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Four Very Peculiar Gersenne Type Sequences: Ball and Reverse Numbers

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Abstract: In this article, we introduce four distinct 10-Gersenne-type sequences with unique and intriguing properties. We present several noteworthy results related to these sequences. In one of them, the sequence includes numbers derived from Ball's algorithm, while in the other three, the terms are multiples of these numbers. Notably, in one sequence, the terms also appear as the reverse numbers of those in the first sequence, and a similar pattern is observed in the other two sequences as well.

Keywords: Gersenne numbers, Ball numbers, reverse numbers

1 Introduction

For any non-negative integer n, consider the Horadam sequence $\{h_n\}_{n\geq 0}$, which is defined by the second-order recurrence relation:

$$h_n = ph_{n-1} + qh_{n-2}, \quad \text{for all } n \geq 2, \tag{1}$$

where p and q are fixed integers, with initial terms $h_0 = a$ and $h_1 = b$. Introduced by Horadam [1,2], this sequence generalizes numerous sequences, with recurrence associated with the characteristic equation $x^2 - px - q = 0$. For further details and comprehensive results on the Horadam sequence, see [2,3].

Recently, the Gersenne sequence was studied in [4] as the class of Horadam type sequences represented by the recurrence Equation (1), where we take $p = k+1$ and $q = -k$, for some integers $k \geq 2$. So, the Gersenne sequence is given by the recurrence relation

$$GM_{(k,n)} = (k+1)GM_{(k,n-1)} - kGM_{(k,n-2)};\ n \geq 2\ , \tag{2}$$

with initial values $GM_{(k,0)} = a$ and $GM_{(k,1)} = b$, where a and b are fixed constant integers.

This recurrence (2) provides the basis for studying their properties and applications in number theory, when $k = 2$, $a = 0$ and $b = 1$,we have the classical Mersenne numbers $\{M_n\}_{n\geq 0} = \{0,1,3,7,15,31,63,127,255,\ldots\}$, which are catalogued as sequence $A000225$ in the OEIS [5]. The Mersenne numbers hold significant importance in the exploration of prime numbers, and various studies on this subject are extensively cited in the literature, see [6,7,8,9] and [10]. Again in recurrence (2), if we let $k = 10; a = 0$; and $b = 1$ in Equation (2) then the Gersenne (Horadam) sequence is specified in the repunit sequence. The Repunit numbers $\{r_n\}_{n\geq 0}$ are the terms of the sequence $\{0,1,11,111,1111,1111,\ldots\}$ where each term satisfies the homogeneous recursive formula $r_{n+1} = 11r_n - 10r_{n-1}$ for all $n \geq 1$ and $r_0 = 0$ and $r_1 = 1$, the sequence $A002275$ in OEIS [5]. Some works to explore the connections of the Repunit sequence $\{r_n\}_{n\geq 0}$ with a Horadam type sequence, see instance [3,11,12], along with the references provided therein.

In this study are presented four families of Gersenne type sequences, in which $k = 10$ in recurrence Equation (2), but with other initial values.

Definition 1. *The sequence of Gersenne-Ball numbers $\{GB_n\}n \geq 0$, the sequence of Gersenne-Ball-Reverse numbers $\{GR_n\}n \geq 0$, the sequence of Gersenne-Double-Ball numbers $\{GD_n\}n \geq 0$, and the sequence of Gersenne-Double-Ball-Reverse numbers $\{GE_n\}n \geq 0$ are given by the second-order recurrence relations and initial conditions, respectively, given by:*

$$GB_n = 11GB_{n-1} - 10GB_{n-2},\ \ GB_0 = 0,\ \ GB_1 = 99\ , \tag{3}$$

$$GR_n = 11GR_{n-1} - 10GR_{n-2},\ \ GR_0 = 0,\ \ GR_1 = 891\ , \tag{4}$$

$$GD_n = 11GD_{n-1} - 10GD_{n-2},\ \ GD_0 = 0,\ \ GD_1 = 198\ , \tag{5}$$

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$$\mathcal{E}_n = 11\mathcal{E}_{n-1} - 10\mathcal{E}_{n-2}, \ \ \mathcal{E}_0 = 0, \ \ \mathcal{E}_1 = 792\,, \quad (6)$$

for all integers $n \geq 2$.

To illustrate the Definition 1, the first seven elements of the Gersenne-Ball numbers are indicated in the following

$$0, 99, 1089, 10989, 109989, 1099989, 10999989, \ldots \ . \quad (7)$$

The first seven elements of the Gersenne-Ball numbers are given below

$$0, 891, 9801, 98901, 989901, 9899901, 98999901, \ldots \ .$$

The first seven terms of the Gersenne-Double-Ball sequence are as follows

$$0, 198, 2178, 21978, 219978, 2199978, 21999978, \ldots \ .$$

Finally, listed below are the first seven terms of the Gersenne-Double-Ball-Reverse sequence.

$$0, 792, 8712, 87912, 879912, 8799912, 87999912, \ldots \ .$$

We will refer to these four sequences as the Ball-type sequence.

The objective of this article is to investigate some connections between elements of these four sequences defined by equations (3), (4), (5) and (6). The present paper is organized as follows. After this introduction, Section 2 reviews the Binet formula for the Gersenne-type sequences given in Definition 1, and explores some results derived from Binet's formula for each of the sequences, which includes the fact that the sequences (4), (5) and (6) are multiples of the sequence (3). Furthermore, in Section 3, determines and analyzes the ordinary, exponential and Poisson generating functions for each of the sequences. Section 4 investigates the properties involving the concept of reverse number in pairs of sequences. More specifically, we prove that the sequences (3) and (4) are reverses, as well as (5) and (6). Finally, in Section 5 some prospects for future studies are pointed out.

2 Binet's Formula and Applications

Let be x_t a positive non-palindromic number with t digits, and for $t \geq 2$, the number of t digits obtained by reversing the position of the digits of x_t is called the reverse number of x_t and is denoted by x_t'. As defined in [13, 14, 15] and among others, the number $B \neq 0$ obtained from algorithm $B = (x_t - x_t') + (x_t - x_t')'$ is called the **Ball's number** with $x_t > x_t'$. All the elements in the list (7) can be obtained with results from algorithm B, as seen in [15, 16, 17].

We recall from [16] some auxiliary results for the Ball-type sequence.

Lemma 1.[16, *Proposition 16] For every integer $n \geq 2$ we obtain* $11 \cdot (10^n - 1) = 10\underbrace{99\ldots99}_{n-2 \text{ times}}89$.

Lemma 2.[16, *Proposition 18] For every integer $n \geq 2$ we obtain* $2 \cdot 11 \cdot (10^n - 1) = 21\underbrace{99\ldots99}_{n-2 \text{ times}}78$.

See that the Horadam-type characteristic equation associated with the Gersenne-Ball, Gersenne-Ball-Reverse, Gersenne-Double-Ball, and Gersenne-Double-Ball-Reverse sequence is

$$x^2 - 11x + 10 = 0\,, \quad (8)$$

and its real distinct roots are $x_1 = 10$ and $x_2 = 1$.

The Binet formula is presented in the next result.

Proposition 1.*(Binet's Formula) For all non-negative integers n, we have*

$$\mathcal{B}_n = 11 \cdot (10^n - 1)\,,$$

$$\mathcal{R}_n = 99 \cdot (10^n - 1)\,,$$

$$\mathcal{D}_n = 22 \cdot (10^n - 1)\,,$$

and

$$\mathcal{E}_n = 88 \cdot (10^n - 1)\,,$$

where $\{\mathcal{B}_n\}n \geq 0$ is the Gersenne-Ball sequence, $\{\mathcal{R}_n\}n \geq 0$ is the Gersenne-Ball-Reverse sequence, $\{\mathcal{D}_n\}n \geq 0$ is the Gersenne-Double-Ball sequence, and $\{\mathcal{E}_n\}n \geq 0$ is the Gersenne-Double-Ball-Reverse sequence.

The proof is immediate since the characteristic equation is (8).

It should be noted that the Proposition 1 is a particular case for the Gersenne-Ball-type sequence of Proposition 1 in [4].

Combining the Proposition 1 with the Lemmas 1 and 2, we get the next result.

Corollary 1.*For all non-negative integers $n \geq 2$, we have*

$$\mathcal{R}_n = 10\underbrace{99\ldots99}_{n-2 \text{ times}}89\,, \quad (9)$$

and

$$\mathcal{D}_n = 21\underbrace{99\ldots99}_{n-2 \text{ times}}78\,, \quad (10)$$

where $\{\mathcal{B}_n\}n \geq 0$ is the Gersenne-Ball sequence and $\{\mathcal{D}_n\}n \geq 0$ is the Gersenne-Double-Ball sequence.

The next is the main result of this section, and is an immediate consequence, by straightforward calculus, of the Proposition 1.

Theorem 1.*For all non-negative integers n, we have*

$$\mathcal{R}_n = 9 \cdot \mathcal{B}_n\,,$$

$$\mathcal{D}_n = 2 \cdot \mathcal{B}_n\,,$$

$$GE_n = 4 \cdot GD_n\ ,$$
$$GE_n = 8 \cdot GB_n\ ,$$

where $\{GB_n\}n \geq 0$ is the Gersenne-Ball sequence, $\{GR_n\}n \geq 0$ is the Gersenne-Ball-Reverse sequence, $\{GD_n\}n \geq 0$ is the Gersenne-Double-Ball sequence, and $\{GE_n\}n \geq 0$ is the Gersenne-Double-Ball-Reverse sequence.

The previous result establishes multiplicity relations between the terms of each sequence, so the sequences $\{GR_n\}n \geq 0$, $\{GD_n\}n \geq 0$ and $\{GE_n\}n \geq 0$ are multiples of the sequence $\{GB_n\}n \geq 0$; being $\{GE_n\}n \geq 0$ a multiple of $\{GB_n\}n \geq 0$ and $\{GD_n\}n \geq 0$.

The next two results will be useful to us later, and can be consulted on [16].

Lemma 3.*[16, Proposition 17] For every integer $n \geq 2$ we obtain* $9 \times 10\underbrace{99\ldots99}_{n-2\ \text{times}}89 = 98\underbrace{99\ldots99}_{n-4\ \text{times}}01$.

Lemma 4.*[16, Proposition 19] For every integer $n \geq 2$ we obtain* $4 \times 21\underbrace{99\ldots99}_{n-2\ \text{times}}78 = 87\underbrace{99\ldots99}_{n-2\ \text{times}}12$.

3 Generating Functions

Considering the literature, the following formal summation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

is called the ordinary generating function for the sequence $\{a_0, a_1, a_2, \ldots\} = \{a_n\}_{n \geq 0}$.

According to the established literature, the exponential generating function, denoted $E_{a_n}(x)$, of a sequence $\{a_n\}_{n \geq 0}$ is a power series of the form

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

In the next result we consider the Binet Equation (Proposition 1) and Theorem 1, and obtain the exponential generating function for the Gersenne-Ball and the Gersenne-Ball-Reverse sequence.

Proposition 2.*For all $n \geq 0$, the exponential generating function for the Gersenne-Ball sequence $\{GB_n\}n \geq 0$, the Gersenne-Ball-Reverse sequence $\{GR_n\}n \geq 0$, the Gersenne-Double-Ball sequence $\{GD_n\}n \geq 0$, and the Gersenne-Double-Ball-Reverse sequence $\{GE_n\}n \geq 0$, respectivelly, is*

$$E_{GB_n}(x) = 11(e^{10x} - e^x)\,,$$
$$E_{GR_n}(x) = 99(e^{10x} - e^x)\,,$$
$$E_{GD_n}(x) = 22(e^{10x} - e^x)\,.$$

and

$$E_{GE_n}(x) = 88(e^{10x} - e^x)\,.$$

The proof is immediate by Binet's formula.

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n \geq 0}$ is given by:

$$P_{a_n}(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} e^{-x} = e^{-x} E_{a_n}(x),$$

where $E_{a_n}(x)$ is the exponential generating function of the sequence $\{a_n\}_{n \geq 0}$. Consequently, the corresponding Poisson generating function is derived from Proposition 2.

Corollary 2.*For all $n > 0$ the Poisson generating function for the Gersenne sequence $\{GM_{(k,n)}\}_{n \geq 0}$ is*

$$P_{GB_n}(x) = \frac{11}{9}(e^{9x} - 1)\,,$$
$$P_{GR_n}(x) = 11(e^{9x} - 1)\,,$$
$$P_{GD_n}(x) = \frac{22}{9}(e^{9x} - 1)\,,$$
$$P_{GE_n}(x) = \frac{88}{9}(e^{9x} - 1)\,,$$

where $\{GB_n\}n \geq 0$ is the Gersenne-Ball sequence, $\{GR_n\}n \geq 0$ is the Gersenne-Ball-Reverse sequence, $\{GD_n\}n \geq 0$ is the Gersenne-Double-Ball sequence, and $\{GE_n\}n \geq 0$ is the Gersenne-Double-Ball-Reverse sequence.

Our next result presents the ordinary generating function for the Gersenne-Ball and the the Gersenne-Ball-Reverse sequence.

Proposition 3.*The ordinary generating function is given by*

$$G_{GB_{(k,n)}}(x) = \frac{99x}{1 - 11x + 10x^2}\ ,$$
$$G_{GR_{(k,n)}}(x) = \frac{891x}{1 - 11x + 10x^2}\ ,$$
$$G_{GD_{(k,n)}}(x) = \frac{198x}{1 - 11x + 10x^2}\ ,$$

and and

$$G_{GE_{(k,n)}}(x) = \frac{792x}{1 - 11x + 10x^2}\ ,$$

where $\{GB_n\}n \geq 0$ is the Gersenne-Ball sequence, $\{GR_n\}n \geq 0$ is the Gersenne-Ball-Reverse sequence, $\{GD_n\}n \geq 0$ is the Gersenne-Double-Ball sequence, and $\{GE_n\}n \geq 0$ is the Gersenne-Double-Ball-Reverse sequence.

It shall be observed that the proposition 3 is a particular case for the Gersenne-Ball-type sequence of Proposition 4 in [4].

To illustrate the last result, we'll show the formal calculation of the generating functions of two sequences. First, the following Figure 1 shows the development of the generating function for the Gersenne-Ball sequence, obtained with the help of the computational resource **Maxima**.

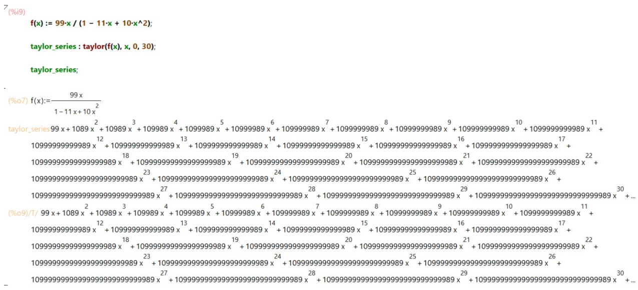

Fig. 1: Generation function GB_n

Now, the following Figure 2 shows the evolution of the generating function for the Gersenne-Ball-Reverse sequence, also obtained using the computational resource **Maxima**.

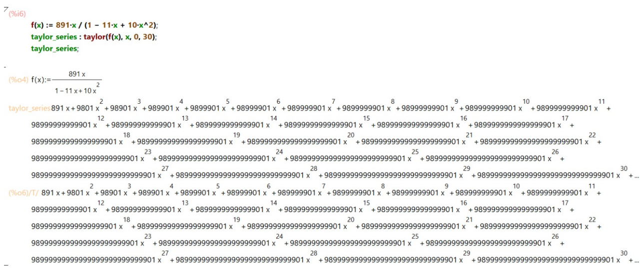

Fig. 2: Generation function GB_n

4 Reverse Sequence

Consider x_n let be a non-palindromic number with $n \geq 2$ digits. The number x_n is a reverse divisor, or reverse multiple, if x_n divides its reverse x'_n. We simply say that x_n and x'_n are reverse divisors.

According [13,16,18,19,20] and among other the numbers 1089 and 9891 are reverse divisors, since $9801 = 9 \cdot 1089$. Similarly, the numbers 10989 and 98901 are reverse divisors, because $98901 = 9 \cdot 10989$. Observe that the numbers 1089 and 10989 are terms of the sequence (3), whereas the numbers 9801 and 98901 are terms of the sequence (4).

In a similar way, following [16,19,20] and the references therein, the numbers 2078 and 8178 are reverse divisors, since $8712 = 4 \cdot 2178$. Similarly, the numbers 21978 and 87912 are reverse divisors, because $87912 = 4 \cdot 21972$. Now, observe that the numbers 2078 and 20978 are terms of the sequence (5), whereas the numbers 8702 and 87902 are terms of the sequence (6).

Definition 2. *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences with integer terms. We say that $\{a_n\}_{n\geq k}$ and $\{b_n\}_{n\geq k}$ are reverse sequences if $b_n = q \cdot a_n$ for all $n \geq k$ with $1 < q \leq 9$ an integer, that is, a_n and b_n are reverse number for all $n \geq k$.*

We are able to present the next result.

Theorem 2. *For all positive integers $n \geq 2$, the following identities hold:*
(a)the Gersenne-Ball sequence $\{Œ_n\}n \geq 2$ and the Gersenne-Ball-Reverse sequence $\{Œ_n\}n \geq 2$ are reverse sequences;
(b)the Gersenne-Double-Ball sequence $\{Œ_n\}n \geq 2$ and the Gersenne-Double-Ball-Reverse sequence $\{Œ_n\}n \geq 2$ are reverse sequences.

Proof.
(a)By combining Theorem 1 and Lemma 3 we have the result.
(a)In similar way, by combining Theorem 1 and Lemma 4.

5 Perspective

This paper was presented four types of Gersenne-Ball sequences, which have the same recurrence as the repunit or 10-Gersenne numbers, but with different initial values. This study aimed to introduce and to systematize the Gersenne-Ball numbers and derivations, namely, Gersenne-Ball-Reverse, Gersenne-Double-Ball and Gersenne-Double-Ball-Reverse numbers, and now we show that each of these numbers belongs to a Gersenne sequence for $k = 10$. We presented recurrence relations, homogeneous equations of these sequences, and their properties. Moreover, the generating functions, as well as the Binet formulas, were provided. It seems to that the results presented here are new in the literature, and these new sequences can be further explored so that we can determine new identities and their connections with other sequences in the literature. As pointed out in [19,20], the numbers in the sequence have unique properties: in base 10 they are all Ball numbers, as pointed out in [13,14,15], and there are only two pairs of 4-digit reverse numbers, namely 1089 and 9801, 2178 and 8712, as recorded in [16,19,20]. Finally, we must point out that none of these four sequences have any prime terms, a fact that is easy to conclude given Proposition 1 and Theorem 1. We believe it constitutes a contribution to the field of mathematics and offers an opportunity for researchers interested in number sequences and specific properties of these numbers to further their studies.

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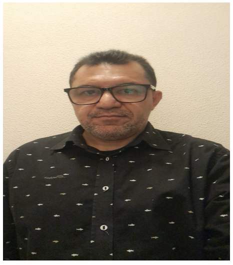

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