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On the Modified Alpha Power Transformed Kumaraswamy Distribution: Properties and Applications

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Abstract: In this paper, we investigate based on the alpha power transformation method, a lifetime distribution known as the modified alpha power transformation Kumaraswamy distribution (MAPTKD) has been examined. The modified alpha power transformed Kumaraswamy distribution is the new distribution's name. The newly created distribution has a various of statistical characteristics, including moments, entropy, order statistics, and the hazard rate function. The parameters have been estimated using the maximum likelihood estimation (MLE) approach. A simulation study was used to determine how the MLEs behaved. The efficiency and flexibility of the new distribution are demonstrated through the analysis of two real-life data sets.

Keywords: Modified alpha power transformation, Kumaraswamy distribution, entropy, Maximum likelihood estimation, Simulation.

1 Introduction

In practical applications, various measurements, including the proportion of specific features, scores from ability assessments, and various indicators and ratios, typically fall within the (0, 1) interval. In these instances, distributions with bounds are crucial for accurately modeling such phenomena. The Kumaraswamy distribution (KD) is particularly significant in the context of bounded distributions. Modeling and analyzing natural phenomena play a crucial role in statistical research across various practical fields, including engineering and science. In the span of three decades, significant efforts have been devoted to developing statistical models that more accurately represent the inherent characteristics of natural phenomena see [1]. Moreover, several new families of probability distributions have been introduced for modeling data across various disciplines, including hydrology, medical science, engineering, and insurance. These distributions offer enhanced flexibility and accuracy in capturing the underlying patterns and variability present in data from these fields. The KDs, including the log-normal, normal, and beta distributions, among others, have been found to provide suboptimal fit for hydrological data, such as daily streamflow and rainfall measurements. Furthermore, this model supports a wide range of applications, including the analysis of test scores, atmospheric temperature, individual height, and other related datasets [2,4,3]. The distribution is characterized by two shape parameters, $\eta > 0$ and $\nu > 0$, with the random variable defined on the interval [0, 1]. Recently, [5] derived the Bayesian estimation of entropy for the Kumaraswamy distribution and applied it to progressively first-failure censored data. Bagci et al. [6] derived the different estimation methods for estimated an alpha power inverted Kumaraswamy distribution. Usman and Haq [7] developed the theoretical framework and applications of the Marshall-Olkin extended inverted Kumaraswamy distribution.
The random variable Z has the two-parameter of KD, denoted by K(η, ν), if its probability density function (PDF) and cumulative distribution function (CDF), are respectively, given as

$$f(z) = \eta \nu z^{\nu-1}(1-z^{\nu})^{\eta-1}, \quad 0<z<1,\ \eta,\nu>0, \tag{1}$$

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and

$$F(z) = 1 - (1 - z^{\nu})^{\eta}; \qquad 0 < z < 1, \eta, \nu > 0. \tag{2}$$

Then, the corresponding reliability rate functions of this distribution at some t, is given by

$$R(t) = (1 - t^{\nu})^{\eta}, \quad 0 < t < 1, \tag{3}$$

as well as, the failure rate functions is

$$H(t) = \eta \nu t^{\nu - 1}(1 - t^{\nu})^{-1}, \quad 0 < t < 1. \tag{4}$$

The two-parameter of KD is unimodal for $\eta\nu > 1$, uniantimodal for $\eta\nu < 1$, increasing for $\eta \leq 1$ and $\nu > 1$, decreasing for $\eta > 1$ and $\nu \leq 1$ and constant for $\eta = \nu = 1$.

In recent years, number of studies have been carried out for the KD. For example, Tian et al. [8] used the methods of the likelihood ratio (LR) test, modified information criterion, and Schwarz information criterion to analyze the change point of the KD. Abo-Kasem et al. [9] derived the statistical inferences and optimal sampling for KD under progressive Type-II censoring schemes. Nanga et al. [10] proposed the secant Kumaraswamy family of distributions and studied the MLE method to obtain estimators of the family of distributions. Giles [11] proposed the new goodness-of-fit tests for the KD. Kumari et al. [12] introduced the reliability estimation for KD under block progressive Type-II censoring. Mahmoud and Saad [13] derived the maximum entropy approach to estimate the parameters of the KD subject to moment constraint.

In the recent past, many generalizations of KD have been studied by authors such as Ogunde et al. [14] proposed the newly developed distribution is called the KGILo distribution, The MLE technique is used to obtain an estimate of the parameters of the new model. Tharu et al. [15] proposed the statistical characteristics of Kumaraswamy uniform distribution and derived the MLE of its model parameters.

On the flip side, Alotaibi et al. [16] proposed a modification to the alpha power distribution and called Modified alpha power transformation (MAPT).
If $F(z)$ be a CDF of any distribution, then the $G_{MAPT}(z)$ is CDF of MAPT expressed by

$$G_{MAPT}(z) = \begin{cases} \frac{\alpha^{F(z)} - 1}{(\alpha - 1)(1 + \alpha - \alpha^{F(z)})}, & \alpha > 0, \alpha \neq 1 \\ F(z), & \alpha = 1, \end{cases} \tag{5}$$

and the corresponding PDF as

$$g_{MAPT}(z) = \begin{cases} \frac{\alpha^{1 + F(z)} \ln(\alpha) f(z)}{(\alpha - 1)(1 + \alpha - \alpha^{F(z)})^2}, & \alpha > 0, \alpha \neq 1 \\ f(z), & \alpha = 1. \end{cases} \tag{6}$$

The main objectives of this study are to contribute to the statistical literature and address some issues about the components for various applications of the new model of the MAPT family. The following reasons are sufficient justification for doing so:

1-Introducing the modified alpha power transformed Kumaraswamy distribution as a novel three-parameter model based on the MAPT family of distributions.
2-The new suggested model is very flexible and it has three sub-models.
3-It is possible to compute several statistical features, including the moments, moment generating function, characteristics function and order statistics model, and so on.
4-The parameters of the MAPTKD can be estimated by utilizing MLE method.

The following sections of the document are organized as follows: In Section 2, we introduce MAPTKD, reliability, hazard and reversed hazard functions. In Section 3, The MAPTK distribution's useful probability properties are discussed. The performance of MLE method is discussed through simulation study in Section 4. The MAPTKD is implemented on a real data set and the results are presented in Section 5.

2 Modified Alpha Power Transformed Kumaraswamy Distribution

By inserting the CDF of KD given by (2) in the CDF of MAPT distribution given by (5), we get the CDF of a new distribution denoted as MAPTKD $(z;\alpha,\eta,\nu)$ given by

$$G_{MAPTK}(z)=\begin{cases}\dfrac{\alpha^{1-(1-z^{\nu})^{\eta}}-1}{(\alpha-1)\left(1+\alpha-\alpha^{1-(1-z^{\nu})^{\eta}}\right)}, & \alpha>0,\alpha\neq 1\\ 1-(1-z^{\nu})^{\eta}, & \alpha=1,\end{cases} \tag{7}$$

its corresponding PDF is given by

$$g_{MAPTK}(z)=\begin{cases}\dfrac{\alpha^{2-(1-z^{\nu})^{\eta}}\ln(\alpha)\eta\nu z^{\nu-1}(1-z^{\nu})^{\eta-1}}{(\alpha-1)\left(1+\alpha-\alpha^{1-(1-z^{\nu})^{\eta}}\right)^{2}}, & \alpha>0,\alpha\neq 1\\ \eta\nu z^{\nu-1}(1-z^{\nu})^{\eta-1}, & \alpha=1.\end{cases} \tag{8}$$

The linear representation for the PDF of MAPTKD is obtained by

$$g_{MAPTK}(z)=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\Upsilon_{m,j}f(z;(1+j)\eta,\nu), \tag{9}$$

where $f(z;(1+j)\eta,\nu)=(1+j)\eta\nu z^{\nu-1}(1-z^{\nu})^{(1+j)\eta-1}$ is the PDF of the KD with two shape parameters $(1+j)\eta$ and ν, and

$$\Upsilon_{m,j}=\sum_{k=0}^{\infty}\frac{(-1)^{j}\alpha(\ln\alpha)^{m+1}(1+k)^{m+1}}{(\alpha-1)(\alpha+1)^{k+2}(m-j)!(1+j)!}.$$

Figure 1 includes a variety of shapes for the PDF of the MAPTKD.

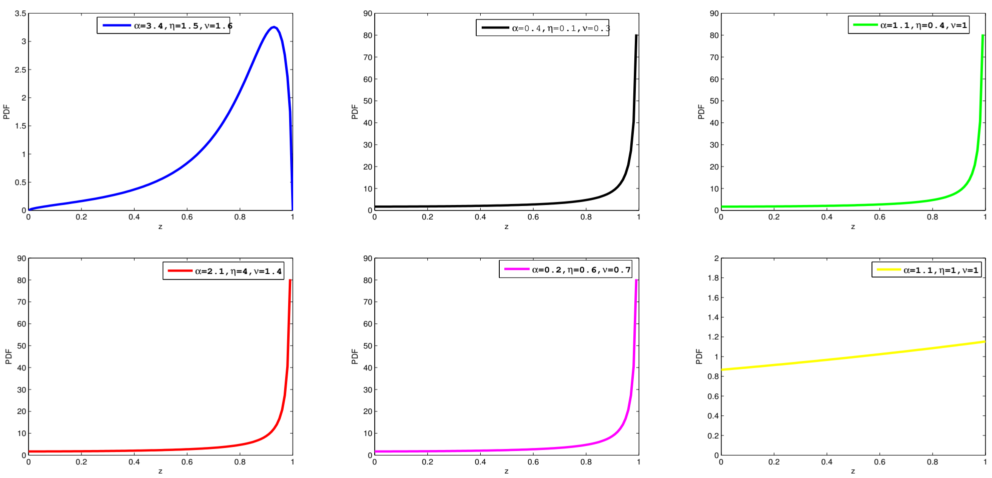

Fig. 1: Different shapes from PDF of MAPTKD for some parameters.

The submodels of the MAPTKD are given in Table 1.

Table 1: Sub-models of the MAPTKD (α, η, ν).

Models	parameters		
Modified alpha power function (MAPF)	α	1	ν
power function (PF)	1	1	ν
Kumaraswamy (K)	1	η	ν

The reliability function (RF) of MAPTKD is determined by

$$R_{MAPTK}(z) = 1 - \frac{\alpha^{1-(1-z^{\nu})^{\eta}} - 1}{(\alpha-1)(1+\alpha-\alpha^{1-(1-z^{\nu})^{\eta}})}. \tag{10}$$

The hazard rate function of MAPTKD is determined by

$$h_{MAPTK}(z) = \frac{\eta\nu\ln(\alpha)z^{\nu-1}(1-z^{\nu})^{\eta-1}\alpha^{1-(1-z^{\nu})^{\eta}}}{(1+\alpha-\alpha^{1-(1-z^{\nu})^{\eta}})(\alpha-\alpha^{1-(1-z^{\nu})^{\eta}})}. \tag{11}$$

Figure 2 includes a variety of shapes for the HRF of the MAPTKD.

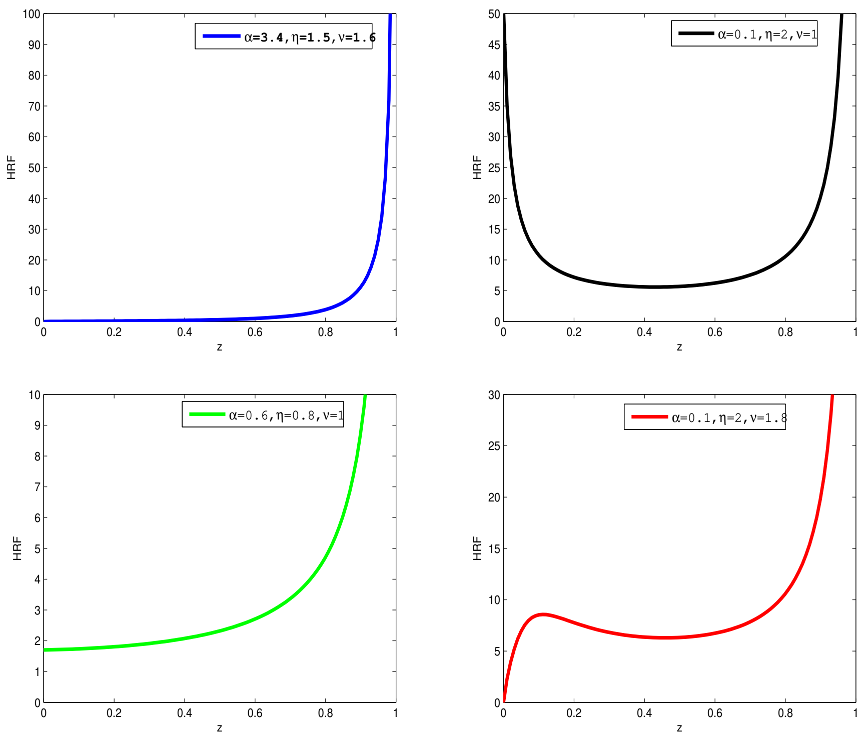

Fig. 2: plot of the HRF of the MAPTKD for some values of parameters.

The reversed hazard rate of MAPTKD is described as follows

$$r_{MAPTK}(z) = \frac{\eta\nu\ln(\alpha)z^{\nu-1}(1-z^{\nu})^{\eta-1}\alpha^{2-(1-t^{\nu})^{\eta}}}{(1+\alpha-\alpha^{1-(1-t^{\nu})^{\eta}})(\alpha^{1-(1-t^{\nu})^{\eta}}-1)}. \tag{12}$$

3 Statistical characteristics

Within this part, we will derive the quantile function, moments, R*é*nyi entropy and order statistics of the MAPTKD.

3.1 Quantile function

The quantile function of any distribution is determined by solving (13)

$$G(z_u) = u, \qquad 0 < u < 1. \tag{13}$$

The quantile function for the MAPTKD can be determined using (7) as follows

$$z_u = \left\{1 - \left[1 - \frac{\ln\left(\frac{u(\alpha^2-1)+1}{u(\alpha-1)+1}\right)}{\ln \alpha}\right]^{\frac{1}{\eta}}\right\}^{\frac{1}{\nu}}, \qquad 0 < u < 1. \tag{14}$$

By substituting u = 0.5 in (14), the median of the MAPTKD can be acquired as follows:

$$M = \left\{1 - \left[1 - \frac{\ln(\frac{\alpha^2+1}{\alpha+1})}{\ln \alpha}\right]^{\frac{1}{\eta}}\right\}^{\frac{1}{\nu}}. \tag{15}$$

3.2 Moments

The r^{th} moments of the MAPTKD is determined by

$$\begin{aligned} E(Z^r) = \int_0^1 z^r g(z) dz &= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \Upsilon_{m,j} \int_0^1 z^r f(z;(1+j)\eta,\nu) dz \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \Upsilon_{m,j} (1+j)\eta B((1+j)\eta, \frac{r}{\nu}+1), \end{aligned} \tag{16}$$

where B(.,.) is a beta function.
The inverse moment (IM) of the MAPTKD is derived from replacing (r) with $(-r)$ in the (16).
Table 2 provides the evaluations of median, mean, variance, coefficient of variation(C.V.), skewness and kurtosis of MAPTKD for some values of parameters.

Table 2: Median and some moments of MAPTKD for some values of parameters α, η and ν.

α	η	ν	Median	Mean	Variance	C.V.	Skewness	Kurtosis
0.8	0.5	0.9	0.6305	0.5874	0.1003	0.5392	-0.2969	1.7442
		1.4	0.7434	0.6773	0.0762	0.4077	-0.6244	2.2286
	1.2	0.9	0.3236	0.3738	0.0752	0.7339	0.4891	2.1146
		1.4	0.4842	0.4919	0.0689	0.5339	0.0695	1.9405
1.5	0.5	0.9	0.8625	0.7452	0.0781	0.3751	-1.0813	3.0049
		1.4	0.9093	0.8075	0.0540	0.2877	-1.4277	4.1974
	1.2	0.9	0.5461	0.5261	0.0796	0.5363	-0.1608	1.8639
		1.4	0.6778	0.6324	0.0635	0.3986	-0.5401	2.3038
2.3	0.5	0.9	0.9436	0.8296	0.0568	0.2872	-1.7211	5.1215
		1.4	0.9634	0.8738	0.0374	0.2213	-2.1174	7.1561
	1.2	0.9	0.6845	0.6265	0.0722	0.4288	-0.5985	2.2994
		1.4	0.7837	0.7184	0.0532	0.3209	-0.9836	3.1853

From Table 2: If α increasing (η and ν are fixed) then median, mean and Kurtosis are increasing while variance, C.V. and Skewness are decreasing.

The moment generating function of the MAPTKD is expressed by

$$M_Z(t) = \int_0^1 e^{tz} g(z) dz = \sum_{r=0}^{\infty} \frac{t^r E(Z^r)}{r!}, \tag{17}$$

where $E(Z^r)$ is the moments of MAPTKD given in (16).

The characteristics function of the MAPTKD is obtained by replacing (t) with (it) in the (17).

$$M_Z(it) = \sum_{r=0}^{\infty} \frac{(it)^r E(Z^r)}{r!}. \tag{18}$$

3.3 Rényi entropy

The Rényi entropy [17] with $q > 0$, $q \neq 1$ of MAPTKD is determined by

$$
\begin{aligned}
H_q &= \frac{1}{1-q} \ln \left(\int_0^1 (g(z))^q dz \right) \\
&= \frac{1}{1-q} \ln \left\{ \left(\frac{\alpha \eta \nu \ln \alpha}{\alpha - 1} \right)^q \int_0^1 \left[\frac{z^{\nu-1}(1-z^\nu)^{\eta-1} \alpha^{1-(1-z^\nu)^\eta}}{(1+\alpha-\alpha^{1-(1-z^\nu)^\eta})^2} \right]^q dz \right\} \\
&= \frac{1}{1-q} \ln \left\{ \left(\frac{\alpha \eta \nu \ln \alpha}{\alpha^2 - 1} \right)^q \sum_{s,v=0}^{\infty} \sum_{u=0}^{v} \frac{(-1)^u ((s+q)\ln \alpha)^v}{(\alpha+1)^{q+s}(v-u)!u!} \binom{2q+s-1}{s} \int_0^1 z^{q(\nu-1)} (1-z^\nu)^{(q+u)\eta - q} dz \right\} \\
&= \frac{1}{1-q} \ln \left\{ \left(\frac{\alpha \eta \nu \ln \alpha}{\alpha^2 - 1} \right)^q \sum_{s,v=0}^{\infty} \sum_{u=0}^{v} \frac{(-1)^u ((s+q)\ln \alpha)^v}{(\alpha+1)^{q+s}(v-u)!u!} \binom{2q+s-1}{s} \frac{1}{\nu} B(u\eta + (\eta-1)q + 1, \frac{q(\nu-1)+1}{\nu}) \right\}.
\end{aligned} \tag{19}
$$

Table 3 disply some values of Rényi entropy for some parameters q, α, η and ν.

Table 3: Some values of Rényi entropy of MAPTKD.

q	α	η	ν	H_q
0.4	0.3	0.7	0.8	-0.196725
			1.4	-0.045422
		1.6	0.8	-0.526035
			1.4	-0.214851
	0.8	0.7	0.8	-0.012519
			1.4	-0.035657
		1.6	0.8	-0.142114
			1.4	-0.035436
	1.5	0.7	0.8	-0.064862
			1.4	-0.176487
		1.6	0.8	-0.029378
			1.4	-0.035681
1.2	0.3	0.7	0.8	-0.644881
			1.4	-0.128788
		1.6	0.8	-1.446190
			1.4	-0.503656
	0.8	0.7	0.8	-0.044619
			1.4	-0.108719
		1.6	0.8	-0.388691
			1.4	-0.084293
	1.5	0.7	0.8	-0.239316
			1.4	-0.552718
		1.6	0.8	-0.071186
			1.4	-0.088994

From Table 3:

–If q increasing then the Rényi entropy is decreasing.
–If α increasing then the Rényi entropy is increasing.

3.4 Order statistics

Given a random sample, $Z_1,...,Z_n$, from MAPTKD$(z;\alpha,\eta,\nu)$ the order statistic is labeled by $Z_{1:n},...,Z_{n:n}$, where $Z_{j:n}$ is call the j^{th} order statistic. It has been known that the PDF of $Z_{j:n}$ is determined by

$$g_{j:n}(z) = \frac{n!}{(j-1)!(n-j)!} g(z)[G(z)]^{j-1}[1-G(z)]^{n-j}$$
$$= \frac{n!}{(j-1)!(n-j)!} g(z)\sum_{d=0}^{n-j}(-1)^d\binom{n-j}{d}[G(z)]^{d+j-1}.$$

The PDF of the j^{th} order statistics $Z_{j:n}$ of MAPTKD is determined as

$$g_{j:n}(z) = \sum_{i=0}^{\infty}\sum_{k=0}^{i}\Upsilon_{i,k} f(z;(1+k)\eta,\nu), \tag{20}$$

where $f(z;(k+1)\eta,\nu)$ is the PDF of the KD with two shape parameters (k + 1)η and ν, and

$$\Upsilon_{i,k} = \sum_{d=0}^{n-j}\sum_{s=0}^{d+j-1}\sum_{v=0}^{\infty}\frac{(-1)^{s+j+k-1}\alpha(\ln\alpha)^{i+1}(s+v+1)^i\binom{n-j}{d}\binom{d+j-1}{s}\Gamma(v+d+1+j)}{B(j,n-1+j)(i-k)!(1+k)!v!\Gamma(d+1+j)(\alpha-1)^{d+j}(\alpha+1)^{d+j+v+1}}.$$

The r^{th} moments of $Z_{j:n}$ may be calculated as follows using (20)

$$E(Z^r_{j:n}) = \int_0^1 z^r g_{j:n}(z)dz = \sum_{i=0}^{\infty}\sum_{k=0}^{i}\Upsilon_{i,k}(1+k)\eta B((1+k)\eta,\frac{r}{\nu}+1). \tag{21}$$

4 Parameter estimation

Let random sample, $Z_1,...,Z_n$, from MAPTK$(z;\alpha,\eta,\nu)$, where α, η and ν are unknown. The maximum likelihood estimate (MLE) procedures will be addressed in this section.

4.1 Maximum likelihood estimation

The log-likelihood function of the parameters $\underline{\Theta} = (\alpha,\eta,\nu)$ of the MAPTKD is presented as

$$\mathscr{L}(z_1,...,z_n|\Theta) = n\ln(\ln\alpha) - n\ln(\alpha-1) + n\ln\eta + n\ln\nu + (\nu-1)\sum_{i=1}^{n}\ln(z_i) + (\eta-1)\sum_{i=1}^{n}\ln(1-z_i^{\nu})$$
$$+\ln(\alpha)\sum_{i=1}^{n}(2-(1-z_i^{\nu})^{\eta}) - 2\sum_{i=1}^{n}\ln(1+\alpha-\alpha^{1-(1-z_i^{\nu})^{\eta}}). \tag{22}$$

Three first partial derivatives of (22) relative to the components of vector parameter, $\Theta = (\alpha,\eta,\nu)$, are respectively expressed by

$$\frac{\partial\mathscr{L}}{\partial\alpha} = \frac{n}{\alpha\ln(\alpha)} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}(2-(1-z_i^{\nu})^{\eta}) - 2\sum_{i=1}^{n}\frac{1-[1-(1-z_i^{\nu})^{\eta}]\alpha^{-(1-z_i^{\nu})^{\eta}}}{1+\alpha-\alpha^{1-(1-z_i^{\nu})^{\eta}}}, \tag{23}$$

$$\frac{\partial \mathscr{L}}{\partial \eta}=\frac{n}{\eta}+\sum_{i=1}^{n}\ln(1-z_i^{\nu})-\ln(\alpha)\sum_{i=1}^{n}(1-z_i^{\nu})^{\eta}\ln(1-z_i^{\nu})-2\sum_{i=1}^{n}\frac{\ln(\alpha)\ln(1-z_i^{\nu})(1-z_i^{\nu})^{\eta}\alpha^{1-(1-z_i^{\nu})^{\eta}}}{1+\alpha-\alpha^{1-(1-z_i^{\nu})^{\eta}}}, \tag{24}$$

and

$$\frac{\partial \mathscr{L}}{\partial \nu}=\frac{n}{\nu}+\sum_{i=1}^{n}\ln(z_i)-(\eta-1)\sum_{i=1}^{n}\frac{z_i^{\nu}\ln z_i}{1-z_i^{\nu}}+\ln(\alpha)\sum_{i=1}^{n}\eta z_i^{\nu}\ln(z_i)(1-z_i^{\nu})^{\eta-1}$$
$$+2\sum_{i=1}^{n}\frac{\eta\ln(\alpha)\ln(z_i)z_i^{\nu}(1-z_i^{\nu})^{\eta-1}\alpha^{1-(1-z_i^{\nu})^{\eta}}}{1+\alpha-\alpha^{1-(1-z_i^{\nu})^{\eta}}}. \tag{25}$$

To estimate the MLEs of Θ, any numerical methods may be utilized to solve the (23) - (25).

4.2 Asymptotic confidence bounds

Utilizing the asymptotic properties of the MLEs, we are able to acquire the asymptotic confidence intervals (ACIs) of α,η and ν. It has been established that
$(\alpha,\eta,\nu)\backsim N\left((\hat{\alpha},\hat{\eta},\hat{\nu}),I_0^{-1}(\alpha,\eta,\nu)\right)$, where I^{-1} is the inverse of the observed information matrix [18] which described as follows

$$\mathbf{I^{-1}}=\begin{pmatrix}-\frac{\partial^2\mathscr{L}}{\partial\alpha^2} & -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\eta} & -\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\nu}\\ -\frac{\partial^2\mathscr{L}}{\partial\eta\partial\alpha} & -\frac{\partial^2\mathscr{L}}{\partial\eta^2} & -\frac{\partial^2\mathscr{L}}{\partial\eta\partial\nu}\\ -\frac{\partial^2\mathscr{L}}{\partial\nu\partial\alpha} & -\frac{\partial^2\mathscr{L}}{\partial\nu\partial\eta} & -\frac{\partial^2\mathscr{L}}{\partial\nu^2}\end{pmatrix}^{-1}=\begin{pmatrix}var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\eta}) & cov(\hat{\alpha},\hat{\nu})\\ cov(\hat{\eta},\hat{\alpha}) & var(\hat{\eta}) & cov(\hat{\eta},\hat{\nu})\\ cov(\hat{\nu},\hat{\alpha}) & cov(\hat{\nu},\hat{\eta}) & var(\hat{\nu})\end{pmatrix}. \tag{26}$$

The second partial derivatives contained in I are shown below.

$$\frac{\partial^2\mathscr{L}}{\partial\alpha^2}=-\frac{n}{\alpha^2\ln(\alpha)}-\frac{n}{\alpha^2(\ln(\alpha))^2}+\frac{n}{(\alpha-1)^2}-\frac{1}{\alpha^2}\sum_{i=1}^{n}\left[2-(1-z_i{}^{\nu})^{\eta}\right]-2\sum_{i=1}^{n}\mathscr{A}_i, \tag{27}$$

$$\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\eta}=-\frac{1}{\alpha}\sum_{i=1}^{n}(1-z_i{}^{\nu})^{\eta}\ln(1-z_i{}^{\nu})-2\sum_{i=1}^{n}\mathscr{M}_i, \tag{28}$$

$$\frac{\partial^2\mathscr{L}}{\partial\alpha\partial\nu}=\frac{1}{\alpha}\sum_{i=1}^{n}(1-z_i{}^{\nu})^{\eta-1}\eta z_i{}^{\nu}\ln(z_i)-2\sum_{i=1}^{n}\mathscr{C}_i, \tag{29}$$

$$\frac{\partial^2\mathscr{L}}{\partial\eta^2}=-\frac{n}{\eta^2}-\ln(\alpha)\sum_{i=1}^{n}(1-z_i{}^{\nu})^{\eta}\ln(1-z_i{}^{\nu})^2+2\sum_{i=1}^{n}\mathscr{D}_i, \tag{30}$$

$$\frac{\partial^2\mathscr{L}}{\partial\eta\partial\nu}=-\sum_{i=1}^{n}\frac{z_i{}^{\nu}\ln(z_i)}{1-z_i{}^{\nu}}+\ln(\alpha)\sum_{i=1}^{n}\frac{(1-z_i{}^{\nu})^{\eta}z_i{}^{\nu}\ln(z_i)\left[\eta\ln(1-z_i{}^{\nu})+1\right]}{1-z_i{}^{\nu}}-2\sum_{i=1}^{n}\mathscr{E}_i, \tag{31}$$

$$\frac{\partial^2\mathscr{L}}{\partial\nu^2}=-\frac{n}{\nu^2}-(\eta-1)\sum_{i=1}^{n}\left(\frac{z_i{}^{\nu}(\ln(z_i))^2}{1-z_i{}^{\nu}}+\frac{(z_i{}^{\nu})^2(\ln(z_i))^2}{(1-z_i{}^{\nu})^2}\right)$$
$$+\ln(\alpha)\sum_{i=1}^{n}\frac{(1-z_i{}^{\nu})^{\eta}\eta^2(z_i{}^{\nu})^2(\ln(z_i))^2\left[\eta^{-1}+\left(\eta z_i{}^{\nu}(1-z_i{}^{\nu})^{\eta}\right)^{-1}-1\right]}{(1-z_i{}^{\nu})^2}+2\sum_{i=1}^{n}\mathscr{H}_i. \tag{32}$$

Where

$$\mathscr{A}_i = \frac{\alpha^{-1-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\left[1-\left(1-z_i{}^{\nu}\right)^{\eta}\right]}{\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)} - \frac{\left(1-\alpha^{-(1-z_i{}^{\nu})^{\eta}}\left[1-\left(1-z_i{}^{\nu}\right)^{\eta}\right]\right)^2}{\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2},$$

$$\mathscr{M}_i = \frac{\alpha^{-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\ln\left(1-z_i{}^{\nu}\right)\ln(\alpha)\left(1-\left(1-z_i{}^{\nu}\right)^{\eta}\right)+\alpha^{-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\ln\left(1-z_i{}^{\nu}\right)}{1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}}$$
$$-\frac{\left(1-\alpha^{-(1-z_i{}^{\nu})^{\eta}}\left(1-\left(1-z_i{}^{\nu}\right)^{\eta}\right)\right)\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\ln\left(1-z_i{}^{\nu}\right)\ln(\alpha)}{\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2},$$

$$\mathscr{C}_i = -\frac{\alpha^{-(1-z_i^{\nu})^{\eta}}(1-z_i^{\nu})^{\eta-1}\eta z_i^{\nu}\ln(z_i)\left[\ln(\alpha)(1-(1-z_i^{\nu})^{\eta})+1\right]}{(1+\alpha-\alpha^{1-(1-z_i^{\nu})\eta})}$$
$$+\frac{\alpha^{1-(1-z_i{}^{\nu})^{\eta}}(1-z_i{}^{\nu})^{\eta-1}\eta\, z_i{}^{\nu}\ln(z_i)\ln(\alpha)(1-\alpha^{-(1-z_i{}^{\nu})^{\eta}}(1-(1-z_i{}^{\nu})^{\eta}))}{(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}})^2},$$

$$\mathscr{D}_i = \frac{\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\left(\ln\left(1-z_i{}^{\nu}\right)\right)^2\ln(\alpha)\left[\left(1-z_i{}^{\nu}\right)^{\eta}\ln(\alpha)+1\right]}{\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)}$$
$$-\frac{\left(\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2\left(\left(1-z_i{}^{\nu}\right)^{\eta}\right)^2\left(\ln\left(1-z_i{}^{\nu}\right)\right)^2\left(\ln(\alpha)\right)^2}{\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2},$$

$$\mathscr{E}_i = \frac{\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\eta\, z_i{}^{\nu}\ln(z_i)\ln\left(1-z_i{}^{\nu}\right)\ln(\alpha)\left[\left(1-z_i{}^{\nu}\right)^{\eta}\ln(\alpha)-\eta\ln\left(1-z_i{}^{\nu}\right)-1\right]}{\left(1-z_i{}^{\nu}\right)\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)}$$
$$+\frac{\left(\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2\left(\left(1-z_i{}^{\nu}\right)^{\eta}\right)^2\ln\left(1-z_i{}^{\nu}\right)\left(\ln(\alpha)\right)^2\eta\, z_i{}^{\nu}\ln(z_i)}{\left(1-z_i{}^{\nu}\right)\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2},$$

and

$$\mathscr{H}_i = \frac{\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\left(1-z_i{}^{\nu}\right)^{\eta}\eta^2\left(z_i{}^{\nu}\right)^2\left(\ln(z_i)\right)^2\ln(\alpha)\left[\left(1-z_i{}^{\nu}\right)^{\eta}\ln(\alpha)-\eta\left(z_i{}^{\nu}\right)-\eta\ln\left(1-z_i{}^{\nu}\right)+1\right]}{\left(1-z_i{}^{\nu}\right)^2\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)}$$
$$-\frac{\left(\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2\left(\left(1-z_i{}^{\nu}\right)^{\eta}\right)^2\eta^2\left(z_i{}^{\nu}\right)^2\left(\ln(z_i)\right)^2\left(\ln(\alpha)\right)^2}{\left(1-z_i{}^{\nu}\right)^2\left(1+\alpha-\alpha^{1-(1-z_i{}^{\nu})^{\eta}}\right)^2}.$$

Consequently, the $(1-\varphi)100\%$ ACIs of α, η and ν can be achieved in the following way:

$$\hat{\alpha} \pm Z_{\frac{\varphi}{2}}\sqrt{var(\hat{\alpha})}, \quad \hat{\eta} \pm Z_{\frac{\varphi}{2}}\sqrt{var(\hat{\eta})}, \quad \hat{\nu} \pm Z_{\frac{\varphi}{2}}\sqrt{var(\hat{\nu})},$$

where $Z_{\frac{\varphi}{2}}$ is the upper $(\frac{\varphi}{2})^{th}$ percentile of the standard normal distribution.

4.3 Simulation

In this section, a monte Carlo simulation based approach were performed utilizing 1000 to study the behavior of distribution parameters. We used varying sample sizes (n =50, 100, 150, 200), varying population parameter values of α, η and ν as (0.3, 0.5, 0.8), (1.2, 0.5, 0.8), (1.2, 1.5, 0.8), (2.0, 0.3, 1.5), (0.8, 0.3, 1.5) and (0.8, 2, 1.5) to explore their effects on varying sample sizes and different population parameter values.
The MLEs $\hat{\alpha}$, $\hat{\eta}$ and $\hat{\nu}$ of parameters α, η and ν of the MAPTK are then determined using the solution of Equations (23)-(25) applying the Newton–Raphson method. We calculate the average estimates, mean squared errors (MSEs), bsias, average lengths and the coverage percentages, for the parameters based on different sample sizes. The simulation algorithm was implemented following these steps:

Step 1:The sample size and initial parameter values were defined.
Step 2:A random sample of size n was generated from the MAPTKD, as outlined in Equation (14).
Step 3:The average estimates were computed, along with their MSEs and relative absolute biases (RABs) for the parameters α, η and ν.
Step 4:The $(1-\varphi)$% CLs for α, η, and ν were obtained.
Step 5:Steps 2-4 were repeated 1000 times.
Step 6:The average values of the maximum likelihood estimates (MLEs), MSEs, RABs, CLs, and coverage percentages for any function α, η and ν (say ψ) were calculated and given as

$$AVMLE = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\psi}_i, \qquad AVMSE = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\psi}_i-\psi_i)^2, \qquad AVRAB = \frac{1}{1000}\sum_{i=1}^{1000}\left|\hat{\psi}_i-\psi_i\right|,$$
$$AVCL = \frac{1}{1000}\sum_{i=1}^{1000}(\psi_i^U-\psi_i^L), \quad and \quad AVCP = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\psi}_i^{CP},$$

where ψ_i^U and ψ_i^L are the upper and lower ACI bounds, respectively and $\hat{\psi}_i^{CP}$ is the coverage probabilities.

Table 4: Average values of the different estimates and the corresponding MSEs, Biases average lengths and coverage probabilities.

n	α	η	ν	α	η	ν	α	η	ν
	(0.3,	0.5,	0.8)	(1.2,	0.5,	0.8)	(1.2,	1.5,	0.8)
50	0.4588	0.7480	0.7694	1.2414	0.5054	0.8522	1.3601	1.5853	0.8343
	0.1047	0.2686	0.0173	0.2089	0.0142	0.0669	0.3855	0.2171	0.0579
	0.2307	0.3686	0.1050	0.3241	0.0923	0.1871	0.4986	0.3845	0.1717
	1.1448	1.4682	0.657	2.3727	0.5541	1.1818	3.5181	1.6414	1.0896
	0.9336	0.9455	0.9235	0.9636	0.9636	0.9818	0.9091	0.9636	0.9818
100	0.3714	0.6045	0.7844	1.3852	0.5322	0.8527	1.2451	1.5452	0.8407
	0.0281	0.0670	0.0144	0.3862	0.0118	0.0579	0.1923	0.0836	0.0299
	0.1124	0.1893	0.0946	0.4281	0.0845	0.2008	0.3518	0.2266	0.1442
	0.6717	0.9849	0.4507	1.9406	0.4014	0.8846	2.1728	1.1857	0.7656
	0.9636	0.9818	0.9273	0.9455	0.9091	0.9455	0.9636	0.9818	0.9818
150	0.3254	0.5264	0.7892	1.2454	0.4910	0.7950	1.2885	1.5469	0.8173
	0.0119	0.0216	0.0083	0.1139	0.0071	0.0275	0.1392	0.0603	0.0206
	0.0857	0.1223	0.0713	0.2262	0.0678	0.1221	0.2890	0.1966	0.1067
	0.4972	0.7663	0.3584	1.3425	0.3104	0.6623	1.9045	0.9515	0.6465
	0.9556	0.9678	0.9273	0.9455	0.9636	0.9636	0.9455	0.9636	0.9636
200	0.3222	0.5452	0.806	1.2175	0.4999	0.8446	1.2338	1.4783	0.8038
	0.0088	0.0322	0.0046	0.0714	0.0049	0.0289	0.1116	0.0432	0.0151
	0.0762	0.1409	0.0493	0.2009	0.0571	0.1329	0.2454	0.1604	0.0980
	0.4283	0.6795	0.3167	1.1343	0.2761	0.6039	1.5264	0.8031	0.5343
	0.9818	0.9636	0.9636	0.9636	0.9818	0.9455	0.9273	0.9636	0.9636

Table 5: Average values of the different estimates and the corresponding MSEs, Biases average lengths and coverage probabilities.

n	α	η	ν	α	η	ν	α	η	ν
	(2.0,	0.3,	1.5)	(0.8,	0.3,	1.5)	(0.8,	2.0,	1.5)
50	2.0612	0.3035	1.7294	0.9823	0.3537	1.6383	1.0391	2.0821	1.4651
	0.3824	0.0034	0.5746	0.2121	0.0158	0.3569	0.3582	0.3113	0.1385
	0.5002	0.0437	0.5949	0.3179	0.1035	0.4607	0.4355	0.4375	0.2881
	3.5580	0.2623	0.6339	2.0554	0.5048	2.3426	2.9968	2.7834	1.7638
	0.9644	0.9802	0.9762	0.9443	0.9273	0.9818	0.9818	0.9818	0.9273
100	2.07994	0.3087	1.6983	0.9121	0.3342	1.5691	0.9711	2.1466	1.4485
	0.3217	0.0025	0.4766	0.1077	0.0092	0.1193	0.1604	0.2352	0.0664
	0.4617	0.0415	0.5219	0.2172	0.0702	0.2882	0.3104	0.4034	0.2083
	2.5528	0.1877	2.5485	1.1465	0.3005	1.4211	1.7588	1.8682	1.1404
	0.9273	0.9636	0.9818	0.9636	0.9273	0.9455	0.9273	0.9818	0.9455
150	2.0037	0.3016	1.7048	0.8423	0.3106	1.5516	0.9072	2.1044	1.5052
	0.1855	0.0013	0.3583	0.0475	0.0042	0.0548	0.1459	0.1248	0.0479
	0.3235	0.0292	0.4436	0.1616	0.0472	0.1963	0.2619	0.2918	0.1772
	1.9596	0.1506	2.0938	0.8563	0.2385	1.1336	1.4327	1.6494	0.9926
	0.9818	0.9636	0.9636	0.9818	0.9273	0.9431	0.9636	0.9273	0.9818
200	2.1170	0.3046	1.5086	1.8777	0.3213	1.4762	0.8811	2.0426	1.4606
	0.2181	0.0010	0.1949	0.0364	0.0028	0.0586	0.0655	0.1556	0.0260
	0.3580	0.0261	0.3300	0.1436	0.0431	0.1926	0.1864	0.3068	0.1222
	1.8151	0.1298	1.6540	0.7641	0.2074	0.9488	1.2173	1.4212	0.8745
	0.9636	0.9636	0.9091	0.9818	0.9818	0.9455	0.9818	0.9091	0.9818

For the parameter estimation, The MLEs, mean square errors (MSEs), absolute biases (ABs), confidence lengths (CLs) and coverage probabilities (CPs) are computed for each parameter of MAPTKD and reported in Tables 4 and 5. All numerical computations are implemented via Mathematica programming language version 14.0, which using Newton-Raphson method of maximization in the computations. From the results of the simulation study several key points become evident from this experiment. Even with relatively small sample sizes, we have noted the following:

(i)In general, it can be seen that the MLEs for different sample sizes are very good in terms of minimum MSEs, ABs and ACLs as well as highest CPs.
(ii)From Tables 4 and 5, as expected for most cases when n increase then the MSEs, ABs and CLs of all investigated estimates decrease.
(iii)In most cases, we observe that the coverage probability is close to the desired level of 0.95 based on different sample sizes.
(iv)We further observe highly consistent coverage probabilities (almost at the standard level). On the flip side, the performance of the MLEs for different sample sizes is acceptable for a limited sample size considering their actual coverage probabilities are generally near the specified nominal levels in most cases.
(v)In most cases, when α and η increase, it is observed that the ACLs and CPs constructed based on MLE approach increase and vice versa.

5 Real data

In this part, we analysis two real data sets to illustrate that the MAPTKD can prove to be an effective lifetime model compared to many known distributions such as alpha power transformed Kumaraswamy (APK), generalized Kumaraswamy (GK), Kumaraswamy (K), inverted Kumaraswamy (IK), alpha power function (AP) and the power function (PF) distributions. The pdfs of those competitive distributions are listed in Table 6.

Table 6: The PDFs of fitted distribution models with $\alpha > 0, \alpha \neq 1, \eta > 0, \nu > 0, 0 < z < 1$.

distribution	PDF
APK	$g(z;\alpha,\eta,\nu) = \frac{\alpha^{1-(1-z^{\nu})^{\eta}} \ln(\alpha)\eta\nu z^{\nu-1}(1-z^{\nu})^{\eta-1}}{\alpha-1}$.
GK	$g(z;\alpha,\eta,\nu) = \alpha\eta\nu z^{\nu-1}(1-z^{\nu})^{\eta-1}(1-(1-z^{\nu})^{\eta})^{\alpha-1}$.
K	$g(z;\eta,\nu) = \eta\nu z^{\nu-1}(1-z^{\nu})^{\eta-1}$.
IK	$g(z;\eta,\nu) = \eta\nu(1+z)^{-(\nu+1)}(1-(1+z)^{-\nu})^{\eta-1}$.
AP	$g(z;\alpha,\nu) = \frac{\alpha^{z^{\nu}} \ln(\alpha)\nu z^{\nu-1}}{\alpha-1}$.
PF	$g(z;\nu) = \nu z^{\nu-1}$.

In addition, certain goodness of fit statistics including: Kolmogorov-Smirnov (K-S) statistic, Akaike information criterion (AIC), correction of AIC (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) values where
$K-S = sup|G(z)-S(z)|, S(z)$ is empirical cumulative distribution,

$$AIC = 2w - 2\mathscr{L}, \quad CAIC = AIC + \frac{2w(w+1)}{\tau - w + 1}, \quad BIC = w\ln(\tau) - 2\mathscr{L}, \quad HQIC = 2w\ln(\ln(\tau)) - 2\mathscr{L},$$

where τ is the sample size and w is the number of parameters. The distribution, which is with the smallest values of $\mathscr{L}$, AIC and K-S (highest p-value), is selected as a best model for the data set considered.

5.1 First Data

The data set is obtained from the UK National Physical Laboratory, the data set represents the lifetime (Hours) of $T8$ fluorescent lamps for 50 devices, [19]. The first data is in Table 7.

Table 7: The first data represents the lifetime of 50 devices.

0.057	0.116	0.117	0.126	0.134	0.149	0.184	0.192	0.198	0.228
0.234	0.238	0.244	0.246	0.285	0.326	0.338	0.366	0.375	0.381
0.384	0.403	0.405	0.412	0.431	0.445	0.458	0.463	0.486	0.486
0.493	0.497	0.511	0.517	0.521	0.552	0.553	0.564	0.570	0.579
0.586	0.588	0.612	0.636	0.690	0.760	0.780	0.796	0.890	0.893

Table 8 gives MLEs of parameters, test statistic K-S and P-Value of the MAPTKD and additional models with respect to the first data.

Table 8: MLEs of parameters, K-S and P-Value for the first data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\nu}$	K-S	p-value
MAPTK	0.5	2.11	2.28	0.08269	0.9836
APK	16.95	2.83	1.06	0.08370	0.9667
GK	4.11	2.10	0.65	0.11523	0.50432
K	–	2.89	1.81	0.08833	0.8900
IK	–	5.18	6.562	0.12848	0.36078
AP	12.883	–	0.396	0.2564	2.17×10^{-3}
PF	–	–	1.008	0.25449	2.4137×10^{-3}

Table 9 gives the $\mathscr{L}$ and some goodness of fit statistics of MAPTKD and some models fitted for the first data.

Table 9: The $\mathscr{L}$ and criterions statistics for the first data.

Model	$\mathscr{L}$	$2\mathscr{L}$	AIC	CAIC	BIC	HQIC
MAPTK	9.939	19.8780	-13.8780	-13.3563	-8.1419	-11.6937
APK	9.251	18.5020	-12.5020	-11.9803	-6.7659	-10.3177
GK	9.792	19.5840	-13.5840	-13.0623	-7.8479	-11.3997
K	9.884	19.7680	-15.7680	-15.5127	-11.9440	-14.3118
IK	4.667	9.3340	-5.3340	-5.0787	-1.5100	-3.8778
AP	-4.654	-9.3080	13.3080	13.5633	17.1320	14.7642
PF	0.0013	0.0026	1.99744	2.0808	3.9095	2.7255

Through compensation the MLE's of the latent parameters α, η, and ν into equation (26), one derives the estimation for the variance-covariance matrix as follows.

$$I_0^{-1} = \begin{pmatrix} 0.011 & 0.017 & -4.199\times10^{-3} \\ 0.017 & 0.111 & 0.011 \\ -4.199\times10^{-3} & 0.011 & 0.019 \end{pmatrix}$$

The ACIs for the parameters α, η, and ν of first data are
$[0.29, 0.71]$, $[1.456, 2.764]$, and $[2.008, 2.552]$, respectively.

To check that the $\mathscr{L}$ has a single solution, we display the shapes of the $\mathscr{L}$ of α, η and ν in Figure 3 for the first data.

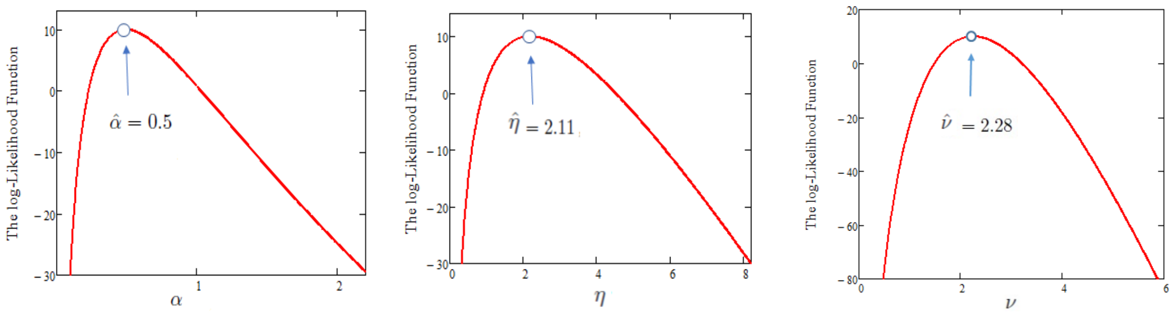

Fig. 3: The shape of the $\mathscr{L}$ of α, η and ν for the first data.

By using the Kaplan-Meier (K-M) method, the graphs of the RF and CDF of MAPTKD and some models are analyzed and illustrated in Figure 4.

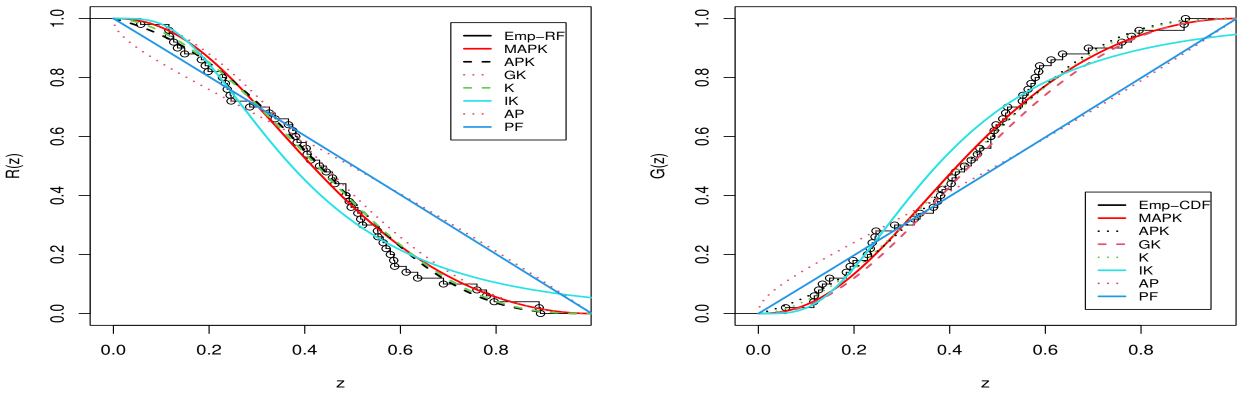

Fig. 4: RF and CDF of MAPTKD and some models for the first data.

5.2 Second Data

The second data set is represents the times of breakdown of a sample of twenty five devices at 180° C and given by Pham [20]. The second data set is in Table 10, after its transmuted to fit models.

Table 10: The second data represents the times of breakdown of size sample = 25.

0.0245	0.0569	0.0652	0.0716	0.0830	0.1066	0.1298	0.1440	0.1534
0.1576	0.1606	0.1611	0.1696	0.2003	0.2132	0.2458	0.2532	0.2685
0.2830	0.2922	0.3222	0.3346	0.3381	0.4441	0.9998	-	-

Table 11 gives MLEs of parameters, test statistic K-S and P-Value of the MAPTKD and other models for the second data.

Table 11: MLEs of parameters, K-S and P-Value for the second data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\nu}$	K-S	p-value
MAPTK	0.086	1.014	2.081	0.12225	0.90481
APK	3.542	3.283	0.819	0.19473	0.26734
GK	367.95	2.346	0.043	0.15217	0.58521
K	–	3.102	1.034	0.21173	0.18527
IK	–	2.815	9.181	0.12375	0.89129
AP	11.466	–	0.223	0.36526	1.6837×10^{-3}
PF	–	–	0.569	0.37873	9.8927×10^{-4}

Table 12 gives the $\mathscr{L}$ and some goodness of fit statistics of MAPTKD and some models fitted for the second data.

Table 12: The $\mathscr{L}$ and criterions statistics for the second data.

Model	$\mathscr{L}$	$2\mathscr{L}$	AIC	CAIC	BIC	HQIC
MAPTK	16.187	32.3740	-26.3740	-25.2311	-22.7174	25.3598
APK	10.1640	20.3280	-14.3280	-13.1851	-10.6714	-13.3138
GK	13.9390	27.8780	-21.8780	-20.7351	-18.2214	-20.8638
K	10.7090	21.4180	-17.4180	-16.8725	-14.9802	-16.7419
IK	15.825	31.6500	-27.6500	-27.1045	-25.2122	-26.9739
AP	1.9940	3.9880	0.0120	0.5575	2.4498	0.6881
PF	4.829	9.658	-7.658	-7.4841	-6.4391	-7.3199

The variance-covariance matrix of second data as follows.

$$I_0^{-1} = \begin{pmatrix} 6.495\times10^{-4} & 3.99\times10^{-3} & 1.172\times10^{-4} \\ 3.99\times10^{-3} & 0.1 & 4.13\times10^{-3} \\ 1.172\times10^{-4} & 4.13\times10^{-3} & 9.084\times10^{-4} \end{pmatrix}$$

The ACIs of second data are $[0.036, 0.136]$, $[0.395, 1.633]$ and $[2.022, 2.14]$, respectively.
The shapes of the $\mathscr{L}$ of α, η and ν in Figure 5 for the second data.

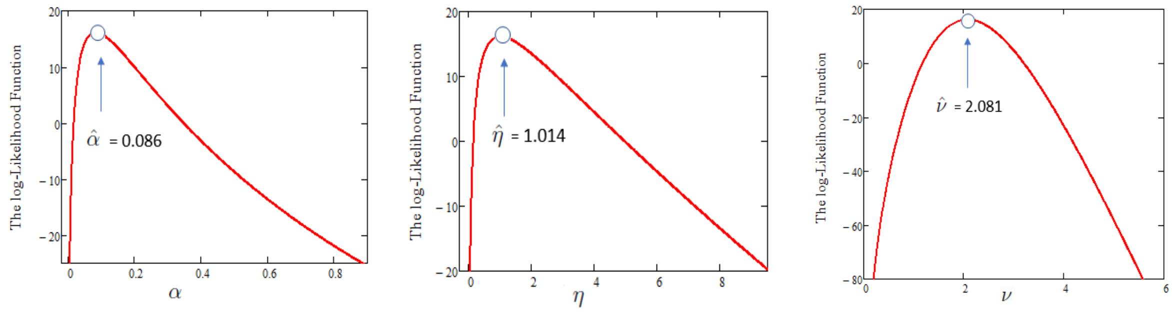

Fig. 5: The shape of the $\mathscr{L}$ of α, η and ν for the second data.

The graphs of the RF and CDF using the K-M method are in the Figure 6 for the second data.

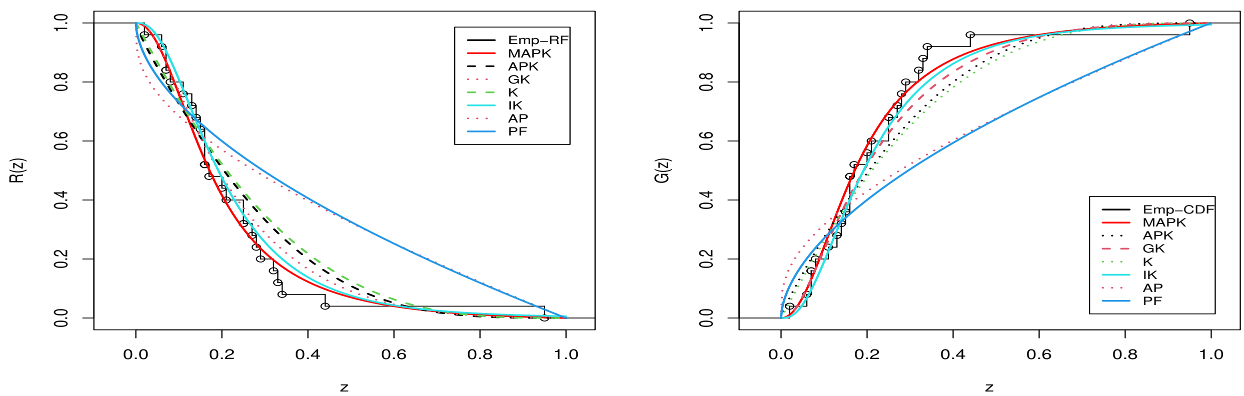

Fig. 6: RF and CDF of MAPTKD and some models for the second data.

6 Conclusion

Current research work has contributed a new extended distribution through the modified alpha power (MAP) transformation on Kumaraswamy (K) distribution. This distribution is a generalization of the Kumaraswamy distribution and called the MAPTKD. The reliability functions and statistical characteristics of MAPTKD have been derived; such as, hazard rate function, moments, entropy and order statistics. The method of MLE was also proposed to estimate the parameters of the MAPTKD and the results of simulation study support that MLE method provide competitive MSE and Bias for estimating the parameters of the MAPTKD. The adequacy of the MAPTKD's fit was substantiated through empirical data analysis, and the findings derived from the data modeling indicate that the MAPTKD provides a superior fit to the data set in comparison to some competing distributions.

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