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Classical Inference for the Five Parameter Exponentiated Weibull Distribution: Properties and Applications in Health and Reliability

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Abstract: Probability distribution has proven its usefulness in almost every discipline of human endeavors. In this paper, the Burr XII? Exponentiated Weibull of (BXIIEW) distributions was introduced, with an increasing hazard rate developed and extensively studied. Several statistical properties of the new model like moment, Skewness, kurtosis, incomplete moment, quantile function, and Renyi entropy were derived. Various estimation techniques, such as maximum likelihood, least squares, weighted least squares, maximum product of space, Anderson darling, and right-tailed Anderson darling, are used to estimate parameters. The simulation study on the proposed model confirmed the reliability and consistency of its parameters. Using three studies in Health and Reliability, the study further emphasizes the significance and practical uses of the BXIIEW. The proposed model shows better goodness-of-fit, according to our empirical results. weighed against competing models that were examined in this investigation.

Keywords: T-X family, exponentiated Weibull distribution, moment, quantile function, classical Inference.

1 Introduction

In probability theory, statistical distributions are developed. The paradigm helps comprehend uncertainty and make predictions and judgments across disciplines, see [1,2,3,4,5,6]. A dataset's optimal mathematical model is sought by statisticians. However, the abundance of heterogeneous data nowadays renders conventional probability distributions unsuitable for modeling them. Their expansions are needed to derive new distribution families. Knowing the traits and linkages of distribution families is crucial for choosing models for various statistical studies and applications. In the literature, probability distributions can be generated by enhancing a baseline distribution with one or more parameters, using the transformation method, using generalized families of distributions, mixing multiple distributions, etc. The odd log-logistic Poisson-G family [7], Half-logistic odd Weibull-Topp-Leone-G [8], McDonald-G [9], Weibull-G [10], Generalized Weibull [11], Kumaraswamy generalized family [12], Fréchet Topp Leone-G [13], Topp-Leone Gompertz-G [14], and Odd Lomax-G Distributions [15] are some of the most well-known families developed in such fashion.

Definition 1.*[16] Let v(t) be the probability density function (pdf) of a random variable, say T, where $T \in [s,a]$ for-$\infty < s < a < \infty$, and let $W[G(x;\xi)]$ be a function of the cumulative distribution function (cdf) of a random variable X, which satisfies the following conditions:*

i. $W(F(x,\zeta)) \in [s,a]$, $-\infty < s < a < \infty$.
ii. $W(F(x,\zeta))$ *is differentiable and monotonically non-decreasing.*
iii. $(F(x,\zeta)) \to s$, *as* $x \to -\infty$, *and* $W(F(x,\zeta)) \to a$, *as* $x \to \infty$, *i.e* $W(F(x,\zeta)) \to 0$, *as* $x \to 0$, *and* $W(F(x,\zeta)) \to 1$, *as* $x \to \infty$.

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The T-X distribution family cdf is

$$G(x,\zeta) = \int_{p}^{W(F(x,\zeta))} v(t)dt \tag{1}$$

where $W(F(x,\zeta))$ satisfies the conditions i, ii, and iii. The pdf of the T-X distribution family is

$$g(x,\zeta) = \left[\frac{\partial}{\partial x} W(F(x,\zeta))\right] v[W(F(x,\zeta))],\ x \in R$$

Using the T-X family principle, various new distribution classes have been developed in the literature. Table 1 lists the $W(F(x,\zeta))$ function for T-X family members.

Table 1: Includes T-X family members.

$W(F(x,\zeta))$	Range of x	Members of the T-X family
$F(x,\zeta)$	[0,1]	[17]
$-\log[1-F(x,\zeta)]$	(0, ∞)	[18]
$-\log[F(x,\zeta)]$	(0, ∞)	[19]
$\frac{F(x,\zeta)}{1-F(x,\zeta)}$	(0, ∞)	[20]
$\log\left[\frac{F(x,\zeta)}{1-F(x,\zeta)}\right]$	(−∞, ∞)	[21]
$\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}$	[0, ∞)	[22]

Khalaf et al. [23] used the definition 1 and $W[G(x;\xi)] = \frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}$ to find the Burr XII-G (BXII-G) family. The cdf and pdf of the BXII-G will be as follows:

$$G(x) = 1 - \left[1 + \left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^d\right]^{-c},\ x > 0,\ c > 0,\ d > 0 \tag{2}$$

The pdf of the BXII-G family is

$$g(x) = cd\, F(x;\xi) f(x;\xi)(2-F(x;\xi))(1-F(x;\xi))^{-2} \times \left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^{d-1}\left[1 + \left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^d\right]^{-(c+1)} \tag{3}$$

where $x > 0, c, d > 0$; c is the scale parameter; d is the shape parameter and $F(x;\xi)$ denotes the baseline cdf distribution function.

Additional iterations of the Exponentiated Weibull distribution have been proposed by various researchers. These include the EW distribution by Pal et al. [24], the EW Poisson distribution by Mahmoudi et al. [25], and the EW power function distribution by Hassan et al. [26], for more information, see [27,28,29,30,31,32,33].

Definition 2. *Consider X a random variable with an Exponentiated Weibull distribution with β as the scale parameter while α and ϑ are the shape parameters, which have the cdf and pdf given in equations (4) and (5) respectively [34]:*

$$F(x) = \left(1 - e^{-\beta x^{\alpha}}\right)^{\vartheta} \tag{4}$$

$$f(x) = \vartheta\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}}\left(1 - e^{-\beta x^{\alpha}}\right)^{\vartheta-1} \tag{5}$$

where $x > 0, \vartheta, \alpha, \beta > 0$. The main objective of this study is to derive the Burr XII-G family of distributions. Adding more parameters to existing distributions improves their modeling capacity, hence the need to generalize Exponentiated Weibull distribution by developing Burr XII Exponentiated Weibull Distribution (BXIIEW) as a sub-model of the family to address the limitations of Exponentiated Weibull distribution in modeling data.

The rest of the article is outlined as follows: sections 2 introduce the BXIIEW distribution. In section 3, some statistical properties of the new distribution are introduced. Section 4 contains six distinct ways to estimate the parameters of the model. A simulation study is conducted on the newly proposed distribution in section 5. Section 6 presents the results of the application of the BXIIEW distribution to three real datasets. While the concluding report is given in section 7.

2 BXII exponentiated Weibull distribution

Definition 3. *A continuous random variable X has the BXIIEW distribution. Substitute the cdf and pdf of the Exponentiated Weibull distribution into equations (6) and (7). Then the cdf and pdf are as follows:*

$$G(x) = 1 - \left[1 + \left[\frac{\left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}, \tag{6}$$

$$\begin{aligned} g(x) =& cd\vartheta\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta-1} \left(2-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}\right) \left(1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}\right)^{-2} \\ &\times \left[\frac{\left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}}\right]^{d-1} \left[1 + \left[\frac{\left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-(c+1)} \end{aligned} \tag{7}$$

The hazard function of BXIIEW distribution is

$$h(x) = cd\vartheta\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta-1} \left(2-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}\right) \times \left(1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}\right)^{-2} \tag{8}$$

where $x > 0, c,\ d,\ \vartheta, \alpha, \beta > 0$.

The survival function for a random variable X is denoted as $s(x)$, and it may be obtained by the following derivation:

$$s(x) = \left[1 + \left[\frac{\left(1-e^{-\beta x^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c} \tag{9}$$

Figure 1 illustrates the pdf of the BXIIEW distribution, which might have a declining trend, a single peak, and a right-skewed shape. Figure 1 demonstrates that the BXIIEW distribution's hazard function $h(x)$ includes J-shaped and increasing forms.

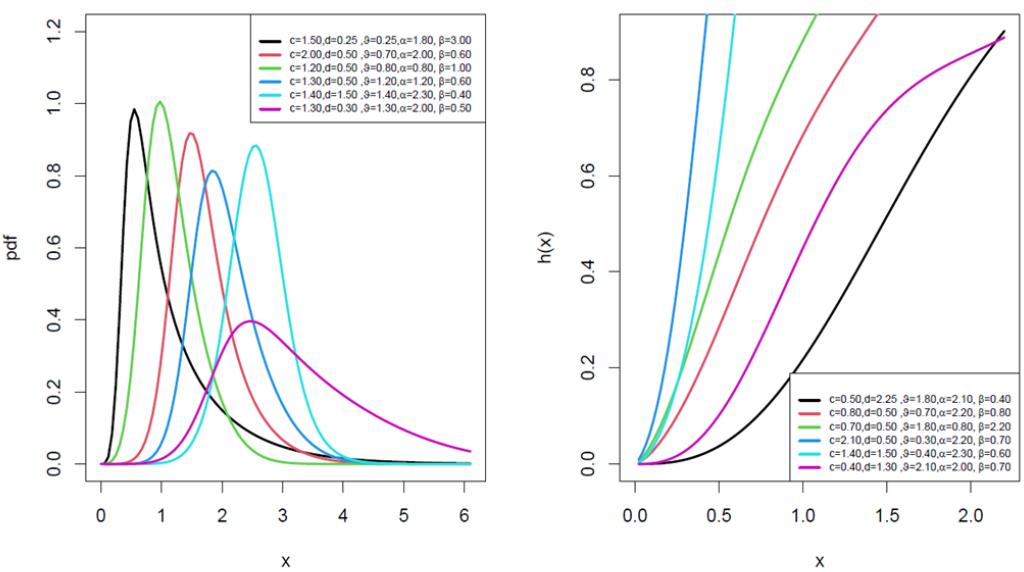

Fig. 1: The pdf and the hazard function plots of the BXIIEW distribution.

3 Statistical properties

3.1 Moment, Skewness and Kurtosis

The r^{th} moment of the BXIIEW distribution is defined as:

$$\acute{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r g(x)\, dx. \tag{10}$$

Equation (3) represents the newly derived family, which can be used to derive the r^{th} moment, by the method of binomial series expansion as given in refs. [35,36,1]: $[1-u]^s = \sum_{r=0}^{\infty} (-1)^r \binom{s}{r} u^r, [1-u]^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(s+j)}{m!\Gamma(p)} u^m \;\; |u| < 1,\; s > 0$ and exponential expansion

$$e^{-a} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a^j \; Then,$$

$$g(x) = \sum_{z=0}^{\infty} (-1)^z \binom{c+1}{z} cdf(x;\xi)(2-F(x;\xi))[F(x;\xi)]^{2dz+2d-1}[1-F(x;\xi)]^{-(dz+d+1)}$$

$$g(x) = \sum_{z=w=0}^{\infty} \frac{(-1)^z cd\Gamma(dz+d+1+w)}{w!\Gamma(dz+d+1)} \binom{c+1}{z} f(x;\xi)(2-F(x;\xi))[F(x;\xi)]^{w+2dz+2d-1}$$

$$g(x) = 2\mathscr{B} f(x;\xi)[F(x;\xi)]^m - \mathscr{B} f(x;\xi)[F(x;\xi)]^{m+1} \tag{11}$$

where

$$\mathscr{B} = \sum_{z=w=i=m=0}^{\infty} \frac{(-1)^{z+i+m} cd\Gamma(dz+d+1+w)}{w!\Gamma(dz+d+1)} \binom{c+r}{z} \binom{w+2dz+2d-1}{i} \binom{i}{m}.$$

By substituting equations (4) and (5) into equation (11),

$$g(x) = 2\mathscr{B}\vartheta\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta(j+1)-1} - \mathscr{B}\vartheta\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta(j+2)-1}$$

$$g(x) = \Omega x^{\alpha-1} e^{-(m+1)\beta x^{\alpha}} - \mathscr{U} x^{\alpha-1} e^{-(k+1)\beta x^{\alpha}} \tag{12}$$

where

$$\Omega = 2\mathscr{B}\vartheta\alpha\beta \sum_{m=0}^{\infty} (-1)^m \binom{\vartheta(j+1)-1}{m}, \; and \; \mathscr{U} = \mathscr{B}\vartheta\alpha\beta \sum_{k=0}^{\infty} (-1)^k \binom{\vartheta(j+2)-1}{k}.$$

By substituting equation (12) into equation (10)

$$\acute{\mu}_r = \Omega \int_0^{\infty} x^{r+\alpha-1} e^{-(m+1)\beta x^{\alpha}} dx - \mathscr{U} \int_0^{\infty} x^{r+\alpha-1} e^{-(k+1)\beta x^{\alpha}} dx$$

$$\acute{\mu}_r = \left(y = (m+1)\beta x^{\alpha} \;\Rightarrow x = \left(\frac{y}{(m+1)\beta}\right)^{\frac{1}{\alpha}} \right)$$

$$\acute{\mu}_r = \left(\Rightarrow dy = \alpha\beta(m+1)x^{\alpha-1}dx \;\Rightarrow dx = \frac{dy}{\alpha\beta(m+1)x^{\alpha-1}} \right)$$

$$\acute{\mu}_r = \frac{\Omega\Gamma\left(\frac{r}{\alpha}+1\right)}{\alpha((m+1)\beta)^{\frac{r}{\alpha}+1}} - \frac{\mathscr{U}\Gamma\left(\frac{r}{\alpha}+1\right)}{\alpha((k+1)\beta)^{\frac{r}{\alpha}+1}}. \tag{13}$$

Using equation (13), we can calculate the $\acute{\mu}_1, \acute{\mu}_2, \acute{\mu}_3$, and $\acute{\mu}_4$ and calculate the variance $V = E(X^2) - (E(X))^2$, skewness $S_1 = \frac{\mu_3}{\mu_2^{\left(\frac{3}{2}\right)}}$, and kurtosis $K_2 = \frac{\mu_4}{\mu_2^2}$ of the BXIIEW distribution.

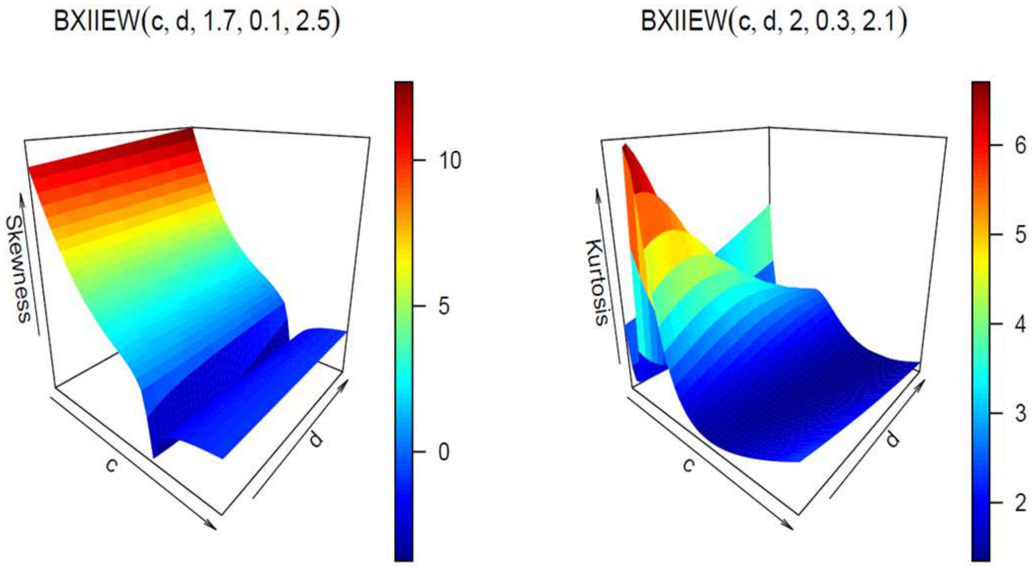

Fig. 2: 3D plots Skewness and Kurtosis for the BXIIEW.

Table 2: The effects of parameter changes on moments, variance (V), Skewness (S_1), and kurtosis (K_2).

c	d	ϑ	α	β	μ_1	μ_2	μ_3	μ_4	V	S_1	K_2
4	3	0.35	0.6	0.02	4.05	20	128	950	3.59	1.431	2.375
				0.03	2.70	9	38	187	1.71	1.407	2.308
			0.7	0.02	5.70	39	308	2780	6.51	1.264	1.827
				0.03	3.80	17	91	549	2.56	1.298	1.899
		0.43	0.6	0.02	7.08	60	589	6601	9.87	1.267	1.833
				0.03	4.72	26	174	1304	3.72	1.312	1.928
			0.7	0.02	9.24	98	1164	15281	12.6	1.199	1.591
				0.03	6.16	43	345	3018	5.05	1.223	1.632
	3.4	0.35	0.6	0.02	3.80	18	104	698	3.56	1.361	2.154
				0.03	2.53	8	30	138	1.59	1.325	2.156
			0.7	0.02	5.41	34	258	2162	4.73	1.301	1.870
				0.03	3.60	15	76	427	2.04	1.308	1.897
		0.43	0.6	0.02	6.71	53	493	5124	7.97	1.277	1.824
				0.03	4.47	23	146	1012	3.01	1.323	1.913
			0.7	0.02	8.84	89	1004	12403	10.8	1.195	1.565
				0.03	5.89	39	297	2450	4.30	1.219	1.610
5	3	0.35	0.6	0.02	4.35	22	129	844	3.07	1.250	1.743
				0.03	2.90	9	38	166	0.59	1.407	2.049
			0.7	0.02	6.11	42	320	2666	4.66	1.175	1.511
				0.03	4.07	18	95	526	1.43	1.243	1.623
		0.43	0.6	0.02	7.58	64	611	6308	6.54	1.193	1.540
				0.03	5.05	28	181	1246	2.49	1.221	1.589
			0.7	0.02	9.84	106	1235	15380	9.17	1.131	1.368
				0.03	6.56	47	365	3038	3.96	1.132	1.375
	3.4	0.35	0.6	0.02	4.15	20	109	669	2.77	1.218	1.672
				0.03	2.76	8	32	132	0.38	1.414	2.062
			0.7	0.02	5.86	38	279	2201	3.66	1.191	1.524
				0.03	3.90	17	82	434	1.79	1.169	1.501
		0.43	0.6	0.02	7.27	59	533	5204	6.14	1.176	1.494
				0.03	4.84	26	157	1028	2.57	1.184	1.520
			0.7	0.02	9.50	98	1101	13108	7.75	1.169	1.364
				0.03	6.33	43	326	2589	2.93	1.156	1.400

Figure 2 illustrates other tendencies by presenting 3D graphs of the skewness and kurtosis of random variable X for different values of c, d, ϑ, α, and β. The graphs reveal multiple non-monotonic patterns, highlighting the flexibility and responsiveness of these statistical measures to changes in the parameters. This malleability suggests that skewness and kurtosis, when given different values, might display complicated behaviors.

3.2 Incomplete moments

Equation (12) gives the $r^{th}(r>0)$ incomplete moments of the BXIIEW distribution as follows: [32]

$$\begin{aligned}\dot{\mu}_r(u) &= \int_{-\infty}^{u} x^r g(x)dx = \Omega \int_0^u x^{r+\alpha-1} e^{-(m+1)\beta x^\alpha} dx - \mathscr{U} \int_0^u x^{r+\alpha-1} e^{-(k+1)\beta x^\alpha} dx \\ &= \left(y = (m+1)\beta x^\alpha \ \Rightarrow x = \left(\frac{y}{(m+1)\beta}\right)^{\frac{1}{\alpha}} \right) \\ &= (x=0 \ \ \Rightarrow t=0 \ , and\ if \ \ x=u) \\ \dot{\mu}_r(u) &= \left(\Rightarrow \ y = (m+1)\beta u^\alpha \ then\ du = \frac{dy}{\alpha\beta(m+1)u^{\alpha-1}} \right)\end{aligned}$$

$$\dot{\mu}_r(u) = \frac{\Omega}{\alpha((m+1)\beta)^{\frac{r}{\alpha}+1}} \gamma\left(\frac{r}{\alpha}+1, (m+1)\beta u^\alpha\right) - \frac{\mathscr{U}}{\alpha((k+1)\beta)^{\frac{r}{\alpha}+1}} \gamma\left(\frac{r}{\alpha}+1, (k+1)\beta u^\alpha\right) \tag{14}$$

3.3 Quantile function (QF)

To derive the QF, use the inverse function of the cdf in equation (6),

$$1 - \left[1 + \left[\frac{\left(1-e^{-\beta x^\alpha}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x^\alpha}\right)^{\vartheta}} \right]^{d} \right]^{-c} = u,$$

hence the QF of BXIIEW distribution for $u \in U(0,1)$ is given by

$$Q(u) = \left(-\frac{1}{\beta} ln \left(1 - \left(\frac{-\left((1-u)^{-\frac{1}{c}}-1\right)^{\frac{1}{d}} + \sqrt{\left((1-u)^{-\frac{1}{c}}-1\right)^{\frac{2}{d}} + 4\left((1-u)^{-\frac{1}{c}}-1\right)^{\frac{1}{d}}}}{2} \right)^{\frac{1}{\vartheta}} \right) \right)^{\alpha} \tag{15}$$

3.4 Rényi entropy

The Rényi entropy of BXIIEW distribution is defined as [3,4]

$$T_R(v) = \frac{1}{1-v} log \int_0^\infty g^v(x)\, dx, v>0, \ v \neq 1, \tag{16}$$

the of the family can be expanded by using binomial series expansion.

$$g^v(x) = \aleph f^v(x) (F(x))^{v+2v(d-1)+2dj+z+i}, \tag{17}$$

where

$$\aleph = \sum_{j=z=m=i=0}^{\infty} \frac{(-1)^{j+i}(cd)^v \Gamma(2v+v(d-1)+dj+z)}{w!\Gamma(2v+v(d-1)+dj)} \binom{v(c+1)}{j} \binom{v}{m} \binom{m}{i}.$$

By substituting the equations (4) and (5) into (17)

$$g^{v}(x) = \aleph\,(\vartheta\alpha)^{v}\beta^{v}x^{v(\alpha-1)}e^{-v\beta x^{\alpha}}\left(1-e^{-\beta x^{\alpha}}\right)^{v(\vartheta-1)}\left(1-e^{-\beta x^{\alpha}}\right)^{\vartheta(v+2v(d-1)+2dj+z+i)}$$
$$= \mathfrak{K}x^{v(\alpha-1)}e^{-(v+s+h)\beta x^{\alpha}} \tag{18}$$

$$T_R(v) = \frac{1}{1-v}\log\left(\int_0^{\infty}\mathfrak{K}x^{v(\alpha-1)}e^{-(v+s+h)\beta x^{\alpha}}dx\right)$$
$$= \left(y=(v+s+h)\beta x^{\alpha}\ \Rightarrow x=\left(\frac{y}{(v+s+h)\beta}\right)^{\frac{1}{\alpha}}\right)$$
$$= \left(\Rightarrow dy=\alpha\beta\,(v+s+h)x^{\alpha-1}dx\ \Rightarrow dx=\frac{dy}{\alpha\beta\,(v+s+h)x^{\alpha-1}}\right)$$
$$T_R(v) = \frac{1}{1-v}\log\left(\frac{\mathfrak{K}\Gamma\left(v-\frac{v}{\alpha}+\frac{1}{\alpha}\right)}{\alpha\left((v+s+h)\beta\right)^{v-\frac{v}{\alpha}+\frac{1}{\alpha}}}\right), \tag{19}$$

where

$$\mathfrak{K} = \aleph\sum_{s=h=0}^{\infty}(-1)^{s+h}(\vartheta\alpha)^{v}\beta^{v}\binom{v(\vartheta-1)}{s}\binom{\vartheta\,(v+2v\,(d-1)+2dj+z+i)}{h}.$$

4 Estimation of parameters

4.1 Maximum likelihood estimation

In this subsection, the parameters of the BXIIEW distribution are estimated using the maximum likelihood method. We have a random sample $x_1,x_2,\ldots,x_n$ of size n with parameters $c,d,\vartheta,\alpha,$ and β from the BXIIEW distribution. Let $\rho=(c,d,\vartheta,\alpha,\beta)^T$ be the parameter vector. The likelihood function can be expressed as [5,6,37]

$$\begin{aligned} l(\rho) =& nlog(c)+nlog(d)+nlog(\vartheta)+nlog(\alpha)+nlog(\beta)+log\sum_{i=1}^{n}x_i^{\alpha-1}-\beta\sum_{i=1}^{n}x_i^{\alpha} \\ &+(2\vartheta-1)\,log\sum_{i=1}^{n}\left(1-e^{-\beta x_i^{\alpha}}\right)+log\sum_{i=1}^{n}\left(2-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right) \\ &-2log\sum_{i=1}^{n}\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)+(d-1)\,log\sum_{i=1}^{n}\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right) \\ &-(c+1)log\sum_{i=1}^{n}\left[\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}. \end{aligned} \tag{20}$$

Now calculate the first partial derivative of equation (20) concerning the parameters $(c,d,\vartheta,\alpha,\beta)$.

$$\frac{\partial l(\rho)}{\partial c} = \frac{n}{c}-log\sum_{i=1}^{n}\left[\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d} \tag{21}$$

$$\begin{aligned} \frac{\partial l(\rho)}{\partial d} =& \frac{n}{d}+\sum_{i=1}^{n}\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)-(c+1)\sum_{i=1}^{n}\ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)^{d} \\ &\times\ln\left(ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)\right) \end{aligned} \tag{22}$$

$$\frac{\partial l(\rho)}{\partial \vartheta} = \frac{n}{\vartheta} + 2ln\sum_{i=1}^{n}\left(1-e^{-\beta x_i^{\alpha}}\right) - \sum_{i=1}^{n}\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(2-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)} \tag{23}$$

$$+2\sum_{i=1}^{n}\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)}$$

$$+(d-1)\sum_{i=1}^{n}\frac{\frac{2\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}} + \frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)^{2}}}{\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}}$$

$$-d(c+1)\sum_{i=1}^{n}\frac{\ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)^{d}\left(\frac{2\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}} + \frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\ln\left(1-e^{-\beta x_i^{\alpha}}\right)}{\left(\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)^{2}}\right)}{\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)\ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)}$$

$$\frac{\partial l(\rho)}{\partial \alpha} = \frac{n}{\rho} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}}{x_i^{\alpha-1}} - \alpha\beta\sum_{i=1}^{n}\frac{x_i^{\alpha-1}}{x_i^{\alpha}} + (2\vartheta-1)\sum_{i=1}^{n}\frac{\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}}{\left(1-e^{-\beta x_i^{\alpha}}\right)} \tag{24}$$

$$-\sum_{i=1}^{n}\frac{\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}{\left(2-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)} + 2\sum_{i=1}^{n}\frac{\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)}$$

$$+(d-1)\sum_{i=1}^{n}\frac{\frac{2\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)\left(1-e^{-\beta x_i^{\alpha}}\right)} + \frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)^{2}\left(1-e^{-\beta x_i^{\alpha}}\right)}}{\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}}$$

$$\frac{\partial l(\rho)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{\alpha} + (2\vartheta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}e^{-\beta x_i^{\alpha}}}{\left(1-e^{-\beta x_i^{\alpha}}\right)} - \sum_{i=1}^{n}\frac{\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\vartheta x_i^{\alpha}e^{-\beta x_i^{\alpha}}}{\left(1-e^{-\beta x_i^{\alpha}}\right)}}{\left(2-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)} \tag{25}$$

$$+2\sum_{i=1}^{n}\frac{\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\vartheta x_i^{\alpha}e^{-\beta x_i^{\alpha}}}{\left(1-e^{-\beta x_i^{\alpha}}\right)}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)}$$

$$+(d-1)\sum_{i=1}^{n}\frac{\frac{2\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\vartheta x_i^{\alpha}e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)\left(1-e^{-\beta x_i^{\alpha}}\right)} + \frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\vartheta x_i^{\alpha}e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)^{2}\left(1-e^{-\beta x_i^{\alpha}}\right)}}{\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}}$$

$$(c+1)\sum_{i=1}^{n}\frac{ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)^{d}\left(\frac{2\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)\left(1-e^{-\beta x_i^{\alpha}}\right)}+\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\vartheta\beta x_i^{\alpha}\ln(x_i)e^{-\beta x_i^{\alpha}}}{\left(1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}\right)^{2}\left(1-e^{-\beta x_i^{\alpha}}\right)}\right)}{\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)\ln\left(\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right)}$$

The Maximum likelihood estimates for parameters c, d, ϑ, α, and β cannot be obtained analytically but numerically by using Newton-Raphson's method. It is calculated by equating equations ((21) to (25)) to zero and solving numerically.

4.2 Least squares estimation (LSE)

The estimation of parameters for the BXIIWE entails the use of the LSE approach. The main objective of this estimating technique is to minimize the following equation:

$$L(\rho) = \sum_{i=1}^{n}\left(G(x_{i;n},\rho) - \frac{i}{n+1}\right)^{2}$$

$$L(\rho) = \sum_{i=1}^{n}\left(1-\left[1+\left[\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c} - \frac{i}{n+1}\right)^{2} \tag{26}$$

4.3 Weighted least squares estimation (WLSE)

The estimation of parameters for the BXIIWE entails the use of the WLSE approach. The main objective of this estimating technique is to minimize the following equation:

$$W(\rho) = \sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(G(x_{i;n},\rho) - \frac{i}{n+1}\right)^{2}$$

$$W(\rho) = \sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(1-\left[1+\left[\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c} - \frac{i}{n+1}\right)^{2} \tag{27}$$

4.4 Maximum product space estimators (MPSE)

The estimation of parameters for the BXIIWE entails the use of the MPSE approach. The main objective of this estimating technique is to minimize the following equation:

$$G_s(\rho) = \frac{1}{n+1}\sum_{i=1}^{n}\ln\left(G(x_{i;n},\rho) - G(x_{i-1;n},\rho)\right)$$

$$G_s(\rho) = \frac{1}{n+1}\sum_{i=1}^{n}\ln\left(1-\left[1+\left[\frac{\left(1-e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c} - \left(1-\left[1+\left[\frac{\left(1-e^{-\beta x_{i-1}{}^{\alpha}}\right)^{2\vartheta}}{1-\left(1-e^{-\beta x_{i-1}^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}\right)\right) \tag{28}$$

4.5 Anderson-Darling estimation (ADE)

The estimation of parameters for the BXIIWE entails the use of the ADE approach. The main objective of this estimating technique is to minimize the following equation:

$$A(\rho) = -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\ln\left(G\left(x_{i;n}, \rho\right)\right) + \ln\left(S\left(x_{i;n}, \rho\right)\right)\right)$$

$$A(\rho) = -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\ln\left(1 - \left[1 + \left[\frac{\left(1 - e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1 - \left(1 - e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}\right) + \ln\left(\left[1 + \left[\frac{\left(1 - e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1 - \left(1 - e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}\right)\right) \tag{29}$$

4.6 Right-Tailed Anderson-Darling estimation (RTADE)

The estimation of parameters for the BXIIWE entails the use of the RTADE approach. The main objective of this estimation technique is to minimize the following equation:

$$R(\rho) = \frac{n}{2} - 2\sum_{i=1}^{n} G\left(x_{i;n}, \rho\right) - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\ln\left(S\left(x_{i;n}, \rho\right)\right)$$

$$R(\rho) = \frac{n}{2} - 2\sum_{i=1}^{n}\left(1 - \left[1 + \left[\frac{\left(1 - e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1 - \left(1 - e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}\right) - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\ln\left(\left[1 + \left[\frac{\left(1 - e^{-\beta x_i^{\alpha}}\right)^{2\vartheta}}{1 - \left(1 - e^{-\beta x_i^{\alpha}}\right)^{\vartheta}}\right]^{d}\right]^{-c}\right) \tag{30}$$

5 Simulation study

In this part, the results of the simulation study on BXIIEW distribution are presented and discussed. The study examines three distinct sets of parameter values ($c = 1.3$, $d = 0.8$, $\vartheta = 0.7$, $\alpha = 0.7$, $\beta = 0.3$), ($c = 1.5$, $d = 0.9$, $\vartheta = 0.9$, $\alpha = 0.8$, $\beta = 0.3$), and ($c = 1.4$, $d = 0.9$, $\vartheta = 0.9$, $\alpha = 0.6$, $\beta = 0.4$). For each set of parameters, 1000 samples are generated, with sample sizes of n =30, 60, 120, 200, and 300. We use these samples to calculate average bias (Bias)), and root mean square error (RMSE). To determine bias and RMSE for the estimated parameter denoted as $\hat{a}$, use the following:

$$bias(\hat{a}) = \frac{\sum_{i=1}^{N}\hat{a}_i}{N} - a, \quad and\ RMSE(\hat{a}) = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{a}_i - a\right)^2}{N}}.$$

The Bias and RMSE estimates are available in Tables 3–5; additionally, the graphs of the MSE for the specified schemes in Tables 3-5 are shown in Figures 3–5; hence, we use ranks related to Tables 3-5, which are compiled in Tables 3-5 to compare the several approaches of estimation.

- With an increase in n, the estimated values of the parameters (c, d, ϑ, α, and β) approach their original values more closely.

- Although there are oscillations in certain schemes, the overall tendency is a decreased one, and the values of Bias and RMSE decrease as n increases.

Table 3: Displays Monte Carlo simulation results for the BXIIEW distribution ($c = 1.3, d = 0.8, \vartheta = 0.7, \alpha = 0.7, \beta = 0.3$).

n	Est.	Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	RMSE	$\widehat{c}$	1.7287	1.3619	1.3660	2.1259	1.2864	1.6107
		$\widehat{d}$	1.3150	0.7852	0.7291	1.2051	0.7206	0.6301
		$\widehat{\vartheta}$	1.6383	0.6792	0.9600	3.1041	0.9584	0.6224
		$\widehat{\alpha}$	0.9262	0.6681	0.6049	1.3007	0.5904	0.5789
		$\widehat{\beta}$	1.6597	0.6105	0.8994	2.5705	1.1265	0.5254
	Bias	$\widehat{c}$	0.4892	0.5480	0.5818	0.5027	0.5485	0.6501
		$\widehat{d}$	0.1085	0.1384	0.1378	0.0293	0.1296	0.0930
		$\widehat{\vartheta}$	0.5095	0.0503	0.0912	1.0873	0.1017	0.1505
		$\widehat{\alpha}$	0.4419	0.2099	0.1569	0.6790	0.1694	0.1433
		$\widehat{\beta}$	0.5484	0.1213	0.1567	0.9424	0.1736	0.1594
60	RMSE	$\widehat{c}$	1.6062	0.9670	1.0532	1.0185	0.9655	1.1067
		$\widehat{d}$	0.9676	0.6752	0.6983	0.9896	0.6024	0.5126
		$\widehat{\vartheta}$	1.1148	0.4279	0.5108	1.7977	0.6406	0.5327
		$\widehat{\alpha}$	0.6047	0.4573	0.4063	0.6947	0.3736	0.3614
		$\widehat{\beta}$	0.9099	0.3597	0.4263	1.2575	0.4087	0.4228
	Bias	$\widehat{c}$	0.3062	0.3515	0.4136	0.1544	0.4192	0.4353
		$\widehat{d}$	0.1051	0.1209	0.1353	0.0115	0.1149	0.0878
		$\widehat{\vartheta}$	0.2894	0.0292	0.0546	0.6029	0.0375	0.1224
		$\widehat{\alpha}$	0.2460	0.1115	0.0736	0.3663	0.0745	0.0441
		$\widehat{\beta}$	0.3098	0.0721	0.0800	0.5238	0.0627	0.1016
120	RMSE	$\widehat{c}$	0.7924	0.7751	0.7475	0.6822	0.7226	0.8420
		$\widehat{d}$	0.6850	0.4632	0.4924	0.6710	0.5305	0.5109
		$\widehat{\vartheta}$	0.8083	0.3893	0.4283	1.0495	0.4049	0.3967
		$\widehat{\alpha}$	0.3558	0.3198	0.2621	0.3240	0.2502	0.2550
		$\widehat{\beta}$	0.6022	0.3225	0.3169	0.7851	0.2940	0.2664
	Bias	$\widehat{c}$	0.2853	0.2981	0.3264	0.0919	0.3309	0.3782
		$\widehat{d}$	0.0867	0.0623	0.0899	0.0310	0.1065	0.0838
		$\widehat{\vartheta}$	0.1761	0.0184	0.0562	0.3339	0.0357	0.0832
		$\widehat{\alpha}$	0.0865	0.0565	0.0335	0.1678	0.0703	0.0165
		$\widehat{\beta}$	0.1677	0.0425	0.0548	0.3271	0.0403	0.0551
200	RMSE	$\widehat{c}$	0.6176	0.6248	0.5680	0.5320	0.5703	0.6185
		$\widehat{d}$	0.6355	0.4076	0.4083	0.6393	0.4329	0.4155
		$\widehat{\vartheta}$	0.7094	0.3263	0.3480	0.7264	0.3447	0.3822
		$\widehat{\alpha}$	0.2410	0.2513	0.2205	0.2530	0.2154	0.2093
		$\widehat{\beta}$	0.4304	0.2550	0.2590	0.5507	0.2327	0.2431
	Bias	$\widehat{c}$	0.2086	0.2144	0.2140	0.0299	0.2283	0.2311
		$\widehat{d}$	0.0811	0.0477	0.0381	0.0049	0.0738	0.0688
		$\widehat{\vartheta}$	0.1415	0.0176	0.0283	0.2350	0.0260	0.0742
		$\widehat{\alpha}$	0.0509	0.0295	0.0187	0.1230	0.0071	0.0159
		$\widehat{\beta}$	0.1176	0.0401	0.0452	0.2268	0.0277	0.0493
300	RMSE	$\widehat{c}$	0.5046	0.4928	0.4752	0.4325	0.4812	0.5045
		$\widehat{d}$	0.6281	0.3867	0.3363	0.5460	0.3943	0.3739
		$\widehat{\vartheta}$	0.6328	0.2925	0.3142	0.6089	0.3141	0.3223
		$\widehat{\alpha}$	0.2079	0.2068	0.1863	0.2105	0.1789	0.1871
		$\widehat{\beta}$	0.3297	0.1958	0.2078	0.3800	0.1989	0.1918
	Bias	$\widehat{c}$	0.1420	0.1510	0.1591	0.0009	0.1744	0.1725
		$\widehat{d}$	0.0803	0.0407	0.03711	0.0047	0.0600	0.0609
		$\widehat{\vartheta}$	0.1191	0.0129	0.0184	0.1820	0.0269	0.0727
		$\widehat{\alpha}$	0.0419	0.0179	0.0074	0.0950	0.0031	0.0145
		$\widehat{\beta}$	0.0832	0.0337	0.0356	0.1588	0.0233	0.0374

Table 4: Displays Monte Carlo simulation results for the BXIIEW distribution ($c = 1.5, d = 0.9, \vartheta = 0.9, \alpha = 0.8, \beta = 0.3$).

n	Est.	Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	RMSE	$\widehat{c}$	1.7565	1.3372	1.4410	2.2534	1.4204	1.7220
		$\widehat{d}$	2.0691	1.1539	1.1231	3.2497	1.0834	0.8798
		$\widehat{\vartheta}$	1.4194	0.6318	0.6798	2.5238	0.5954	0.7660
		$\widehat{\alpha}$	1.4999	0.7409	0.6952	1.6212	0.6745	0.7551
		$\widehat{\beta}$	0.6406	0.3096	0.3495	0.8369	0.3003	0.3910
	Bias	$\widehat{c}$	0.4366	0.5723	0.6623	0.3930	0.6480	0.6739
		$\widehat{d}$	0.3183	0.2481	0.2109	0.4455	0.1749	0.0827
		$\widehat{\vartheta}$	0.3782	0.0542	0.0477	0.8230	0.0561	0.1895
		$\widehat{\alpha}$	0.7017	0.2286	0.1707	0.9652	0.2192	0.2165
		$\widehat{\beta}$	0.2147	0.0474	0.0689	0.3102	0.0479	0.1211
60	RMSE	$\widehat{c}$	1.0680	1.1226	1.1531	1.0024	1.1112	1.3043
		$\widehat{d}$	1.3207	0.8947	0.9772	2.1643	0.8010	0.7702
		$\widehat{\vartheta}$	1.4184	0.4723	0.5098	1.5508	0.4724	0.7367
		$\widehat{\alpha}$	0.6834	0.5418	0.5362	0.9440	0.4978	0.5280
		$\widehat{\beta}$	0.4675	0.2216	0.2279	0.4718	0.2067	0.2870
	Bias	$\widehat{c}$	0.2196	0.3976	0.4651	0.0983	0.4576	0.5417
		$\widehat{d}$	0.1126	0.1562	0.1352	0.1464	0.1452	0.1072
		$\widehat{\vartheta}$	0.3185	0.0152	0.0397	0.5752	0.0414	0.1018
		$\widehat{\alpha}$	0.3477	0.1500	0.1349	0.5800	0.1394	0.1134
		$\widehat{\beta}$	0.1607	0.0360	0.0311	0.2328	0.0266	0.0542
120	RMSE	$\widehat{c}$	1.0204	0.8398	0.8202	0.7251	0.8613	0.9775
		$\widehat{d}$	0.9484	0.6700	0.6589	1.0526	0.6786	0.6096
		$\widehat{\vartheta}$	1.0503	0.4330	0.4149	1.0832	0.4116	0.4219
		$\widehat{\alpha}$	0.4656	0.4365	0.4249	0.5536	0.3353	0.3519
		$\widehat{\beta}$	0.3355	0.1626	0.1577	0.3547	0.1538	0.1778
	Bias	$\widehat{c}$	0.2067	0.2771	0.2945	0.0883	0.3256	0.3962
		$\widehat{d}$	0.0886	0.0966	0.0808	0.0921	0.1005	0.0824
		$\widehat{\vartheta}$	0.2626	0.0086	0.0260	0.3742	0.0213	0.0441
		$\widehat{\alpha}$	0.1829	0.1068	0.0764	0.3321	0.0541	0.0373
		$\widehat{\beta}$	0.1085	0.0174	0.0236	0.1613	0.0183	0.0238
200	RMSE	$\widehat{c}$	0.8079	0.6874	0.7214	0.6057	0.7232	0.8452
		$\widehat{d}$	0.7403	0.5094	0.5085	1.0163	0.5637	0.5167
		$\widehat{\vartheta}$	0.9895	0.3322	0.3411	1.1301	0.3424	0.4163
		$\widehat{\alpha}$	0.3628	0.3593	0.3207	0.4240	0.3041	0.3041
		$\widehat{\beta}$	0.3061	0.1288	0.1296	0.3126	0.1254	0.1380
	Bias	$\widehat{c}$	0.1910	0.1802	0.2435	0.0847	0.2667	0.3413
		$\widehat{d}$	0.0138	0.0630	0.0590	0.0318	0.0878	0.0773
		$\widehat{\vartheta}$	0.2056	0.0074	0.0151	0.3108	0.0205	0.0406
		$\widehat{\alpha}$	0.1003	0.0803	0.0426	0.2295	0.0211	0.0023
		$\widehat{\beta}$	0.0726	0.0111	0.0120	0.1298	0.0119	0.0158
300	RMSE	$\widehat{c}$	0.7446	0.6451	0.6557	0.5251	0.2970	0.7940
		$\widehat{d}$	0.5691	0.4351	0.4031	0.8335	0.2683	0.3688
		$\widehat{\vartheta}$	0.7803	0.2836	0.2904	0.8485	0.1053	0.2953
		$\widehat{\alpha}$	0.3063	0.2935	0.2877	0.3217	0.7940	0.2689
		$\widehat{\beta}$	0.2485	0.0994	0.1044	0.2581	0.1068	0.1124
	Bias	$\widehat{c}$	0.1030	0.1151	0.2369	0.0647	0.0076	0.3292
		$\widehat{d}$	0.0103	0.0559	0.0355	0.0072	0.0097	0.0266
		$\widehat{\vartheta}$	0.1670	0.0004	0.0150	0.2330	0.0030	0.0145
		$\widehat{\alpha}$	0.0563	0.0313	0.0206	0.1636	0.0129	0.0014
		$\widehat{\beta}$	0.0591	0.0017	0.0092	0.0960	0.0106	0.0067

Table 5: Displays Monte Carlo simulation results for the BXIIEW distribution ($c = 1.4, d = 0.9, \vartheta = 0.9, \alpha = 0.6, \beta = 0.4$).

n	Est.	Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	RMSE	$\widehat{c}$	2.1323	1.3451	2.2221	2.2123	1.4267	1.7017
		$\widehat{d}$	1.8111	1.0419	0.9867	2.5379	0.9588	0.8427
		$\widehat{\vartheta}$	2.8027	0.8794	0.9003	2.4772	0.6543	0.8620
		$\widehat{\alpha}$	1.0835	0.6011	0.5739	1.2547	0.5323	0.5857
		$\widehat{\beta}$	3.0620	0.9884	0.7767	1.8853	0.6087	0.8595
	Bias	$\widehat{c}$	0.5004	0.5525	0.6479	0.4405	0.6488	0.7361
		$\widehat{d}$	0.3223	0.2307	0.1829	0.2986	0.1859	0.1303
		$\widehat{\vartheta}$	0.6547	0.0686	0.0401	0.8373	0.0280	0.1004
		$\widehat{\alpha}$	0.5498	0.1873	0.1733	0.7362	0.1759	0.1639
		$\widehat{\beta}$	0.6809	0.1307	0.1371	1.8853	0.1111	0.1842
60	RMSE	$\widehat{c}$	1.0913	1.0455	1.0996	1.0018	0.9982	1.3032
		$\widehat{d}$	1.2172	0.8011	0.9299	1.1446	0.7617	0.6622
		$\widehat{\vartheta}$	1.4055	0.4172	0.7011	1.7963	0.5635	0.5536
		$\widehat{\alpha}$	0.5955	0.4589	0.4154	0.6469	0.3565	0.3544
		$\widehat{\beta}$	1.1151	0.3586	0.5883	1.2097	0.4235	0.4378
	Bias	$\widehat{c}$	0.2552	0.3948	0.4052	0.0719	0.4192	0.5492
		$\widehat{d}$	0.1070	0.1439	0.1656	0.0802	0.1366	0.0782
		$\widehat{\vartheta}$	0.3333	0.0431	0.0329	0.6005	0.0130	0.0375
		$\widehat{\alpha}$	0.2670	0.1331	0.1045	0.4075	0.0939	0.0754
		$\widehat{\beta}$	0.3602	0.0417	0.0909	0.5018	0.0577	0.0748
120	RMSE	$\widehat{c}$	0.8728	0.7543	0.8071	0.6798	0.8359	0.9844
		$\widehat{d}$	0.7903	0.6410	0.6019	0.8817	0.6273	0.5450
		$\widehat{\vartheta}$	0.9377	0.3666	0.3879	1.2615	0.4310	0.4368
		$\widehat{\alpha}$	0.3305	0.3019	0.2951	0.3901	0.2678	0.2555
		$\widehat{\beta}$	0.8771	0.2966	0.3288	0.8735	0.3088	0.3362
	Bias	$\widehat{c}$	0.2398	0.2436	0.3164	0.0642	0.3274	0.4090
		$\widehat{d}$	0.0700	0.0980	0.0795	0.0735	0.1114	0.0755
		$\widehat{\vartheta}$	0.1497	0.0166	0.0363	0.3996	0.0126	0.0401
		$\widehat{\alpha}$	0.1337	0.0799	0.0440	0.2287	0.0410	0.0203
		$\widehat{\beta}$	0.1967	0.0326	0.0495	0.3545	0.0266	0.0525
200	RMSE	$\widehat{c}$	0.7346	0.7217	0.6689	0.5431	0.6577	0.7331
		$\widehat{d}$	0.6015	0.4865	0.4835	0.7928	0.4875	0.4205
		$\widehat{\vartheta}$	0.7570	0.3204	0.3350	0.9122	0.3758	0.3814
		$\widehat{\alpha}$	0.2457	0.2468	0.2139	0.2924	0.2058	0.2074
		$\widehat{\beta}$	0.4979	0.2283	0.2507	0.6077	0.2421	0.2554
	Bias	$\widehat{c}$	0.1812	0.2195	0.2182	0.0594	0.2369	0.2851
		$\widehat{d}$	0.0383	0.0698	0.0664	0.0402	0.0671	0.0505
		$\widehat{\vartheta}$	0.1379	0.0160	0.0268	0.2542	0.0114	0.0157
		$\widehat{\alpha}$	0.0843	0.0415	0.0225	0.1714	0.0220	0.0139
		$\widehat{\beta}$	0.1357	0.0215	0.0368	0.2413	0.0203	0.0245
300	RMSE	$\widehat{c}$	0.6237	0.6129	0.6008	0.4998	0.5943	0.6732
		$\widehat{d}$	0.6045	0.4490	0.4347	0.7156	0.4435	0.3872
		$\widehat{\vartheta}$	0.7435	0.2911	0.3135	0.8274	0.3195	0.3302
		$\widehat{\alpha}$	0.2026	0.2105	0.1961	0.2350	0.1838	0.1866
		$\widehat{\beta}$	0.4382	0.2054	0.2260	0.4710	0.2206	0.2304
	Bias	$\widehat{c}$	0.1679	0.1931	0.2117	0.0448	0.2275	0.2666
		$\widehat{d}$	0.0729	0.0667	0.0647	0.0335	0.0589	0.0464
		$\widehat{\vartheta}$	0.1339	0.0104	0.0229	0.2333	0.0107	0.0115
		$\widehat{\alpha}$	0.0418	0.0199	0.0031	0.1126	0.0028	0.0058
		$\widehat{\beta}$	0.1100	0.0189	0.0267	0.1801	0.0162	0.0242

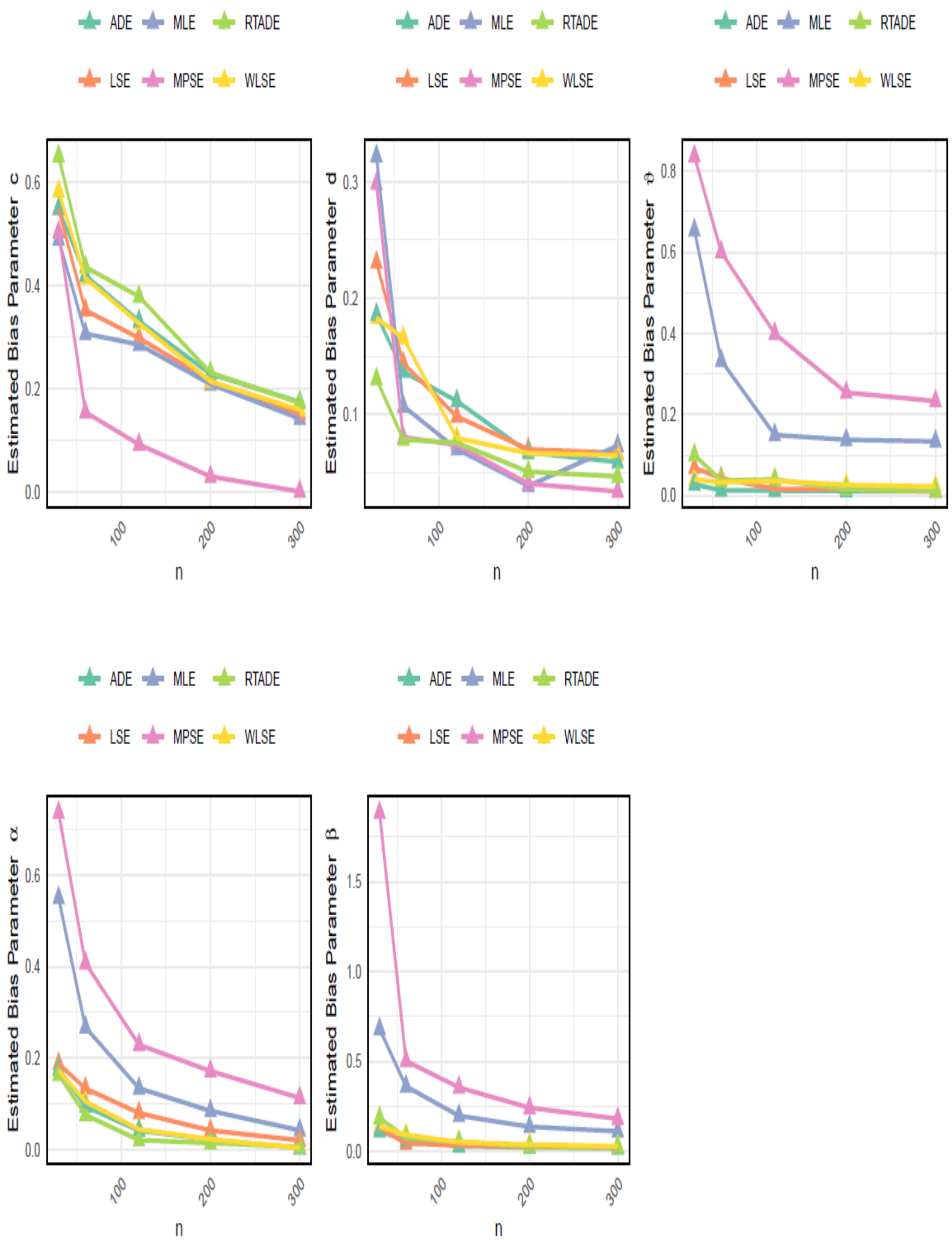

Fig. 3: Bias for c, d, ϑ, α, and β for the schemes in Table 3.

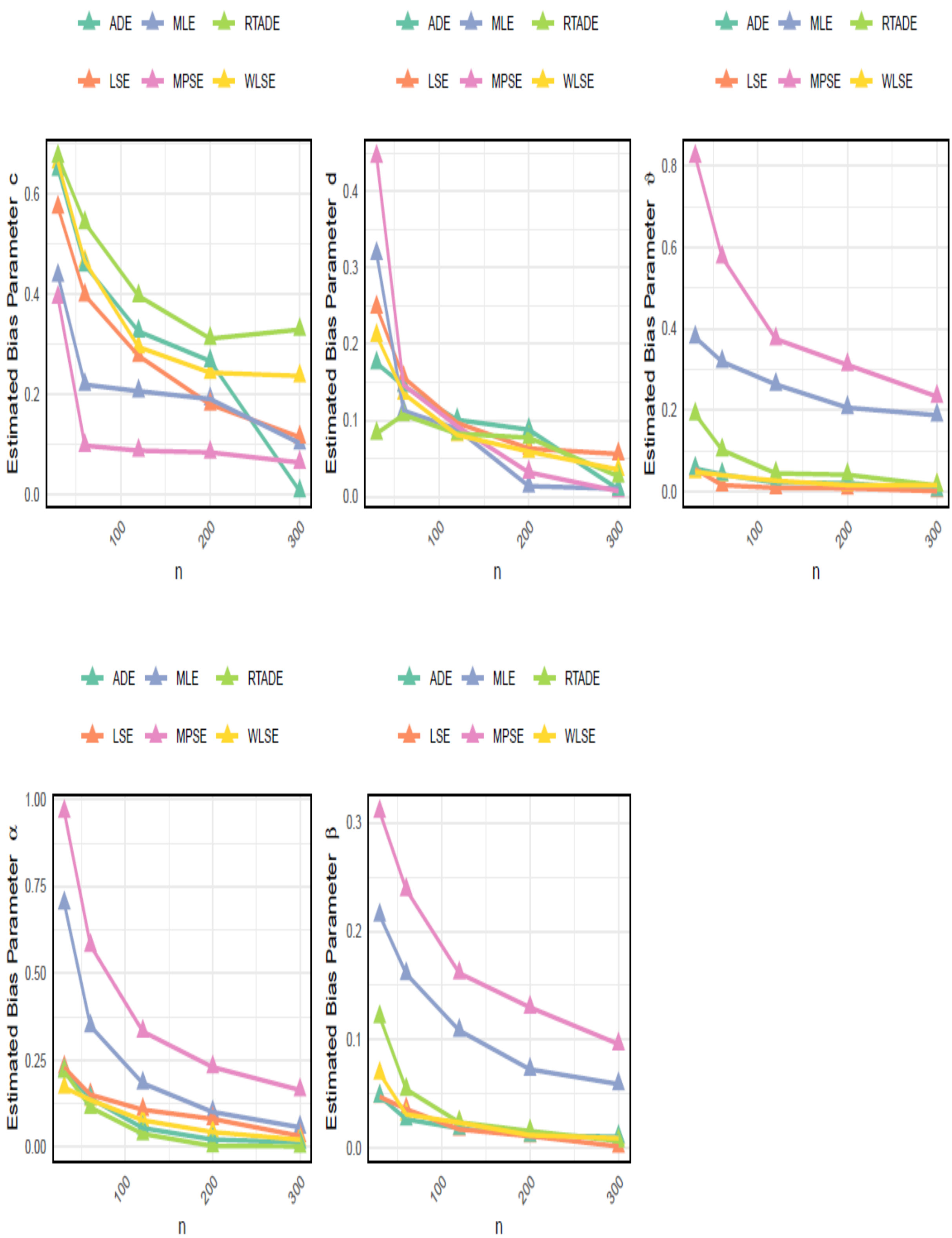

Fig. 4: Bias for c, d, ϑ, α, and β for the schemes in Table 4.

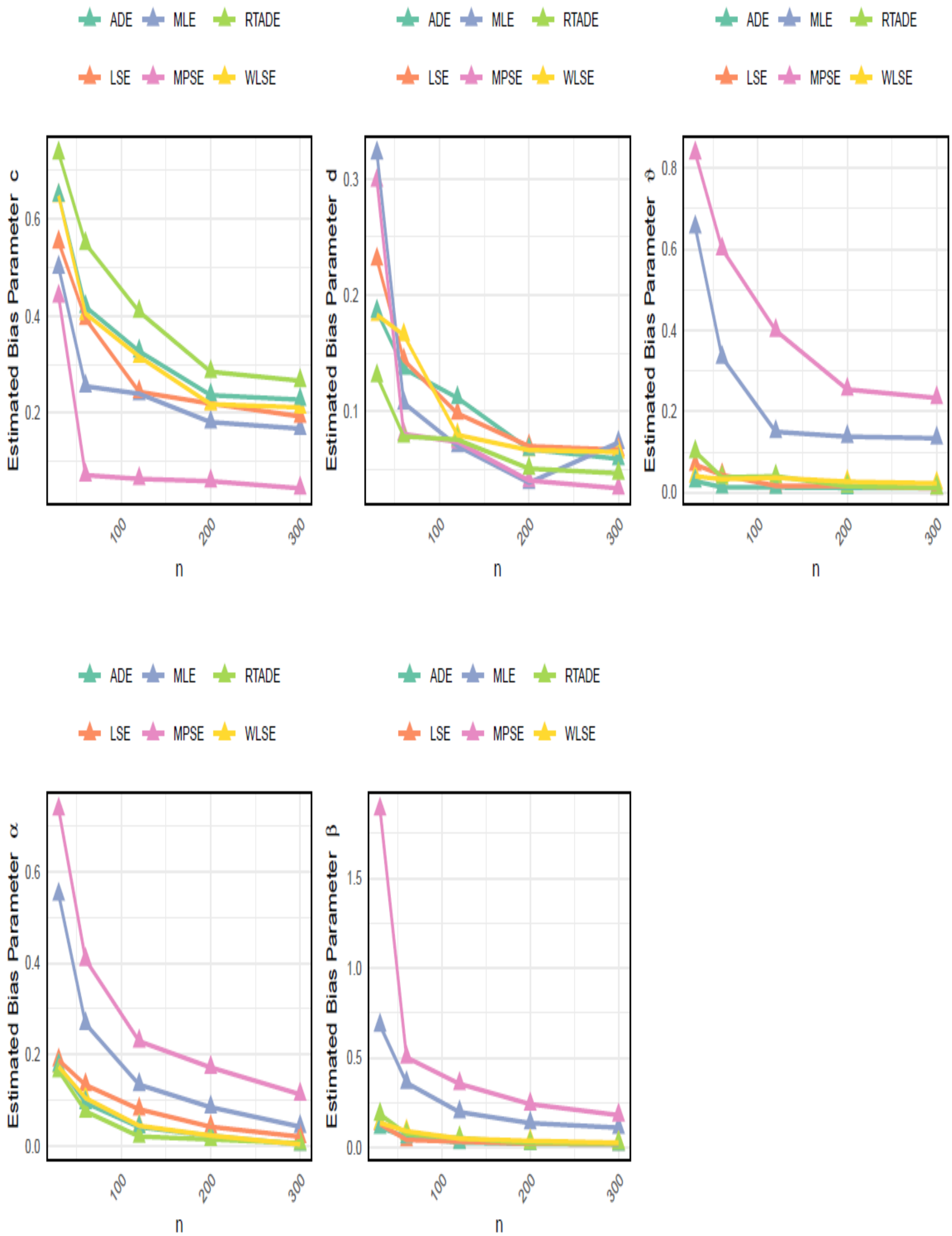

Fig. 5: Bias for c, d, ϑ, α, and β for the schemes in Table 5.

6 Application

In this part, BXIIEW distribution is applied to three datasets to show its flexibility. The goodness-of-fit criteria and maximum likelihood estimates (MLEs) of the proposed model are compared to other competing models. Table 6 shows the various descriptive statistics of the datasets. We use nine known goodness-of-fit metrics to compare these models, including the log-likelihood, Akaike information criterion (AIC), consistent AIC (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (K-S), Anderson-Darling (A), and Cramér-von Mises (W) values, as well as largest P-values (P-v).

Figures 6, 8, and 10 include the Box plot, violin plot, histogram, Q-Q plot, and TTT plot. Figures 7, 9, and 11 show the fitted densities, empirical cdf, and Probability plot for the three datasets, demonstrating the superiority of the BXIIEW model over its counterparts: Truncated Exponentiated Exponential Exponentiated Weibull distribution (TEEEW), Beta Exponentiated Weibull distribution (BeEW) [38], Kumaraswamy Exponentiated Weibull distribution (KuEW), Exponential Generalized Exponentiated Weibull distribution (EGEW), Weibull Exponentiated Weibull distribution (WeEW) [39], Exponentiated Weibull (WE) [27].

Tables (7-9) summarise the MLs of the parameters, the log-likelihood, and various goodness of fit for each model. The results in Tables (7-9) indicate that the BXIIWE has the lowest CAIC, AIC, BIC, HQIC, K-S, A, and W measures, as well as the best p-values among all the fitted models. Furthermore, Figures 7, 9, and 11 demonstrate that BXIIWE best matches the actual distribution of the examined datasets. Consequently, when compared to competing distributions, the BXIIWE is the most appropriate model for the analyzed data.

Table 6: Descriptive analysis of the three data.

Dataset	N	Min.	Median	Mean	SD	Max.	SK	KU
Data I	108	1.04	5.19	5.76	3.25	16.5	0.97	0.61
Data II	59	3	6.92	6.98	1.62	11.04	0.18	0.02
Data III	40	0.32	3.35	3.14	1.36	5.38	0.4	0.84

Table 6 suggests that the three data sets are positively skewed as well as the graph of the pdf of the BXIIEW distribution. In each of the three data sets kurtosis (KU) is less than 3, indicating that the datasets are platykurtic.

6.1 First dataset I

The first data represents the mortality rate in Mexico on account of COVID-19. It spans through 108 days, beginning on March 4th to July 20th, 2020. The dataset has been previously used by El-Saeed et al. [40,41]:

(8.826, 6.105 ,10.383, 7.267 ,13.220, 6.015 ,10.855, 6.122 ,10.685, 10.035, 5.242 ,7.630 ,14.604 ,7.903 ,6.327 ,9.391 ,14.962 ,4.730 ,3.215 ,16.498, 11.665 ,9.284, 12.878, 6.656, 3.440 ,5.854, 8.813 ,10.043, 7.260, 5.985, 4.424 ,4.344 ,5.143 ,9.935 ,7.840 ,9.550 ,6.968 ,6.370 ,3.537 ,3.286 ,10.158, 8.108 ,6.697 ,7.151 ,6.560 ,2.988 ,3.336 ,6.814 ,8.325 ,7.854 ,8.551 ,3.228, 3.499 ,3.751 ,7.486 ,6.625 ,6.140 ,4.909 ,4.661 ,1.867 ,2.838 ,5.392, 12.042, 8.696 ,6.412 ,3.395 ,1.815 ,3.327 ,5.406 ,6.182 ,4.949 ,4.089 ,3.359 ,2.070, 3.298 ,5.317 ,5.442 ,4.557 ,4.292 ,2.500 ,6.535 ,4.648 ,4.697 ,5.459 ,4.120, 3.922 ,3.219, 1.402 ,2.438, 3.257 ,3.632, 3.233 ,3.027, 2.352 ,1.205 ,2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923).

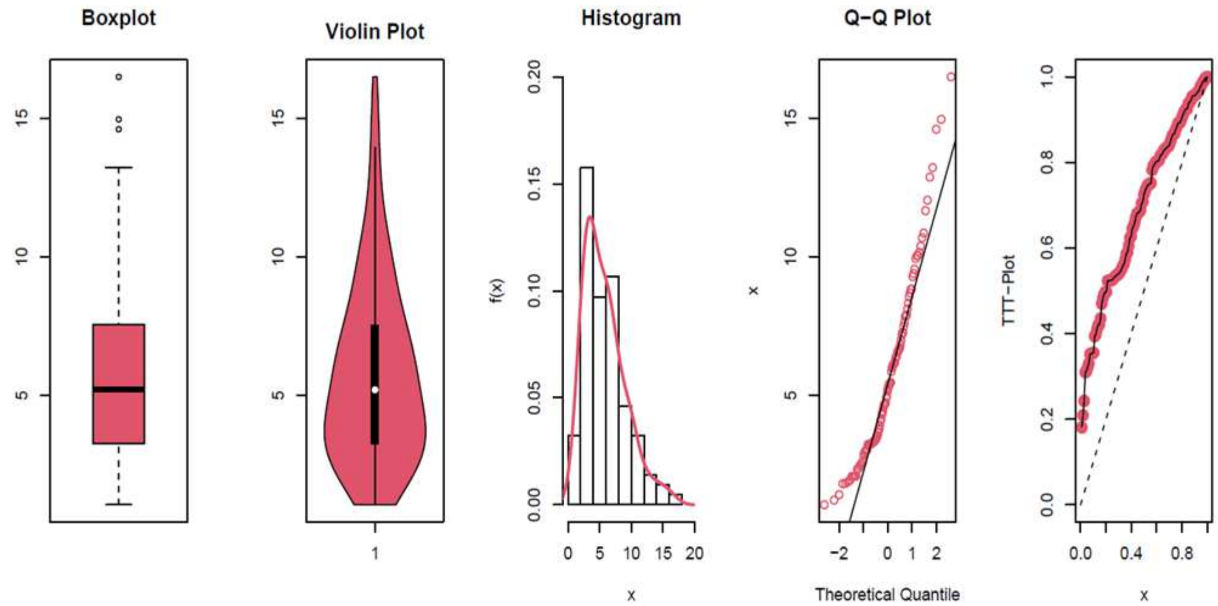

Fig. 6: Shows data I visualizations.

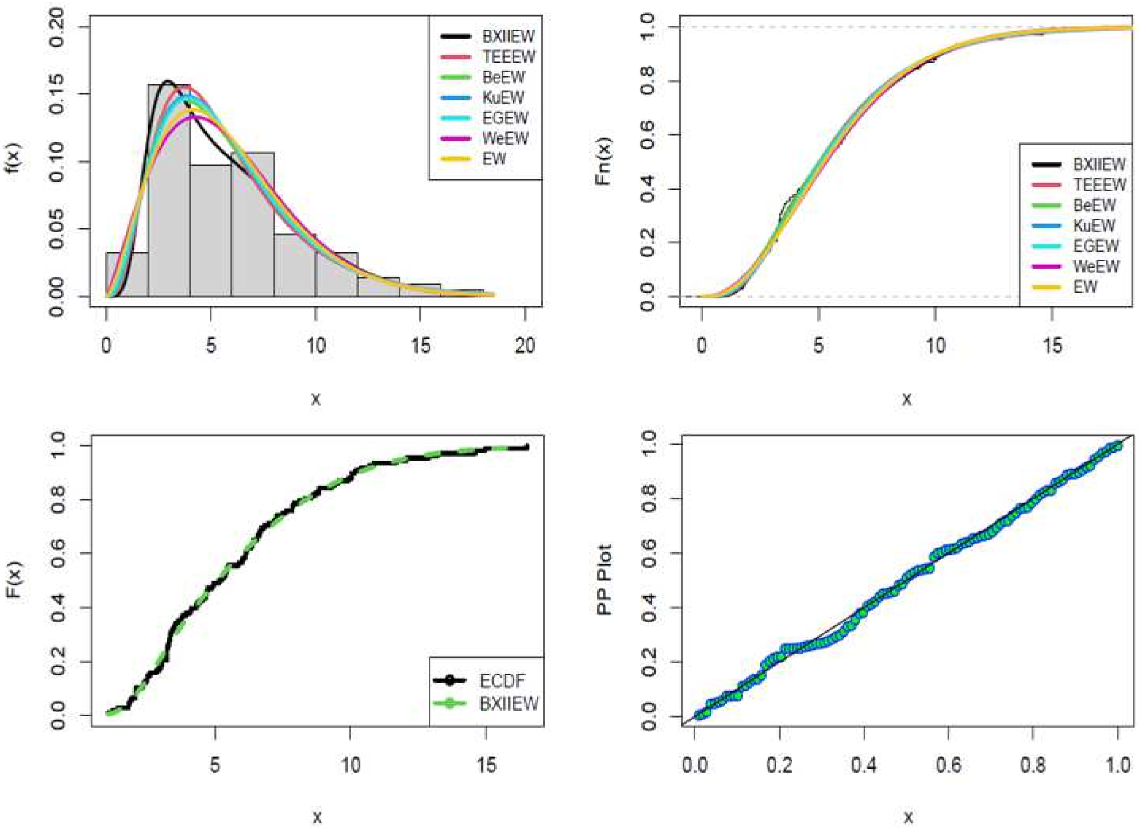

Fig. 7: The fitted densities, Empirical cdf, and Probability Plot, respectively for Data I.

Table 7: The goodness-of-fit criteria, p-values, and the information criteria results for data I.

Dist.	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	P-v
BXIIEW	$\hat{c}$:0.4390 $\hat{d}$:0.6036 $\hat{\vartheta}$:1.1648 $\hat{\alpha}$:1.5378 $\hat{\beta}$:0.4375	264.77	539.55	540.13	552.96	544.98	0.0275	0.1687	0.0450	0.9808
TEEEW	$\hat{c}$:0.5180 $\hat{d}$:1.9611 $\hat{\vartheta}$:2.0812 $\hat{\alpha}$:1.0314 $\hat{\beta}$:0.3246	266.06	542.12	542.71	555.53	547.56	0.0579	0.3161	0.0666	0.7226
BeEW	$\hat{c}$:1.6683 $\hat{d}$:0.1627 $\hat{\vartheta}$:1.4534 $\hat{\alpha}$:1.5480 $\hat{\beta}$:0.5723	265.70	541.40	541.99	554.81	546.84	0.0491	0.2813	0.0694	0.6747
KuEW	$\hat{c}$:1.7539 $\hat{d}$:0.5914 $\hat{\vartheta}$:1.5583 $\hat{\alpha}$:1.2745 $\hat{\beta}$:0.3452	266.13	542.28	542.87	555.69	547.72	0.0585	0.3301	0.0710	0.6472
EGEW	$\hat{c}$:0.7700 $\hat{d}$:1.7473 $\hat{\vartheta}$:1.5868 $\hat{\alpha}$:1.1879 $\hat{\beta}$:0.3288	266.32	542.65	543.24	556.06	548.09	0.0617	0.3514	0.0720	0.6297
WeEW	$\hat{c}$:1.2397 $\hat{d}$:1.0210 $\hat{\vartheta}$:1.4057 $\hat{\alpha}$:1.2882 $\hat{\beta}$:0.1906	267.82	545.65	546.23	559.06	551.08	0.0893	0.5584	0.0760	0.5596
EW	$\hat{c}$:1.7166 $\hat{d}$:1.4474 $\hat{\vartheta}$:0.2040	267.13	540.27	540.50	548.32	543.54	0.0763	0.4634	0.0749	0.5795

6.2 Second dataset II

An accelerated life test was performed on 59 conductors to investigate microcircuit failures induced by electromigration, or the movement of atoms within the circuit's conductors. The test results, showing the failure times in hours, are shown below. The dataset has been used by Chaudhary et al. [42]

9.289, 6.545, 6.956, 7.543, 5.459, 6.492, 4.706, 8.120, 2.997, 8.687, 6.129, 8.591, 5.381, 11.038, 4.288, 6.958, 4.137, 6.522, 7.495, 7.459, 6.538, 6.573, 6.087, 5.589, 6.725, 5.807, 8.532, 6.369, 9.663, 7.024, 9.218, 8.336, 7.945, 6.869, 4.700, 6.352, 9.254, 6.948, 7.489, 5.009, 6.033, 7.398, 7.496, 10.092, 7.974, 4.4531, 7.683, 8.799,7.365, 7.224, 5.640, 6.923, 5.434, 7.937, 6.476, 6.515, 6.071, 5.923, 10.491.

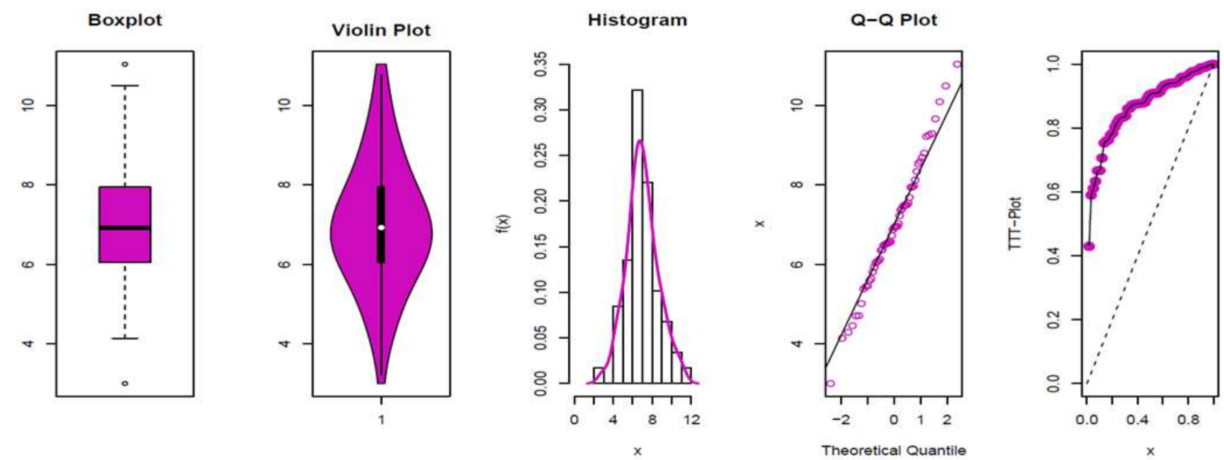

Fig. 8: Shows data II visualizations.

Table 8: The goodness-of-fit, p-values, and the information criteria results for data II.

Dist.	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	P-v
BXIIEW	$\widehat{c}$:1.8859 $\widehat{d}$:0.5297 $\widehat{\vartheta}$:0.2237 $\widehat{\alpha}$:1.6776 $\widehat{\beta}$:0.1123	110.96	231.92	233.06	242.31	235.98	0.0262	0.1512	0.0553	0.9889
TEEEW	$\widehat{c}$:0.3534 $\widehat{d}$:2.1598 $\widehat{\vartheta}$:2.2354 $\widehat{\alpha}$:2.3268 $\widehat{\beta}$:0.1909	111.76	233.52	234.65	243.91	237.58	0.0398	0.2363	0.0697	0.9169
BeEW	$\widehat{c}$:1.8834 $\widehat{d}$:1.0574 $\widehat{\vartheta}$:1.8570 $\widehat{\alpha}$:2.5795 $\widehat{\beta}$:0.1763	111.43	232.87	234.00	243.25	236.92	0.0351	0.2021	0.0594	0.9770
KuEW	$\widehat{c}$:1.8559 $\widehat{d}$:1.1113 $\widehat{\vartheta}$:1.8464 $\widehat{\alpha}$:2.5859 $\widehat{\beta}$:0.1738	111.41	232.83	233.96	243.22	236.88	0.0350	0.2007	0.0606	0.9724
EGEW	$\widehat{c}$:1.0057 $\widehat{d}$:1.9016 $\widehat{\vartheta}$:1.8550 $\widehat{\alpha}$:2.5959 $\widehat{\beta}$:0.1784	111.44	232.88	234.01	243.27	236.94	0.0351	0.2024	0.0594	0.9773
WeEW	$\widehat{c}$:1.9274 $\widehat{d}$:0.9943 $\widehat{\vartheta}$:1.5672 $\widehat{\alpha}$:1.9367 $\widehat{\beta}$:0.1550	112.02	234.04	235.17	244.43	238.09	0.0644	0.3591	0.0868	0.7322
EW	$\widehat{c}$:2.0971 $\widehat{d}$:2.1953 $\widehat{\vartheta}$:0.1679	118.36	242.82	243.26	249.06	245.26	0.03629	0.2080	0.1677	0.0639

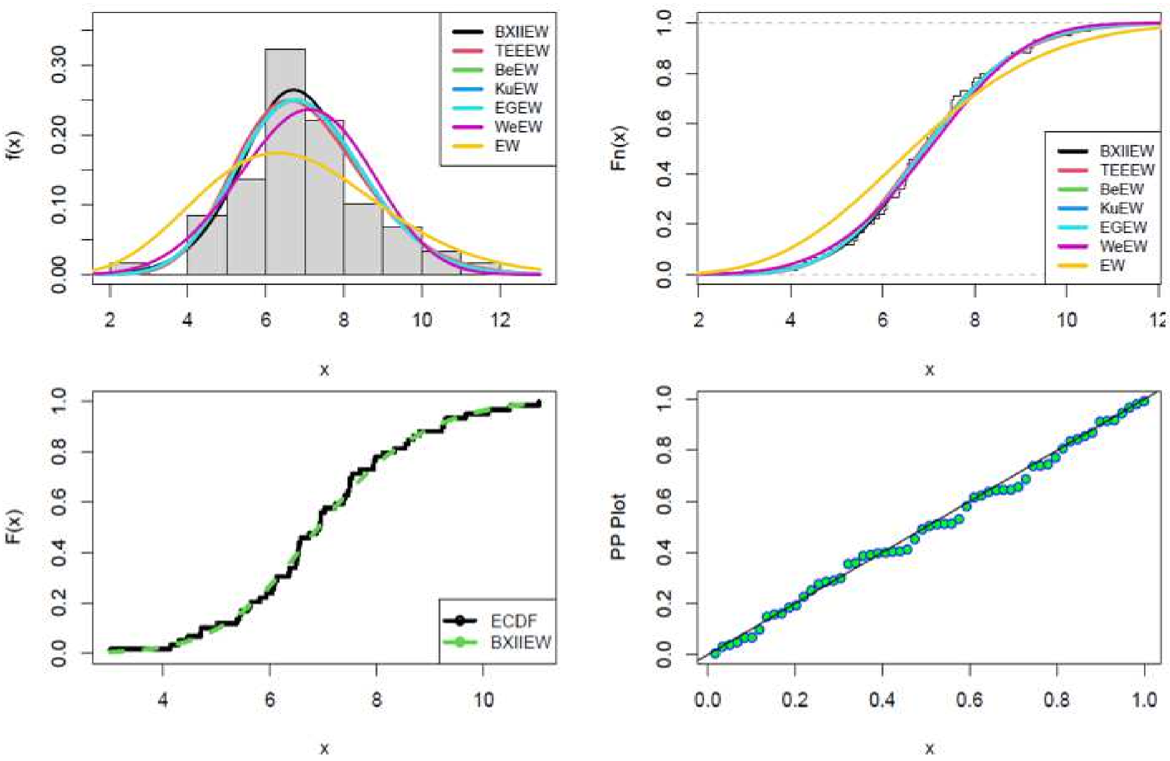

Fig. 9: The fitted densities, empirical cdf, and probability plot, respectively for data II.

6.3 Third dataset III

The dataset provided below indicates the duration of survival for 40 patients who were diagnosed with blood cancer and participated in a study done at one of Saudi Arabia's Ministry of Health hospitals. Abouammoh et al. [43] studied the data earlier.

0.315, 0.496 ,0.616, 1.145, 1.208, 1.263 ,1.414, 2.025 ,2.036, 2.162, 2.211, 2.37 ,2.532, 2.693 ,2.805 ,2.91 ,2.912 ,3.192 ,3.263 ,3.348 ,3.348 ,3.427 ,3.499 ,3.534 ,3.767 ,3.751, 3.858 ,3.986, 4.049, 4.244, 4.323 ,4.381, 4.392 ,4.397, 4.647 ,4.753 ,4.929 ,4.973, 5.074, 5.381.

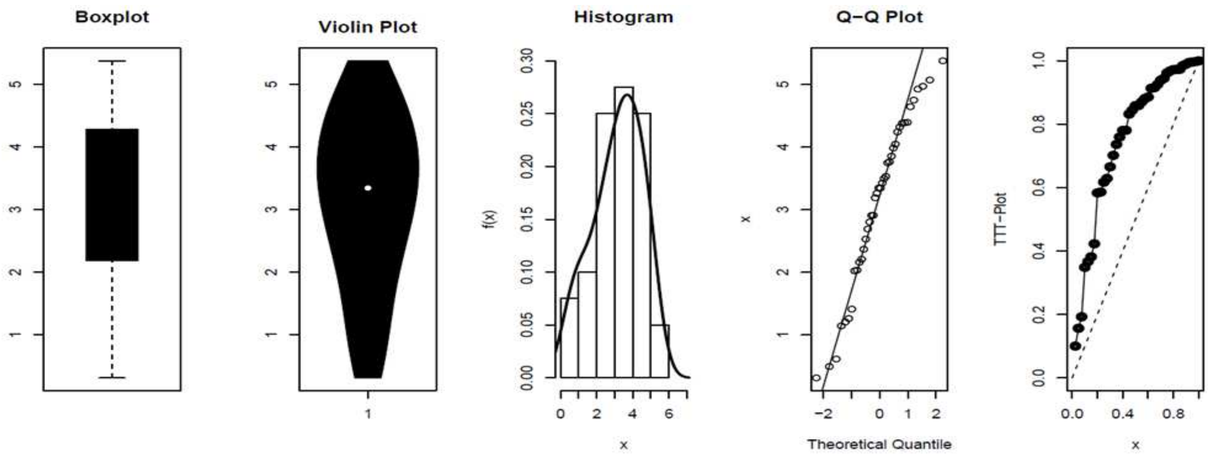

Fig. 10: Shows data III visualizations.

Table 9: The goodness-of-fit, p-values, and the information criteria results for data III.

Dist.	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	P-v
BXIIEW	$\widehat{c}$:0.4508 $\widehat{d}$:1.2298 $\widehat{\vartheta}$:0.2317 $\widehat{\alpha}$:2.9826 $\widehat{\beta}$:0.2245	65.60	141.20	142.97	149.65	144.26	0.0361	0.2316	0.0758	0.9754
TEEEW	$\widehat{c}$:0.0601 $\widehat{d}$:0.7383 $\widehat{\vartheta}$:0.9874 $\widehat{\alpha}$:2.8869 $\widehat{\beta}$:0.2569	68.50	147.28	149.04	155.72	150.33	0.1005	0.6628	0.1233	0.5767
BeEW	$\widehat{c}$:1.0055 $\widehat{d}$:1.0520 $\widehat{\vartheta}$:0.6620 $\widehat{\alpha}$:3.0975 $\widehat{\beta}$:0.2489	68.25	146.54	148.30	154.98	149.59	0.0896	0.5964	0.1254	0.5546
KuEW	$\widehat{c}$:0.6014 $\widehat{d}$:0.1627 $\widehat{\vartheta}$:0.6015 $\widehat{\alpha}$:3.3842 $\widehat{\beta}$:0.4454	66.40	142.86	144.63	151.31	145.92	0.0339	0.2522	0.0779	0.9681
EGEW	$\widehat{c}$:0.3203 $\widehat{d}$:1.1797 $\widehat{\vartheta}$:0.3857 $\widehat{\alpha}$:3.2598 $\widehat{\beta}$:0.3695	66.71	143.52	145.29	151.97	146.57	0.0460	0.3283	0.0860	0.9287
WeEW	$\widehat{c}$:1.9605 $\widehat{d}$:1.3332 $\widehat{\vartheta}$:0.4420 $\widehat{\alpha}$:2.0918 $\widehat{\beta}$:0.2337	67.65	145.33	147.09	153.77	148.38	0.0714	0.4846	0.0961	0.8533
EW	$\widehat{c}$:1.3001 $\widehat{d}$:1.9778 $\widehat{\vartheta}$:0.3156	70.66	147.34	148.01	152.41	149.17	0.1557	0.9923	0.1320	0.4881

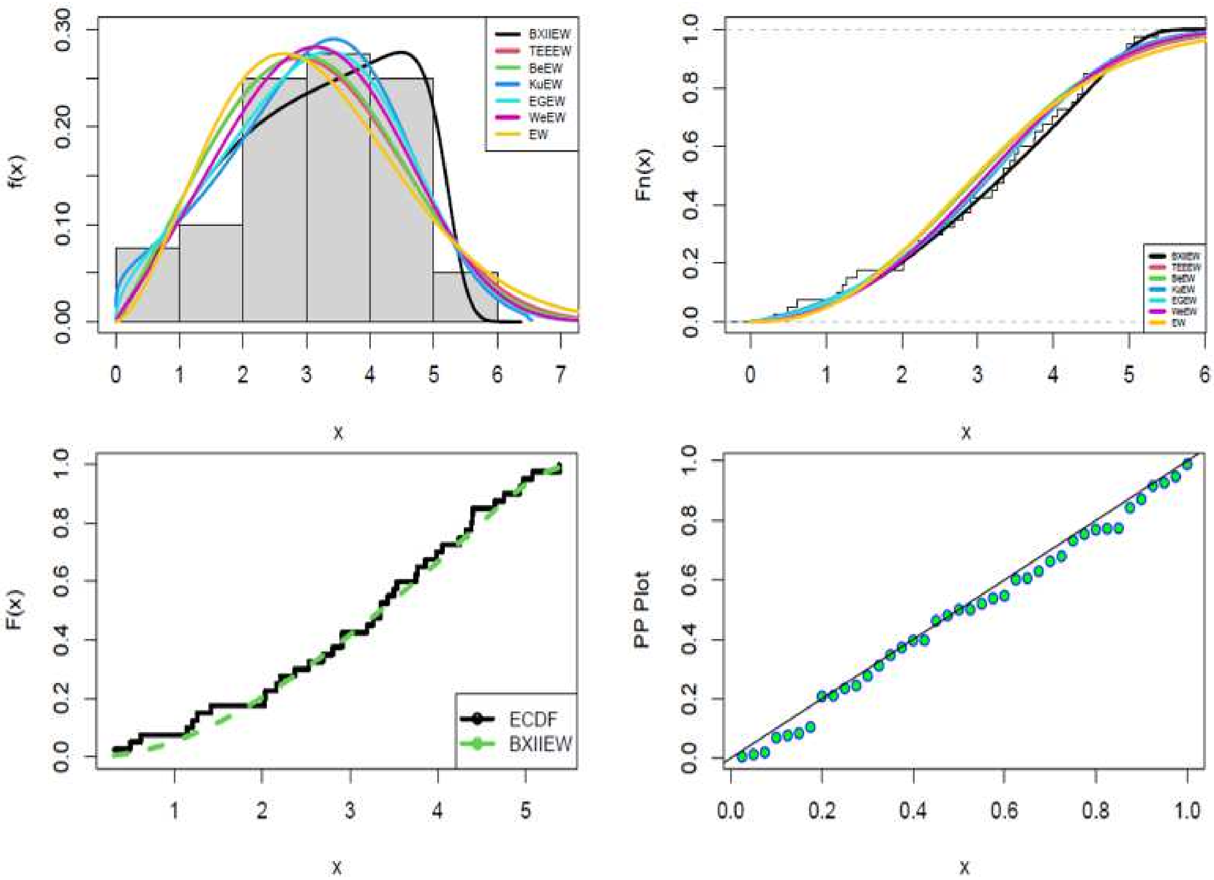

Fig. 11: The fitted densities, empirical cdf, and probability plot, respectively for data III.

7 Conclusions

This article introduces an enhanced version of the Exponentiated Weibull model that integrates the Burr-G family of distributions. The primary objective of this research is to explore the statistical properties of the proposed BXIIEW distribution, which includes the calculation of moments, incomplete moments, the quantile function, and Rényi entropy. Various estimating approaches that account for sex differences were employed to analyze the behavior of the parameters within the BXIIEW model. Additionally, Monte Carlo simulations were utilized to identify the most efficient estimators for the model parameters.

Through an empirical examination of actuarial statistics, the study demonstrates that the BXIIEW distribution exhibits a greater degree of tail heaviness when compared to the TEEEW, BeEW, KuEW, EGEW, WeEW, and WE distributions. This characteristic indicates that the BXIIEW distribution is particularly well-suited for modeling data with heavier tails.

Overall, the BXIIEW distribution outperforms the six other models in real-world applications, showcasing its robustness and adaptability in diverse scenarios.

Declaration of competing interest

All authors participated in carrying out this research.

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