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Review on the Pseudo-Spectral Method and their Associated Differentiation and Integration Matrices of Both Integer and Fractional Order

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Abstract: This review concerns one of the spectral methods, which is the pseudo-spectral method. The pseudo-spectral method can be considered adequate for solving several types of IBVP of integer and fractional order. This method expresses the unknown variable of the differential equation in terms of the unknown variables at the chosen collocation points. Two traditional numerical integration techniques and the Gauss-Lobatto Quadrature method have been used for that purpose. In addition, several differentiation and integration matrices constructed using the pseudo-spectral method for some orthogonal polynomials have been presented.

Keywords: Chebyshev basis; Legendre basis; pseudo-spectral method; pseudo-spectral differentiation matrices; pseudo-spectral integration matrices; Gauss–Lobatto quadrature; Clenshaw–Curtis.

1 Introduction

Differential Equations (DEs) are important in many fields like physics [1, 2], engineering [3], chemistry [4, 5], fluid [6], nanofluid [7, 8], and biology [9, 10]. However, in several cases, the exact solution of differential equations is not obtained using specific methods. So, we use the numerical and the approximated methods to solve the differential equations. There are many numerical methods, such as finite element [11], finite difference [12], and runge kutta methods. However, the spectral techniques are used to get semi-analytic or approximated solutions. The spectral methods are used to find the approximate solutions in many applications as nanofluid flow [13], and three lakes pollution system [14]. As observed, these methods can deal with initial and boundary problems of fractional order [15] and integer order [16]. In this review, we will focus only on the pseudo-spectral method. The pseudo-spectral method was first introduced by El-gendi [17]. He introduced a matrix that was used as an integral operator and used it to solve differential, integral, and integro-differential equations via Chebyshev polynomials.

The idea of spectral methods is to expand the unknown function of the given differential problem into a finite sum of basis functions multiplied by unknown constants [18]. Then, substitute into the differential problem to convert it into an algebraic system of the unknown constants. Hence, solving that system to get the values of the constants and substituting them back into the finite expansion to get an approximate solution to the differential problem. However, in the pseudo-spectral method, the obtained algebraic system's unknowns are the values of the unknown function itself at chosen extreme points. So, we obtain the approximate solution directly.

The primary purpose of the pseudo-spectral method is to construct pseudo-spectral matrices for differentiation [19] and integration [20–22]. Besides the several advantages of the pseudo-spectral matrices, they can be used as differentiation and integration tools to get the derivatives and integrals of specific given functions. This aim can be achieved by expanding the unknown function of the given problem in terms of the unknown function itself at specific extreme points.

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Unfortunately, to the best of our knowledge, the pseudo-spectral method can be used only via orthogonal polynomials. This is due to the usage of the orthogonal relations of the used polynomials in the construction process. Consequently, the challenge is converting the inner product of the used orthogonal relations to the discrete inner product. Several authors used different approaches to calculate the desired relations, which will be presented throughout the review.

The rest of the review consists of the following: in section 2, we discuss the principle of the pseudo-spectral method. The transformation methodologies from definite integration into finite summation have been introduced in section 3. Two methodologies depend on regular numerical integration, and the third on Gauss–Lobatto quadrature. In section 4, the construction process of the pseudo-spectral matrices is presented. Problem formulation and the solution Methodology are shown in section 5. Section 6 is devoted to presenting several forms of pseudo-spectral matrices for differentiation and integration in the cases of integer and fractional order. Conclusion and future work directions are presented in section 7.

In the following section, the basic concepts of the pseudo-spectral method will be introduced.

2 Pseudo-spectral method

This section is devoted to explaining the main principle of the pseudo-spectral method. As known, the spectral expansion of a well-defined continuous function $\mathcal{G}(s)$ on the interval $[\mathrm{a}, \mathrm{b}]$ is [23]:

$$\mathcal{G}(s) = \sum_{n=0}^{\infty} \gamma_n \mathcal{Q}_n(s), \tag{1}$$

where $\{\mathcal{Q}_n(s)\}_{n\geq 0}$ is a set of orthogonal polynomials that defined on the interval $[\mathrm{a}, \mathrm{b}]$, and $\{\gamma_n\}_{n\geq 0}$ are unknown constants.

The above expansion can be approximated to $N+1$ to be:

$$\mathcal{G}(s) \approx \mathcal{G}_N(s) = \sum_{n=0}^{N} \gamma_n \mathcal{Q}_n(s). \tag{2}$$

Classically, the constants can be determined by multiplying both sides of Eq. (2) by $\mathcal{Q}_n(s)$, $\omega_{\mathcal{Q}}(s)$, and integrating over the interval $[\mathrm{a}, \mathrm{b}]$ to get:

$$\gamma_n = \frac{1}{\|\mathcal{Q}_n(s)\|^2} \int_{\mathrm{a}}^{\mathrm{b}} \mathcal{G}(s)\, \mathcal{Q}_n(\mathrm{s})\, \omega_{\mathcal{Q}}(s)\, \mathrm{d}s, \tag{3}$$

where $\|\mathcal{Q}_n(s)\|^2$ is the inner product that defined as:

$$\|\mathcal{Q}_n(s)\|^2 = \int_{\mathrm{a}}^{\mathrm{b}} (\mathcal{Q}_n(s))^2\, \omega_{\mathcal{Q}}(\mathrm{s})\, \mathrm{d}s, \tag{4}$$

such that $\omega_{\mathcal{Q}}(s)$ is the weight function of the orthogonal polynomials $\{\mathcal{Q}_n(s)\}_{n\geq 0}$, and $n = 0, 1, \cdots, N$.

However, in the case of differential problems, the function $\mathcal{G}_N(s)$ is an unknown function. Hence, the integration (3) can not be calculated, and the constants can not be determined.

The forthcoming section will present various methods to calculate the integral (3).

3 Weighted Residual methods

As mentioned above, the problem of the integration (3) is the unknown function $\mathcal{G}_N(s)$. The main purpose of the following is to convert that integration (3) to a finite summation. However, the function is still unknown. This process will expand the integration in terms of the unknown function at specific points $\mathcal{G}(s_n) = \mathcal{G}_n$; $n = 0, 1, \cdots, N$. The points $\{s_n\}_0^N$ will be defined and determined depending on the method used. We will introduce three approaches to converting the integral into finite summation in the three subsections.

3.1 Numerical Integration

In this subsection, any classical numerical method can be used. The authors in [16, 20] used the traditional trapezoidal rule:

$$\int_{a}^{b} \mathcal{F}(s)\,ds \approx h\sum_{n=0}^{N} \frac{\mathcal{F}(s_n)}{\theta_n}, \tag{5}$$

where $h = \frac{b-a}{N}$, and

$$\theta_n = \begin{cases} 1, \; n = 1, 2, \ldots, N-1, \\ 2, \qquad n = 0, N. \end{cases} \tag{6}$$

3.2 Clenshaw–Curtis quadrature formula

To reduce the truncation error caused by round-off, the authors in [23] developed a weighted method to enhance the value of the approximated integration as follows:

$$\int_{a}^{b} \mathcal{F}(s)\,ds \approx \sum_{n=0}^{N} \mathcal{W}_n \,\mathcal{F}(s_n), \tag{7}$$

where,

$$\mathcal{W}_0 = \mathcal{W}_N = \begin{cases} \frac{1}{N^2}, & N \text{ odd}, \\ \frac{1}{N^2-1}, & N \text{ even}, \end{cases} \tag{8}$$

$$\mathcal{W}_n = \frac{2}{N\theta_n}\left(1 - \sum_{i=1}^{[N/2]} \frac{2}{\theta_{2i}\left(4i^2-1\right)} \cos\frac{2ni\pi}{N}\right), \quad 1 \le n \le N-1, \tag{9}$$

and θ_n defined as in Eq. (6).

3.3 Gauss–Lobatto quadrature

The third type that will be presented is the Gauss–Lobatto quadrature. In this method, the integration will be expanded exactly to a finite summation [18]. First, the extreme collocation points of the used orthogonal polynomials are discussed. As defined in [18], the authors introduce the following polynomial:

$$\mathcal{Z}_{N-1}(s) = \frac{\mathcal{Q}_{N+1}(s) + \alpha_N \mathcal{Q}_N(s) + \beta_N \mathcal{Q}_{N-1}(s)}{(s-a)(b-s)}, \tag{10}$$

where α_N and β_N are the solution of the equations:

$$\mathcal{Q}_{N+1}(a) + \alpha_N \mathcal{Q}_N(a) + \beta_N \mathcal{Q}_{N-1}(a) = 0, \tag{11}$$

and

$$\mathcal{Q}_{N+1}(b) + \alpha_N \mathcal{Q}_N(b) + \beta_N \mathcal{Q}_{N-1}(b) = 0. \tag{12}$$

The Gauss–Lobatto quadrature points are the ends points a and b and the zeros of the polynomial $\mathcal{Z}_{N-1}(s)$, i.e.,:

$$\mathcal{Z}_{N-1}(s) = 0. \tag{13}$$

Also, Gauss–Lobatto quadrature points $\{s_n\}_0^N$ are the roots of the equation:

$$(s-a)\;(b-s)\;\mathcal{Q}'_N(s) = 0. \tag{14}$$

Finally, the integration can expanded as follows:

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathcal{F}(s)\mathcal{W}(s)\mathrm{d}s = \sum_{n=0}^{N} \mathcal{F}(s_n)\mathcal{W}_n; \qquad \forall \mathcal{F} \in \mathcal{P}_{2N-1}, \tag{15}$$

where $\{s_n\}_0^N$ are the Gauss–Lobatto quadrature points defined in (13), and

$$\mathcal{W}_n = \begin{cases} \frac{1}{(\mathrm{b}-\mathrm{a})\mathcal{Z}_{N-1}(\mathrm{a})} \int_{\mathrm{a}}^{\mathrm{b}} (\mathrm{b}-s)\, \mathcal{Z}_{N-1}(s)\omega_{\mathcal{Q}}(s)\mathrm{d}s, & n = 0, \\ \frac{\mathrm{k}_{N+1} \|\mathcal{Z}_{N-2}(s)\|^2_{\hat{\mathcal{W}}}}{(s_\mathrm{j}-\mathrm{a})(\mathrm{b}-s_\mathrm{j})\, \mathrm{k}_N\, \mathcal{Z}_{N-2}(s_\mathrm{j})\, \mathcal{Z}'_{N-1}(s_\mathrm{j})}, & 1 \le \mathrm{j} \le N-1, \\ \frac{1}{(\mathrm{b}-\mathrm{a})\mathcal{Z}_{N-1}(\mathrm{b})} \int_{\mathrm{a}}^{\mathrm{b}} (s-\mathrm{a})\, \mathcal{Z}_{N-1}(s)\omega_{\mathcal{Q}}(s)\mathrm{d}s, & n = N, \end{cases} \tag{16}$$

such that k_{N+1} is the leading coefficient of $\mathcal{Z}_{N-1}$, and

$$\hat{\mathcal{W}}(s) = (s-\mathrm{a})(\mathrm{b}-s)\omega_{\mathcal{Q}}(s). \tag{17}$$

Unfortunately, the degree of the unknown function, $\mathcal{G}(s)$, is not guaranteed to satisfy the condition of the integration (15). So, according to the definition of the discrete inner product:

$$\|\mathcal{Q}_n\|^2_{N,\mathcal{W}} = \sum_{i=0}^{N} (\mathcal{Q}_n(s_i))^2\, \mathcal{W}_i, \quad 0 \le n \le N. \tag{18}$$

Moreover, since the degree of the orthogonal polynomials, $\mathcal{Q}_n(s)$, is known, the discrete inner product can be determined in terms of the inner product itself (4) with the aid Eq. (15). Hence, by returning to the expansion (2) and summing it from the 0 to N after multiplying it by Gauss–Lobatto quadrature $\mathcal{W}n$ to get:

$$\gamma_n = \frac{1}{\|\mathcal{Q}_n\|^2_{N,\mathcal{W}}} \sum_{i=0}^{N} \mathcal{G}(s_i)\, \mathcal{Q}_n(s_i)\, \mathcal{W}_i. \tag{19}$$

After converting the integration into a finite summation, it is easy to setup the pseudo-spectral expansion. The following section will discuss pseudo-spectral expansion, problem formulation, and the solution method.

4 Pseudo-spectral matrices

In this section, we will start with theoretically constructing the pseudo-spectral expansion, which enables us to create the pseudo-spectral matrices. Consider the expansion (2) with the integration (3). Then, according to section 3, the integration (3) will be converted into:

$$\gamma_n = \sum_{i=0}^{N} \Lambda_{n\,i}\, \mathcal{G}(s_i)\, \mathcal{Q}_n(s_i), \tag{20}$$

where $\{\Lambda_{n\,i}\}_0^N$ are constants that are determined according to the methods described in Section 3.

Substitute from Eq. (20) into Eq. (2) to get the following:

$$\mathcal{G}(s) = \sum_{i=0}^{N} \Gamma_i(s)\mathcal{G}(s_i), \tag{21}$$

where

$$\Gamma_i(s) = \sum_{n=0}^{N} \Lambda_{n\,\mathrm{i}}\, \mathcal{Q}_n(s_i)\mathcal{Q}_n(s). \tag{22}$$

Differentiating (22) with respect to s to get:

$$\frac{\mathrm{d}^{\nu}\Gamma_i(s)}{\mathrm{d}s^{\nu}} = \sum_{n=0}^{N} \Lambda_{n\,i}\, \mathcal{Q}_n(s_i) \frac{\mathrm{d}^{\nu}\mathcal{Q}_n(s)}{\mathrm{d}s^{\nu}}, \tag{23}$$

where ν may be integer or fractional order.

In the fractional order case, the interval [a, b] must be shifted to [0, 1].

Differentiating Eq. (21), collocating it chosen $N+1$ points according to Section 3, and substituting from Eq. (23) to obtain:

$$\mathcal{G}^{(\nu)}(s_k) = \sum_{i=0}^{N} \delta_{k\,i}^{(\nu)} \mathcal{G}(s_i), \tag{24}$$

where:

$$\delta_{k\,i}^{(\nu)} = \left.\frac{\mathrm{d}^{\nu}\Gamma_i(s)}{\mathrm{d}s^{\nu}}\right|_{s=s_k}, \quad i,k = 0,1,\cdots,N. \tag{25}$$

The matrix form of Eq. (24) can be written as:

$$\mathcal{G}^{(\nu)}(\mathbf{s}) = \mathcal{D}^{(\nu)}\mathcal{G}(\mathbf{s}), \tag{26}$$

where

$$\mathcal{G}(\mathbf{s}) = \begin{pmatrix} \mathcal{G}(s_0) \\ \mathcal{G}(s_1) \\ \vdots \\ \mathcal{G}(s_N) \end{pmatrix}, \mathcal{G}^{(\nu)}(\mathbf{s}) = \begin{pmatrix} \mathcal{G}^{(\nu)}(s_0) \\ \mathcal{G}^{(\nu)}(s_1) \\ \vdots \\ \mathcal{G}^{(\nu)}(s_N) \end{pmatrix}, \mathcal{D}^{(\nu)} = \begin{pmatrix} \delta_{00}^{(\nu)} & \delta_{01}^{(\nu)} & \cdots & \delta_{0N}^{(\nu)} \\ \delta_{10}^{(\nu)} & \delta_{11}^{(\nu)} & \cdots & \delta_{1N}^{(\nu)} \\ \vdots & \cdots & \cdots & \vdots \\ \delta_{N0}^{(\nu)} & \delta_{N1}^{(\nu)} & \cdots & \delta_{NN}^{(\nu)} \end{pmatrix}, \tag{27}$$

and $\mathbf{s} = (\mathrm{s}_0, \mathrm{s}_1, \cdots, \mathrm{s}_N)^{\mathrm{T}}$.

The matrix $\mathcal{D}^{(\nu)}$ is called the pseudo-spectral differentiation matrix or the D-matrix.

A similar procedure can be achieved to get the pseudo-spectral integration matrix or the B-matrix. From Eq. (21):

$$\int_{\mathrm{a}}^{\mathrm{s}_k} \mathcal{G}(s)\,\mathrm{d}s = \sum_{i=0}^{N} \beta_{k\,i}\, \mathcal{G}(s_i), \tag{28}$$

where:

$$\beta_{k\,i} = \int_{\mathrm{a}}^{\mathrm{s}_k} \Gamma_i(s)\,\mathrm{d}s, \quad i,k = 0,1,\cdots,N. \tag{29}$$

While, the matrix form of Eq. (28) can be written as:

$$\int_{\mathrm{a}}^{\mathbf{s}} \mathcal{G}(s)\,\mathrm{d}s = \mathcal{B}\,\mathcal{G}(\mathbf{s}), \tag{30}$$

where

$$\int_{\mathrm{a}}^{\mathbf{s}} \mathcal{G}(s)\,\mathrm{d}s = \begin{pmatrix} \int_{\mathrm{a}}^{s_0} \mathcal{G}(s)\,\mathrm{d}s \\ \int_{\mathrm{a}}^{s_1} \mathcal{G}(s)\,\mathrm{d}s \\ \vdots \\ \int_{\mathrm{a}}^{s_N} \mathcal{G}(s)\,\mathrm{d}s \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \beta_{10} & \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \cdots & \cdots & \vdots \\ \beta_{N0} & \beta_{N1} & \cdots & \beta_{NN} \end{pmatrix}. \tag{31}$$

The above integration can be of fractional order, but the interval must also be considered.

The problem formulation and the solution method will be discussed in the next section.

5 Problem formulation and the solution Methodology

In this section, the application of the pseudo-spectral matrices will be introduced. It will be shown that the method is reliable and easy to apply. As a sample, an IBVP of integer order will be introduced.

We consider the following IBVP of integer order ν:

$$\mathbf{L}\left(s\,,\,\mathcal{G}(s)\,,\,\mathcal{G}'(s)\,,\,\mathcal{G}''(s),\cdots,\mathcal{G}^{(\nu)}(s)\right)=0,\qquad s\in[\mathrm{a},\mathrm{b}], \tag{32}$$

under the appropriate initial boundary conditions. The conditions can be Dirichlet, Neumann, or Robin boundary conditions.

Without any loss of generality, consider the following initial boundary conditions:

$$\begin{array}{ll} \mathcal{G}(\mathrm{a})=\mathrm{c}_{\mathrm{a}_0}, & \mathcal{G}(\mathrm{b})=\mathrm{c}_{\mathrm{b}_0},\\ \mathcal{G}'(\mathrm{a})=\mathrm{c}_{\mathrm{a}_1}, & \mathcal{G}'(\mathrm{b})=\mathrm{c}_{\mathrm{b}_1},\\ \vdots & \vdots\\ \mathcal{G}^{(\mathrm{n})}(\mathrm{a})=\mathrm{c}_{\mathrm{a}_\mathrm{n}}, & \mathcal{G}^{(\mathrm{m})}(\mathrm{b})=\mathrm{c}_{\mathrm{b}_\mathrm{m}}, \end{array} \tag{33}$$

where $\mathrm{c}_{\mathrm{a}_\mathrm{i}},\ \mathrm{c}_{\mathrm{b}_\mathrm{j}}\in\mathbb{R}$, $\mathrm{a}_\mathrm{i},\ \mathrm{b}_\mathrm{j}\in\{0\}\cup\mathbb{N}$.

Substitute form Eq. (26) into Eq. (32) and collocated it to get:

$$\mathbf{L}\left(\mathbf{s}\,,\,\mathcal{G}(\mathbf{s})\,,\,\mathcal{D}^{(1)}\mathcal{G}(\mathbf{s})\,,\,\mathcal{D}^{(2)}\mathcal{G}(\mathbf{s}),\cdots,\mathcal{D}^{(\nu)}\mathcal{G}(\mathbf{s})\right)=0. \tag{34}$$

While, the initial boundary conditions will be:

$$\begin{array}{ll} \sum_{i=0}^{N}\delta_{0\,i}^{(0)}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{a}_0}, & \sum_{i=0}^{N}\delta_{N\,i}^{(0)}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{b}_0},\\ \sum_{i=0}^{N}\delta_{0\,i}^{(1)}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{a}_1}, & \sum_{i=0}^{N}\delta_{N\,i}^{(1)}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{b}_1},\\ \vdots & \vdots\\ \sum_{i=0}^{N}\delta_{0\,i}^{(\mathrm{n})}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{a}_\mathrm{n}}, & \sum_{i=0}^{N}\delta_{N\,i}^{(\mathrm{m})}\,\mathcal{G}(s_i)=\mathrm{c}_{\mathrm{b}_\mathrm{m}}, \end{array} \tag{35}$$

The system of algebraic equations (34) and (35) will be solved to get the values of the unknown function $\mathcal{G}$ at the Gauss–Lobatto quadrature.

Similar steps can be performed to obtain the solution of the integral, intgro-differential equations of integer or fractional order.

The next section will present several forms of the pseudo-spectral matrices introduced by different researchers.

6 Pseudo-spectral Matrices Examples

Many researchers investigated different forms of pseudo-spectral matrices. These forms developed via several orthogonal polynomials and different techniques, as shown in Section 3. We collected some types of pseudo-spectral matrices and classified them using orthogonal polynomials.

6.1 *Chebyshev basis functions of the First Kind*

Herein, in this subsection, the first kind of Chebyshev basis will be introduced in several forms to construct the pseudo-spectral matrices.

The well-known Chebyshev polynomials of the first kind $\mathcal{T}_n(s)$ of degree n (CH-Ps) satisfy the recurrence relation [24]:

$$\mathcal{T}_{n+1}(s)=2s\mathcal{T}_n(s)-\mathcal{T}_{n-1}(s), \tag{36}$$

where $\mathcal{T}_0(s)=1$, $\mathcal{T}_1(s)=s$, $s\in[-1,1]$, and $n=1,2,3,\cdots$.

The CH-Ps are orthogonal polynomials under the relation:

$$\int_{-1}^{1}\mathcal{T}_n(s)\mathcal{T}_m(s)\frac{1}{\sqrt{1-s^2}}\mathrm{d}s=\begin{cases}0, & n\neq m,\\ \pi, & n=m,\\ \frac{\pi}{2}, & n=m\neq 1.\end{cases} \tag{37}$$

The CH-Ps Gauss–Lobatto quadrature points are:

$$s_n = \cos\frac{\pi n}{N}, \quad n = 0, 1, \cdots, N. \tag{38}$$

On the other hand, another orthogonal polynomial generated from CH-Ps called the monic Chebyshev polynomial (Monic-CH-P) is defined as [24]:

$$\mathcal{T}_n^{\mathcal{M}}(s) = \begin{cases} 1, & n = 0, \\ \frac{1}{2^{n-1}}\mathcal{T}_n(s), & n \in \mathbb{N}. \end{cases} \tag{39}$$

The Monic-CH-Ps satisfy the recurrence relation:

$$\mathcal{T}_n^{\mathcal{M}}(s) = s\mathcal{T}_{n-1}^{\mathcal{M}}(s) - \frac{1}{4}\mathcal{T}_{n-2}^{\mathcal{M}}(s), \quad n = 3, 4, \cdots, \tag{40}$$

with the initials $\mathcal{T}_1^{\mathcal{M}}(s) = s$ and $\mathcal{T}_2^{\mathcal{M}}(s) = s^2 - \frac{1}{4}$.

6.1.1 Chebyshev Pseudo-spectral B-matrix

In [17], the author used the principle of Clenshaw and Curtis [23] to approximate the function $\mathcal{G}(s)$ in terms of CH-Ps as follows:

$$\mathcal{G}(s) = \sum_{n=0}^{N} \frac{1}{\theta_n}\gamma_{\mathrm{n}}\, \mathcal{T}_n(s), \tag{41}$$

where θ_n as defined in (6),

$$\gamma_n = \frac{2}{\mathrm{N}} \sum_{i=0}^{N} \frac{1}{\theta_i}\mathcal{G}(s_i)\, \mathcal{T}_n(s_i), \tag{42}$$

and s_i are the CH-Ps Gauss–Lobatto quadrature points.

The elements of the Chebyshev pseudo-spectral B-matrix can be obtained by simplifying the coefficient of $\mathcal{G}(s_{\mathrm{n}})$ for the equation:

$$\int_{-1}^{s_k} \mathcal{G}(s)\, \mathrm{d}s = \sum_{i=0}^{N+1} C_i\, \mathcal{T}_i(s_k), \tag{43}$$

where

$$C_i = \begin{cases} \gamma_0 - \frac{1}{4}\gamma_1 + \sum_{j=2}^{N} \frac{(-1)^{j+1}}{\theta_j\,(\mathrm{j}^2-1)}\gamma_j, & i = 0, \\ \frac{\gamma_{i-1}-\gamma_{i+1}}{2i}, & i = 1, 2, \cdots, N-2, \\ \frac{\gamma_{N-2}-0.5\gamma_N}{2(N-1)}, & i = N-1, \\ \frac{\gamma_{N-1}}{2N}, & i = N, \\ \frac{0.5\gamma_N}{2(N+1)}, & i = N+1, \end{cases} \tag{44}$$

such that γ_n denied as in Eq. (42). Later, this method and its associated matrix are called El-gendi method [25].

6.1.2 CH-Ps Higher order pseudo-spectral D-matrix

The authors in [19], based on Clenshaw and Curtis [23] and an explicit formula for higher derivatives of Chebyshev polynomials, investigate the following higher-order D-matrix:

$$\delta_{k\,i}^{(\nu)} = \frac{2}{N\theta_i} \sum_{j=\nu}^{N} \sum_{\substack{n=0,\\ (n+j-\nu)\mathrm{even}}}^{j-\nu} \frac{1}{\theta_j}\mathrm{p}_{n,j}^{(\nu)}\mathcal{T}_j(s_i)\, \mathcal{T}_n(s_k), \tag{45}$$

where $\nu \in \mathbb{N}$, θ_n is defined in Eq. (6), and

$$\mathrm{p}_{n,j}^{(\nu)} = \frac{2^\nu\, j\, \mathrm{c}_n}{(\nu-1)!}\frac{(x-n+\nu-1)!(x+\nu-1)!}{x!\,(x-n)!}, \tag{46}$$

such that $2x = j + n - \nu$, and

$$\mathrm{c}_n = \begin{cases} \frac{1}{2}, & n = 0, \\ 1, & n = 1, 2, \cdots. \end{cases} \tag{47}$$

6.1.3 Monic-CH-Ps Higher order pseudo-spectral D-matrix

Another higher-order differentiation matrix was investigated in the [26] via Monic-CH-Ps (39). This matrix was constructed using Gauss–Lobatto quadrature principals. They proved and determined Gauss–Lobatto quadrature items as the following:

$$\mathcal{Z}_{N-1}(s) = \frac{\mathcal{T}^{\mathcal{M}}_{N+1}(s) - \frac{1}{4}\mathcal{T}^{\mathcal{M}}_{N-1}(s)}{1-s^2}, \tag{48}$$

and

$$\mathcal{W}_n = \frac{\pi}{\theta_n N}, \tag{49}$$

where θ_n as defined in Eq. (6). The elements of that matrix can be considered as follows:

$$\delta^{(\nu)}_{k\,i} = \sum_{j=\nu}^{N} \sum_{\substack{n=0,\\ (n+j-\nu)\text{even}}}^{j-\nu} \frac{2^{2j-1} c_j}{N\,\theta_i} \mathrm{p}^{(\nu)}_{n,j} \mathcal{T}^{\mathcal{M}}_j(s_k)\, \mathcal{T}^{\mathcal{M}}_n(s_i), \tag{50}$$

where $\nu \in \mathbb{N}$,

$$c_j = \begin{cases} 2, \; j = 0, \\ 1, \; j = 1, 2, \cdots, N, \\ \frac{1}{2}, \; j = N, \end{cases} \tag{51}$$

and

$$\mathrm{p}^{(\nu)}_{n,j} = \frac{2^{2n-2x}\, j}{(\nu-1)!} \frac{(x-n+\nu-1)!(x+\nu-1)!}{x!\,(x-n)!}, \tag{52}$$

such that $2x = j + n - \nu$.

6.1.4 Monic-CH-Ps pseudo-spectral B-matrix

The authors in [27] also use the monic-CH-Ps. However, they used them to generate the integration matrix. The B-matrix was created by Clenshaw and Curtis method. The following represents the elements of $\nu \in N$ successive integrations B-matrix:

$$\beta^{(\nu)}_{k\,i} = \frac{s_k - s_i}{(\nu-1)!}\beta_{k\,i}, \quad 0 \le k, i \le N, \tag{53}$$

$$\beta_{k\,j} = \frac{1}{N\theta_j}(s_k+1) + \frac{1}{N\theta_j} s_j(s_k^2+1) + \frac{1}{N\theta_j}\sum_{n=2}^{N} c_n 2^n \mathcal{T}^{\mathcal{M}}_n(s_j)$$
$$\left(\frac{\mathcal{T}_{n+1}(s_k)}{2(n+1)} - \frac{\mathcal{T}_{n-1}(s_k)}{2(n-1)} + \frac{(-1)^{n+1}}{n^2-1}\right), \qquad k, j = 0, 1, \ldots, N.$$

$$c_n = \begin{cases} 1, \; 0 \le n < N, \\ \frac{1}{2}, \; n = N, \end{cases}, \tag{54}$$

and θ_n is defined as in Eq.(6).

6.1.5 Shifted fractional CH-Ps pseudo-spectral differentiation matrix

The authors in [28] investigated a novel differentiation matrix of non-integer order. Therefore, according to the domain of the fractional derivatives operators, the shifted polynomials $\mathcal{T}^*_n(s)$; s $\in [0, 1]$, must be used instead of the normal ones $\mathcal{T}_n(s)$, where:

$$\mathcal{T}^*_n(s) = \mathcal{T}_n(2s-1), \tag{55}$$

such that $n \in \{0\} \cup \mathbb{N}$.

Hence, necessary reformulations for the relations also have been shifted [24].

The elements of the shifted fractional CH-Ps pseudo-spectral differentiation matrix of fractional order ν in the sense of Clenshaw and Curtis method can be represented as:

$$\delta_{kj}^{(\nu)} = \frac{2}{N\theta_j}\sum_{n=0}^{N}\frac{1}{\theta_n}\mathcal{T}_n^*(s_j)\mathcal{T}_n^{*\,(\nu)}(s_k), \tag{56}$$

where θ_n as defined in Eq. (6), $s_n = \frac{1}{2}(1+\cos\frac{\pi n}{N})$, and:

$$\mathcal{T}_n^{*\,(\nu)}(s) = \begin{cases} n\displaystyle\sum_{\mathrm{k}=\lceil\nu\rceil}^{n}\frac{(-1)^{n-\mathrm{k}}2^{2\mathrm{k}}\,(n+\mathrm{k}-1)!\,\mathrm{k}!}{(n-\mathrm{k})!\,(2\mathrm{k})!\,\Gamma(\mathrm{k}+1-\nu)}s^{k-\nu}, & n=\lceil\nu\rceil\,\lceil\nu\rceil+1,\cdots,N,\\ 0, & n=0,1,\cdots,\lceil\nu\rceil-1, \end{cases} \tag{57}$$

such that $\lceil\nu\rceil$ is the smallest integer that grater than ν.

The forthcoming subsection is devoted to presenting the pseudo-spectral matrices that are constructed via Legendre basis functions.

6.2 Legendre basis functions

In this subsection, several pseudo-spectral matrices for both integration and differentiation will be introduced via different approaches. Those matrices are constructed via Legendre polynomials (L-Ps) as basis functions. Furthermore, the derivatives of the Legendre polynomials will also be introduced as basis functions.

The L-Ps, $\mathcal{L}_{n+1}(s)$ of degree n, that defined on the interval $[-1,1]$ and its derivatives can be determined via the recurrence relation as follow [18]:

$$\mathcal{L}_{n+1}(s) = \frac{2n+1}{n+1}s\mathcal{L}_n(s) - \frac{2n}{n+1}\mathcal{L}_{n-1}(s), \tag{58}$$

and

$$\mathcal{L}_{n+1}(s) = \frac{1}{2n+1}\left(\mathcal{L}'_{n+1}(s) - \mathcal{L}'_{n-1}(s)\right), \tag{59}$$

such that $\mathcal{L}_0(s) = 1$, $\mathcal{L}_1(s) = s$ and $n\in\mathbb{N}$.

L-Ps and their derivatives are orthogonal polynomials according to the relations:

$$\int_{-1}^{1}\mathcal{L}_n(s)\mathcal{L}_m(s)\mathrm{d}s = \begin{cases} 0, & n\neq m,\\ \frac{2}{2n+1}, & n=m, \end{cases} \tag{60}$$

$$\int_{-1}^{1}\mathcal{L}'_n(s)\mathcal{L}'_m(s)(1-s^2)\mathrm{d}s = \begin{cases} 0, & n\neq m,\\ \frac{2n(n+1)}{2n+1}, & n=m. \end{cases} \tag{61}$$

Several relations and properties can be found in ([18]).

In the next subsections, we will present some pseudo-spectral matrices for the integration and differentiation of integer and non-integer orders.

6.2.1 L-Ps pseudo-spectral higher-order differentiation matrix

In [16], the authors used the trapezoidal rule for the integration and the explicit form for higher derivatives of the polynomial to get the following differentiation matrix of integer order ν:

$$\delta_{k,j}^{(\nu)} = \sum_{l=\nu}^{N}\sum_{r=0}^{[(j-\nu)/2]}\frac{2l+1}{N}\mathcal{L}_l(s_k)a_{r,\nu}^{(l)}s_j^{l-2r-\nu}, \tag{62}$$

where

$$a_{r+1,\nu}^{(l)} = -\frac{(l-2r-\nu)(l-2r-\nu-1)}{2(r+1)(2l-2r-1)}a_{r,\nu}^{(l)}, \tag{63}$$

and

$$a_{0,\nu}^{(l)} = \frac{(2l)!}{2^l l!(l-\nu)!}. \tag{64}$$

6.2.2 Shfited fractional L-Ps pseudo-spectral differentiation matrix

Herein, a fractional matrix will be presented. Therefore, the basis functions must first be shifted to be compatible with the domain of the fractional operators. Thus:

$$\mathcal{L}_n^*(s) = \mathcal{L}_n(2s-1), \quad n \in \{0\} \cup \mathbb{N}. \tag{65}$$

Accordingly, the recurrence relations, orthogonal relations, and properties must be modified.

Using the shifted functions, the authors in [29] constructed the shifted fractional L-Ps pseudo-spectral differentiation matrix in the sense of Gauss-Lobatto Quadrature. Consequently, they calculated the shifted Legendre Gauss-Lobatto Quadrature as follows:

$$\mathcal{Z}_{N-1}(s) = \frac{\mathcal{L}_{N+1}^*(s) - \mathcal{L}_{N-1}^*(s)}{s(s-1)}, \tag{66}$$

and

$$\mathcal{W}_n = \frac{1}{N(N+1)\left[\mathcal{L}_N^*(s_n)\right]^2}, \tag{67}$$

where $s_n = 2t_n - 1$ are the shifted Legendre Gauss-Lobatto Quadrature points, such that Legendre Gauss-Lobatto Quadrature points, t_n, are the solution of:

$$\left(1-t^2\right)\mathcal{L}_N(t) = 0. \tag{68}$$

Hence, the elements of the fractional D-matrix of order ν will be:

$$\delta_{kj}^{(\nu)} = \frac{1}{[\mathcal{L}_N^*(s_j)]^2} \sum_{n=0}^{N} \theta_n \mathcal{L}_n^*(s_j) \mathcal{L}_n^{*(\nu)}(s_k), \tag{69}$$

where

$$\theta n = \begin{cases} \frac{2n+1}{N(N+1)}, & n = 0, 1, 2, \dots, N-1, \\ \frac{1}{N+1}, & n = N, \end{cases} \tag{70}$$

and

$$\mathcal{L}_n^{*\,(\nu)}(s) = \begin{cases} \sum\limits_{\mathrm{k}=\lceil\nu\rceil}^{n} \frac{(-1)^{n+k}(n+k)!}{(n-k)!k!\,\Gamma(k+1-\nu)} s^{k-\nu}, & n = \lceil\nu\rceil\ \lceil\nu\rceil+1, \cdots, N, \\ 0, & n = 0, 1, \cdots, \lceil\nu\rceil - 1. \end{cases} \tag{71}$$

6.2.3 Shfited fractional L-Ps pseudo-spectral integration matrix

The author here used the same shifted L-Ps and the same approach, which is shifted Legendre Gauss-Lobatto Quadrature. However, the pseudo-matrix is an integration matrix of fractional order ν [30]. The elements of the matrix are:

$$\beta_{kj}^{(\nu)} = \frac{1}{[\mathcal{L}_N^*(s_j)]^2} \sum_{n=0}^{N} \theta_n \mathcal{L}_n^*(s_j)\, \mathcal{I}^{(\nu)} \mathcal{L}_n^*(s_k), \tag{72}$$

where

$$\mathcal{I}^{(\nu)} \mathcal{L}_n^*(s) = \sum_{\mathrm{k}=\lceil\nu\rceil}^{n} \frac{(-1)^{n+k}(n+k)!}{(n-k)!k!\,\Gamma(k+1+\nu)} s^{k+\nu}. \tag{73}$$

6.2.4 pseudo-spectra matrices via first derivative Legendre polynomials (FDLPs)

Elbarbary first introduced the derivatives of the orthogonal polynomials as basis functions in the spectral expansion, and he has since introduced the derivatives of the Chebyshev polynomials [31]. Hence, the authors in [32] assigned the derivative of the L-Ps as new basis orthogonal polynomials. Thus, consider the derivative of $\mathcal{L}_{n+1}(s)$ to be:

$$\mathcal{L}'_{n+1}(s) = \frac{\mathrm{d}\,\mathcal{L}_{n+1}(s)}{\mathrm{d}\,s}. \tag{74}$$

Furthermore, they determined that the first derivative of Legendre Gauss-Lobatto quadrature as follow:

$$\mathcal{Z}_{N-1}(s) = \frac{\mathcal{L}'_{N+2}(s) - \frac{(N+2)(N+3)}{N(N+1)}\mathcal{T}^{\mathcal{M}}_{N-1}(s)}{1-s^2}, \tag{75}$$

and

$$\mathcal{W}_n = \frac{N(N+1)(N+2)}{(N+3)\left[\mathcal{L}''_N(s_n)\right]^2}\mathcal{E}_n, \tag{76}$$

$$\mathcal{E}_n = \begin{cases} \frac{(N-1)^2}{4}, & n = 0\,, N, \\ 2, & 0 < n < N. \end{cases} \tag{77}$$

In addition, the authors investigated two pseudo-matrices. The first one is for integer differentiation of order ν, whose elements are:

$$\delta^{(\nu)}_{kj} = \sum_{n=0}^{N} \epsilon_n\,\mathcal{W}_j\,\mathcal{L}'_{n+1}(s_j)\,\mathcal{L}^{(\nu+1)}_{n+1}(s_k), \quad k,j = 0,1,\ldots,N. \tag{78}$$

On the other hand, the elements of the first derivative Legendre pseudo-spectral integration matrix (FDLPs B-Matrix) are:

$$\beta_{kj} = \sum_{n=0}^{N} \epsilon_n\,\mathcal{W}_j\,\mathcal{L}'_{n+1}(s_j)[\mathcal{L}_{n+1}(s_k) + (-1)^n], \quad k,j = 0,1,\ldots,N, \tag{79}$$

where

$$\epsilon_n = \begin{cases} \frac{(2n+3)}{2(n+1)(n+2)}, & n = 0,1,2,\ldots,N-1, \\ \\ \frac{1}{\sum_{i=0}^{N}[\mathcal{L}'_{N+1}(s_i)]^2 \mathrm{W}_i}, & n = N, \end{cases} \tag{80}$$

and s_n are the first derivative Legendre Gauss-Lobatto quadrature, which are the zeros of:

$$\left(1-s^2\right)\mathcal{L}''_{N+1}(s) = 0. \tag{81}$$

6.3 Mixed Pseudospectral Chebyshev and Legendre differentiation matrices

As the best of our knowledge, all pseudo-spectral matrices via Gauss-Lobatto quadrature are built via orthogonal polynomials and one kind of polynomial for each matrix. However, the authors [33] mixed two polynomials, CH-Ps and L-Ps, to generate another one. They developed polynomials whose even terms are CH-Ps and whose odd terms are L-Ps, i.e.:

$$\Phi_n(x) = \begin{cases} \mathcal{T}_n(s)\,, & n = 2m, \\ \mathcal{L}_n(s)\,, & n = 2m+1, \end{cases} \quad m = 0,1,\cdots. \tag{82}$$

These polynomials are called (CH-L)Ps.

Other conjugate polynomials, (L-CH)Ps, are created by assigning the even terms for L-Ps and the odd terms for (CH-L)Ps to be:

$$\Psi_n(x) = \begin{cases} \mathcal{L}_n(s)\,, & n = 2m, \\ \mathcal{T}_n(s)\,, & n = 2m+1, \end{cases} \qquad m = 0, 1, \cdots . \tag{83}$$

While their recurrence relations are:

$$\begin{aligned} \Phi_n(s) = {} & \left[2(2s^2-1) + \delta_{g1}\left(\frac{-4n+3}{n(n-1)}s^2 + \frac{4n^2-14n+9}{n(n-1)(2n-5)}\right)\right]\Phi_{n-2}(s) \\ & - \left[1 - \delta_{g1}\frac{4n^2-12n+6}{n(n-1)(2n-5)}\right]\Phi_{n-4}(s), \qquad n = 4, 5, \ldots, N, \end{aligned} \tag{84}$$

where g =GCD $(n, 2)$, $\Phi_0(s) = 1$, $\Phi_1(s) = s$, $\Phi_2(s) = 2s^2 - 1$ and $\Phi_3(s) = \frac{1}{2}(5s^3 - 3s)$, and

$$\begin{aligned} \Psi_n(s) = {} & \left[2(2s^2-1) + \delta_{g2}\left(\frac{-4n+3}{n(n-1)}s^2 + \frac{4n^2-14n+9}{n(n-1)(2n-5)}\right)\right]\Psi_{n-2}(\mathrm{z}) \\ & - \left[1 - \delta_{g2}\frac{4n^2-12n+6}{n(n-1)(2n-5)}\right]\Psi_{n-4}(s), \qquad n = 4, 5, \ldots, N, \end{aligned} \tag{85}$$

where g =GCD $(n, 2)$, $\Psi_0(s) = 1$, $\Psi_1(s) = s$, $\Psi_2(s) = \frac{1}{2}(3s^2 - 1)$ and $\Psi_3(s) = 4s^3 - 3s$.

Unfortunately, the two polynomials are not orthogonal. However, they satisfy the following relations:

$$\int_{-1}^{1}\Phi_n(s)\Phi_m(s)\omega(s)\mathrm{d}s = \begin{cases} 0, & |n-m| = 2i-1\,, \quad \omega(s) = \dfrac{1}{\sqrt{1-s^2}} \text{ or } \omega(s) = 1\,, \\ \pi, & n = m = 0\,, \quad \omega(s) = \dfrac{1}{\sqrt{1-s^2}}\,, \\ \dfrac{\pi}{2}, & n = m = 2i\,, \quad \omega(s) = \dfrac{1}{\sqrt{1-s^2}}, \\ \dfrac{2}{2n+1}, & n = m = 2i-1\,, \quad \omega(s) = 1\,, \end{cases} \tag{86}$$

$$\int_{-1}^{1}\Psi_n(s)\Psi_m(s)\omega(s)\mathrm{d}s = \begin{cases} 0, & |n-m| = 2i+1\,, \quad \omega(s) = \dfrac{1}{\sqrt{1-s^2}} \text{ or } \omega(s) = 1\,, \\ \dfrac{2}{2n+1}, & n = m = 2i, \quad \omega(s) = 1\,, \\ \dfrac{\pi}{2}, & n = m = 2i+1, \quad \omega(s) = \dfrac{1}{\sqrt{1-s^2}}\,. \end{cases} \tag{87}$$

where $i = 1, 2, \cdots$.

As usual, the pseudo-spectral matrices are $N+1 \times N+1$. Nevertheless, in the case of the mixed polynomials, the matrices will be $2N+2 \times 2N+2$. In addition, two sets of Gauss-Lobatto quadrature points are used, Chebyshev $\{x_i\}_{i=0}^{N}$ and Legendre $\{y_i\}_{i=0}^{N}$ Gauss-Lobatto quadrature points.

6.3.1 CH-L pseudo-spectral differentiation matrices

Here, we will introduce the pseudo-spectral differentiation that was constructed by (CH-L)Ps, $\Phi_n(s)$. The derivative of the function $\mathcal{G}(x)$ at the Chebyshev Gauss-Lobatto quadrature points $\{x_i\}_{i=0}^{N}$ cab be expressed as:

$$\mathcal{D}\mathcal{G}(\mathbf{x}) = \left[\mathcal{D}1_{\mathrm{CH}}\right] \cdot \begin{bmatrix} \mathcal{G}(\mathbf{x}) \\ \mathcal{G}(\mathbf{y}) \end{bmatrix}, \tag{88}$$

where $\mathcal{D}\mathcal{G}(\mathbf{x}) = \left(\mathcal{G}'(x_0),\, \mathcal{G}'(x_1),\, \cdots,\, \mathcal{G}'(x_N)\right)^T$, $\mathcal{G}(\mathbf{x}) = (\mathcal{G}(x_0),\, \mathcal{G}(x_1),\, \cdots,\, \mathcal{G}(x_N))^T$,

$\mathcal{G}(\mathbf{y}) = (\mathcal{G}(y_0), \mathcal{G}(y_1), \cdots, \mathcal{G}(y_N))^T,$

$$\mathcal{D}1_{\text{CH}} = \left[\delta 11 \ \delta 12\right], \tag{89}$$

$$\delta 11 = \begin{pmatrix} \delta 11_{00} & \delta 11_{01} & \cdots & \delta 11_{0N} \\ \delta 11_{10} & \delta 11_{11} & \cdots & \delta 11_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ \delta 11_{N0} & \delta 11_{N1} & \cdots & \delta 11_{NN} \end{pmatrix}, \tag{90}$$

$$\delta 11_{k,j} = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{2}{\theta_{2n}\theta_{jN}} \Phi_{2n}(x_j)\Phi'_{2n}(x_k), \tag{91}$$

$$\delta 12 = \begin{pmatrix} \delta 12_{00} & \delta 12_{01} & \cdots & \delta 12_{0N} \\ \delta 12_{10} & \delta 12_{11} & \cdots & \delta 12_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ \delta 12_{N0} & \delta 12_{N1} & \cdots & \delta 12_{NN} \end{pmatrix}, \tag{92}$$

and

$$\delta 12_{k,j} = \sum_{n=1}^{\lceil N/2 \rceil} \frac{1}{r_{2n-1}} \frac{2}{N(N+1)\mathcal{L}_N^2(y_j)} \Phi_{2n-1}(y_j)\Phi'_{2n-1}(x_k), \tag{93}$$

such that:

$$\theta_n = \begin{cases} 2, \ n = 0, N, \\ 1, \ 0 < n < N, \end{cases} \quad \theta_n = \begin{cases} \frac{2}{N}, & n = N, \\ \frac{2}{2n+1}, & 0 \le n < N. \end{cases} \tag{94}$$

Another matrix can be achieved via (CH-L)Ps, $\Phi_n(s)$. The derivative of the function $\mathcal{G}(y)$ will be determined at the Legendre Gauss-Lobatto quadrature points $\{y_i\}_{i=0}^{N}$ according to the following matrix:

$$\mathcal{D}\mathcal{G}(\mathbf{y}) = \left[\mathcal{D}1_{\text{L}}\right] \cdot \begin{bmatrix} \mathcal{G}(\mathbf{x}) \\ \mathcal{G}(\mathbf{y}) \end{bmatrix}, \tag{95}$$

where $\mathcal{D}\mathcal{G}(\mathbf{y}) = \left(\mathcal{G}'(y_0), \mathcal{G}'(y_1), \cdots, \mathcal{G}'(y_N)\right)^T,$

$$\mathcal{D}1_{\text{L}} = \left[\delta 13 \ \delta 14\right], \tag{96}$$

$$\delta 13 = \begin{pmatrix} \delta 13_{00} & \delta 3_{01} & \cdots & \delta 13_{0N} \\ \delta 13_{10} & \delta 13_{11} & \cdots & \delta 13_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ \delta 13_{N0} & \delta 13_{N1} & \cdots & \delta 13_{NN} \end{pmatrix}, \tag{97}$$

$$\delta 13_{k,j} = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{2}{\theta_{2n}\theta_j N} \Phi_{2n}(x_j)\Phi'_{2n}(y_k), \tag{98}$$

$$\delta 14 = \begin{pmatrix} \delta 14_{00} & \delta 14_{01} & \cdots & \delta 14_{0N} \\ \delta 14_{10} & \delta 14_{11} & \cdots & \delta 14_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ \delta 14_{N0} & \delta 14_{N1} & \cdots & \delta 14_{NN} \end{pmatrix}, \tag{99}$$

$$\delta 14_{k,j} = \sum_{n=1}^{\lceil N/2 \rceil} \frac{1}{r_{2n-1}} \frac{2}{N(N+1)\mathcal{L}_N^2(y_j)} \Phi_{2n-1}(y_j)\Phi'_{2n-1}(y_k). \tag{100}$$

such that θ_n and r_n are s defined in (94).

6.3.2 L-CH pseudo-spectral differentiation matrix

Similar steps will be introduced to construct the two pseudo-spectral differentiation matrices that are constructed by (L-CH)Ps, $\Psi_n(s)$. The first matrix to find the derivative of the function $\mathcal{G}(x)$ at the Chebyshev Gauss-Lobatto quadrature points $\{x_i\}_{i=0}^{N}$ can be expressed as:

$$\mathcal{D}\mathcal{G}(\mathbf{x}) = \left[\mathcal{D}2_{\text{CH}}\right] \cdot \begin{bmatrix}\mathcal{G}(\mathbf{x})\\ \mathcal{G}(\mathbf{y})\end{bmatrix}, \tag{101}$$

where,

$$\mathcal{D}2_{\text{CH}} = \left[\delta 21\ \delta 22\right], \tag{102}$$

$$\delta 21_{k,j} = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{1}{r_{2n}} \frac{2}{N(N+1)\mathcal{L}_N^2(y_j)} \Psi_{2n}(y_j)\Psi'_{2n}(x_k), \tag{103}$$

$$\delta 22_{k,j} = \sum_{n=1}^{\lceil N/2 \rceil} \frac{2}{\theta_{2n-1}\theta_j N} \Psi_{2n-1}(x_j)\Psi'_{2n-1}(x_k). \tag{104}$$

While the derivative at Legendre Gauss-Lobatto quadrature points $\{y_i\}_{i=0}^{N}$ is:

$$\mathcal{D}\mathcal{G}(\mathbf{y}) = \left[\mathcal{D}2_{\text{L}}\right] \cdot \begin{bmatrix}\mathcal{G}(\mathbf{x})\\ \mathcal{G}(\mathbf{y})\end{bmatrix}, \tag{105}$$

where,

$$\mathcal{D}2_{\text{L}} = \left[\delta 23\ \delta 24\right], \tag{106}$$

$$\delta 23_{k,j} = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{1}{r_{2n}} \frac{2}{N(N+1)\mathcal{L}_N^2(y_j)} \Psi_{2n}(y_j)\Psi'_{2n}(y_k), \tag{107}$$

$$\delta 24_{k,j} = \sum_{n=1}^{\lceil N/2 \rceil} \frac{2}{\theta_{2n-1}\theta_j N} \Psi_{2n-1}(x_j)\Psi'_{2n-1}(y_k). \tag{108}$$

7 Conclusion

This review discusses the differentiation and integration Matrices developed using the pseudo-spectral method. Choosen trial basis functions have been selected, such as Chebyshev polynomials of the first kind and their shifted type, monic Chebyshev polynomials, Legendre polynomials, shifted Legendre polynomials and first derivatives of Legendre polynomials. Also, the mixed Chebyshev-Legendre and Legendre-Chebyshev polynomials have been presented. Consequently, some matrices have been presented for differentiation and integration in the case of integer order. Meanwhile, the shifted polynomials are used to create fractional-order matrices. The problem in the construction process is how to transform an integration into finite summation. Two methodologies are presented for this transformation. The first one depends on classical numerical integration. So, we delivered the trapezoidal rule and Clenshaw–Curtis quadrature formula. The second methodology is Gauss–Lobatto quadrature. As future work, the survey will be extended to cover other polynomials and methodologies rather than the Gauss–Lobatto quadrature, such as the Gauss-Radau quadrature.

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