi}{\varphi}y}\right]\Big|_{\mu}^{t} \\
&= \frac{\psi}{\psi-\varphi}\lambda\left[1-e^{\frac{\psi-\varphi}{\varphi}(\mu-t)}\right]
\end{aligned}$$

6 Application to V-Ordinary Differential Equations of order (φ,ψ)

It is important to highlight that writing

$$V^{\varphi,\psi}g(t) = \frac{\varphi}{\psi}g'(t) + \frac{\psi-\varphi}{\psi}g(t)$$

makes it possible to convert V-ordinary differential equations of order (φ,ψ). In fact, if we consider the V-differential equation of order (φ,ψ).

$$V^{\varphi,\psi}(g(t)) + R(t)g(t) = 0$$

where $R(t)$ is continuous, then the equation can be written as

$$\frac{\varphi}{\psi}g'(t) + \frac{\psi-\varphi}{\psi}g(t) + R(t)g(t) = 0,$$

that is

$$\frac{\varphi}{\psi}Dz + \frac{\psi-\varphi}{\psi}z + Rz = 0,$$

which implies

$$Dz + \frac{(\psi-\varphi)+\psi Rz}{\varphi} = 0$$

which is a first-order differential equation, whose solution is

$$z = ce^{-\left(\frac{\psi-\varphi}{\psi}t+\frac{\psi}{\varphi}\int R(t)\,\mathrm{d}t\right)}$$

Example 5. Given the equation $V^{\frac{1}{4},\frac{1}{2}}z+z=te^{-t}$, to find the solution, it is necessary to

$$\frac{\frac{1}{4}}{\frac{1}{2}}z'+\frac{\frac{1}{2}-\frac{1}{4}}{\frac{1}{2}}z+z=te^{-t}$$

which implies,

$$z'+z+2z=2te^{-t}$$

then $z'+3z=2te^{-t}$ has the solution

$$z=ce^{-3t}+\left(t-\frac{1}{2}\right)e^{-t}$$

Example 6. For $V^{\frac{1}{9},\frac{1}{3}}z+3z=4te^{-5t}$, we have to

$$\frac{\frac{1}{9}}{\frac{1}{3}}z'+\frac{\frac{1}{3}-\frac{1}{9}}{\frac{1}{3}}z+3z=4te^{-5t}$$
$$\frac{1}{3}z'+\frac{2}{3}z+3z=4te^{-5t}$$
$$z'+2z+9z=12te^{-5t}$$
$$z'+11z=12te^{-5t}$$

Whose solution is

$$z=ce^{-11t}+\frac{12}{6}\left(t-\frac{1}{6}\right)e^{-5t}$$
$$z=ce^{-11t}+2\left(t-\frac{1}{6}\right)e^{-5t}.$$

7 Conclusion

In this article, a new generalized derivative has been proposed. The set of differentiable functions and the set of V-differentiable of order (φ,ψ) functions coincide; however, the values of the derivatives are different. Properties of the V-derivative of order (φ,ψ) were also studied, and an integral operator, which is the inverse of the V-derivative of order (φ,ψ), was introduced. This allows us to consider in the future the study of the partial V-derivative of order (φ,ψ), of integral transforms associated with the defined integral operator, to extend the definition of V-derivative of order (φ,ψ) to the field of complex numbers, among other potential works.

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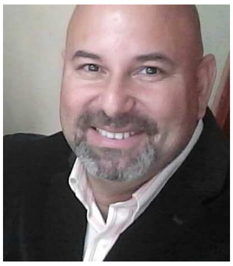

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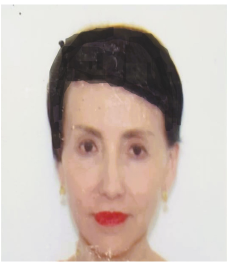

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