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Geometric Analysis and Enhanced Approach for Addressing the Generalized M-Truncated Space-Time Fractional Burgers Model

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Abstract: We use the fractional Hirota bilinear method to derive analytical solutions for the hyperbolic generalized M-truncated space-time Burgers model. By constructing double soliton waves for this fractional differential model, we demonstrate the efficacy of symbolic computation tools like Maple. This approach highlights the Hirota bilinear method as a promising and straightforward technique for tackling nonlinear differential equations of both integer and fractional orders. Our results confirm that this method is not only easy to apply but also effective and versatile for various engineering and physics problems. We explore fundamental concepts related to surfaces using M-truncated fractional analysis. This involves computing differential geometrical properties such as the M-truncated fractional Gaussian curvature and the M-truncated fractional mean curvature , which offer new physical insights into the problem. The ability to select arbitrary fractional orders allows us to create more complex structures. Variations in soliton behavior due to changes in fractional order extend its applicability in applied sciences. The dynamic behavior of the solutions is depicted through 2D and 3D graphical representations, highlighting variations across different fractional orders.

Keywords: M-truncated fractional derivative, fractional calculus, fractional models, generalized Burger's equation.

1 Introduction

The concept of fractional derivatives can be traced back to the notable correspondence between Leibniz and L'Hospital in 1695. Over the past sixty years, fractional calculus (FC) has made a substantial impact across various fields, including physics, chemistry, electrical engineering, biology, economics, image processing, and aerodynamics [1,2,3]. Through the past decade, FC has become an essential tool for modeling long-memory processes, garnering interest from engineers, physicists, and mathematicians alike. Gaining insight into solutions of fractional differential equations is essential for advancing our understanding of physical processes with fractional orders, and it has significant implications for practical applications and real-world impacts. Partial and ordinary differential equations are extensively employed in fields such as fluid dynamics [4], system identification [5,6], control theory [7,8], and image processing [9], among others, to model complex phenomena [10]-[24].

FC is a field of study that extends traditional calculus, which is usually limited to integer-order derivatives, to include fractional orders [1,2,3]. This extension gives rise to various preparations of fractional derivatives, including the Riemann-Liouville (RL) [18], Caputo [20], He's [19], conformable [21], and local fractional derivatives [22,23, 24]. The concepts of RL fractional derivative is a fundamental approach based on integrals, while He's fractional explanation employs He's polynomials for its definition. The fractional derivative definition by Caputo integrates integer order differentials with the RL approach, making it particularly effective for analyzing initial value systems. The more recent conformable fractional derivative applies ordinary product rules and is effective for functions with singularities. Every one of these definitions offers distinct advantages and is utilized across diverse fields, such as physics, engineering, and signal processing, to tackle problems involving fractional order models and natural phenomena.

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The truncated Mittag-Leffler function (MLF) can be defined as [25]:

$$_{I}E_{\beta}(\varepsilon s^{\alpha})=\sum_{j=0}^{I}\frac{(\varepsilon s^{\alpha})^{j}}{\Gamma(\beta j+1)},\quad \beta>0,\quad s\,\varepsilon C, \tag{1}$$

Definition 1: Let Let $\psi:[0,\infty)\rightarrow\Re$ be a function, the local truncated M-fractional differential (MFD) of with respect to y is given [25]:

$$_{I}D_{M,t}^{\alpha,\beta}\psi(s)=\lim_{\delta\rightarrow 0}\frac{\psi(s\,_{I}E_{\beta}(\delta s^{-\alpha}))-\psi(s)}{\delta},\quad \forall\,\beta,s>0,\ 0<\alpha<1, \tag{2}$$

The MFD adheres to the following axioms:

$$_{I}D_{M}^{\alpha,\beta}t^{m}=\frac{m}{\Gamma(\beta+1)}t^{m-\alpha},\ m\,\varepsilon\,\Re,\ \ _{I}D_{M}^{\alpha,\beta}c=0,\ \ \forall\,\psi(t)=c, \tag{3}$$

$$_{I}D_{M}^{\alpha,\beta}(A\psi+B\varphi)=A_{I}D_{M}^{\alpha,\beta}\psi+B_{I}D_{M}^{\alpha,\beta}\varphi,\ \ \forall\,A,B\,\varepsilon\,\Re, \tag{4}$$

$$_{I}D_{M}^{\alpha,\beta}(\varphi\psi)=\varphi_{I}D_{M}^{\alpha,\beta}\psi+\psi_{I}D_{M}^{\alpha,\beta}\varphi, \tag{5}$$

$$_{I}D_{M}^{\alpha,\beta}\left(\frac{\varphi}{\psi}\right)=\frac{\psi_{I}D_{M}^{\alpha,\beta}\varphi-\varphi_{I}D_{M}^{\alpha,\beta}\psi}{\psi^{2}}, \tag{6}$$

$$_{I}D_{M}^{\alpha,\beta}\varphi(\psi)=\frac{d\varphi}{d\psi}{}_{I}D_{M}^{\alpha,\beta}\psi, \tag{7}$$

$$_{I}D_{M}^{\alpha,\beta}\varphi(t)=\frac{d\varphi}{dt}\frac{t^{1-\alpha}}{\Gamma(\beta+1)}, \tag{8}$$

With φ,ψ represents two α-differentiable functions of a dependent variable, the above relations are proved in reference [25].

Choosing $\beta=1$ and $i=1$ on the two sides of Eq.(4), we have

$$_{1}D_{M,t}^{\alpha,1}\psi(s)=\lim_{\delta\rightarrow 0}\frac{\psi(s\,_{1}E_{1}(\delta s^{-\alpha}))-\psi(s)}{\delta},\quad \forall\ s>0,\ 0<\alpha<1,$$

But, it is know that

$$_{1}E_{1}(\delta s^{-\alpha})=\sum_{r=0}^{1}\frac{(\delta s^{\alpha})^{r}}{\Gamma(2)}=1+\delta s^{-\alpha},$$

Thus, we conclude that

$$_{1}D_{M,t}^{\alpha,1}\psi(s)=\lim_{\delta\rightarrow 0}\frac{\psi(s+\delta s^{1-\alpha})-\psi(s)}{\delta}=D_{t}^{\alpha}\psi(s),\quad \forall\ s>0,\ 0<\alpha<1,$$

which is exactly the conformable fractional derivative. Simply we write $_{1}D_{M}^{\alpha,\beta}$ as $D_{M}^{\alpha,\beta}$. The MFD of some functions [25]

$$D_{M,s}^{\alpha,\beta}e^{cs}=\frac{cs^{1-\alpha}}{\Gamma(\beta+1)}e^{cs},$$
$$D_{M,s}^{\alpha,\beta}\sin(cs)=\frac{cs^{1-\alpha}}{\Gamma(\beta+1)}\cos(cs),$$
$$D_{M,s}^{\alpha,\beta}\cos(cs)=-\frac{cs^{1-\alpha}}{\Gamma(\beta+1)}\sin(cs),$$

$$D_{M,s}^{\alpha,\beta}e^{cs^{\alpha}}=\frac{c\alpha}{\Gamma(\beta+1)}e^{cs^{\alpha}},$$
$$D_{M,s}^{\alpha,\beta}\sin(cs^{\alpha})=\frac{c\alpha}{\Gamma(\beta+1)}\cos(cs^{\alpha}),$$
$$D_{M,s}^{\alpha,\beta}\cos(cs^{\alpha})=-\frac{c\alpha}{\Gamma(\beta+1)}\sin(cs^{\alpha}).$$

The MFD can be used for non-differentiable functions, making it suitable for applications involving discontinuous media. Currently, FC generalizes the concepts of integer-order integration and differentiation to include fractional orders. Recently, nonlinear fractional models have become a prominent area of research, drawing interest from physicists, mathematicians, astronomers, and engineers. These models have numerous applications across various scientific fields, including plasma physics, condensed matter physics, biomathematics, chemistry, biology, communication, and astronomy. Fractional calculus plays a crucial role in engineering and physics, with applications in areas such as fractal wave propagation, particle physics, electrical systems, and wave mechanics.

The fractional Burgers equation, a simplified form of the fractional model, effectively describes the interplay between dissipative effects and nonlinear propagation. This model finds applications in a range of fields, including hydrodynamics, fluid dynamics, wave propagation in thermoelastic media, acoustic transmission, plasma physics, traffic flow, magnetohydrodynamics, shock waves, supersonic flow around airfoils, diffusion-affected waves, liquid dynamics, and information sciences. In this manuscript, we will explore the double soliton and solitary wave solutions and examine their interactions.

Additionally, we introduce a detailed characterization of wavefront interactions for the hyperbolic generalized fractional Burgers model (GFBM). This mathematical model can be viewed as a fractional Navier-Stokes system with a hyperbolic modification. Recent research on the generalized fractional Burgers model (GBM) has primarily focused on numerical methods. Our work aims to provide an explicit characterization of the double soliton solution for this equation.

This manuscript is rearranged as follows: Section 2 introduced the fundamental formulas for the hyperbolic GFBM and explores its solutions using Hirota's technique. Sections 3 and 4, delve into the double soliton solution, applying both classical and modified Hirota procedures. Section 5 discusses the spatial fractional geometric demonstration. The manuscript concludes with a summary of findings, prospects, and a discussion.

2 Explanation of the model and solution methodology

We introduce the following hyperbolic GFBM in the form

$$\tau D_{M,t}^{\alpha\alpha,\beta}u + D_{M,t}^{\alpha,\beta}u + uD_{M,x}^{\alpha,\beta}u - \kappa D_{M,x}^{\alpha\alpha,\beta}u = \sum_{n=0}^{3} a_n u^n. \quad (9)$$

with $u = u(t^\alpha, x^\alpha)$ real function of time and space since τ, κ are arbitrary positive constants, a_n are random constants and $D_{M,x}^{\alpha\alpha,\beta} = D_{M,x}^{\alpha,\beta}(D_{M,x}^{\alpha,\beta})$ is the twice fractional differentiation regarding x. When $\alpha = \beta = 1$ equation (9) become the well-known hyperbolic GBM studied in [26] as

$$\tau u_{tt} + u_t + uu_x - \kappa u_{xx} = \sum_{n=0}^{3} a_n u^n.$$

Given that the polynomial on the equation (9) has real roots, it can be rewritten in the equivalent form:

$$\tau D_{M,t}^{\alpha\alpha,\beta}u + D_{M,t}^{\alpha,\beta}u + uD_{M,x}^{\alpha,\beta}u + BD_{M,x}^{\alpha,\beta}u - \kappa D_{M,x}^{\alpha\alpha,\beta} = \lambda(s-u)(q-u). \quad (10)$$

with s, q, B, λ are arbitrary constants. In the next, using equation (10) to express hyperbolic GFBM.

As discussed earlier, the primary goal of this manuscript is to deliver a detailed clarification of single and double soliton solutions using Hirota's technique. Employing this method leads to a complex and challenging nonlinear algebraic system that cannot be easily solved without additional information about the model's parameters. The technique for solving these algebraic equations, arising from Hirota's procedure, becomes more manageable when we seek solutions with certain known properties. Consequently, we will examine how the double soliton approaches the corresponding traveling wave as a parameter tends to a specific value. To achieve this, we will begin with an analytical explanation of traveling wave solutions, which are crucial for understanding the asymptotic behavior. To get it, we put

$$u = \frac{D_{M,x}^{\alpha,\beta}f}{f}, \quad (11)$$

we assume that

$$f = 1 + \varepsilon\varphi(x^\alpha, t^\alpha), \quad (12)$$

with $\varphi(x^\alpha, t^\alpha)$ is function will be calculated, ε is parameter. Therefore, by inserting equation (11) into (10), multiplying the results by ε^3. We have a three order series of ε. Putting each coefficient of ε by zero we can get system of ordinary fractional differential in $\varphi(x^\alpha, t^\alpha)$ for example from the coefficient of ε we have

$$\tau D_{M,t}^{\alpha\alpha,\beta}\varphi + D_{M,t}^{\alpha,\beta}(D_{M,x}^{\alpha,\beta}\varphi) + BD_{M,x}^{\alpha\alpha,\beta}\varphi \\ \kappa D_{M,x}^{\alpha\alpha\alpha,\beta}\varphi + qs\lambda D_{M,x}^{\alpha,\beta}\varphi = 0, \quad (13)$$

assuming that the solution of equation (13) takes the form

$$\varphi = e^{\Gamma(\beta+1)(ax^\alpha - vt^\alpha + c)/\alpha}, \quad (14)$$

where a, v, c are constants will be computed. Using equation (14) transformed the system of fractional differential equations to algebraic system as

$$aB - v - a^2\kappa + qs\lambda + v^2\tau = 0, \quad (15)$$

$$-v + a^2(1+\kappa) + 2qs\lambda + a[B + (s-q)\lambda] - v^2\tau = 0, \quad (16)$$

$$\lambda(a+q)(a-s(=0. \quad (17)$$

From equation (17), since $\lambda \neq 0$ so the parameter a is either equal to s or q. Firstly, the case when we have

$$\varphi_1 = e^{\Gamma(\beta+1)(sx^\alpha - v_1t^\alpha + c_1)/\alpha}, \quad (18)$$

$$v_1 = \frac{s[s(\lambda+1) + 2B + 2\lambda q]}{2}, \quad (19)$$

Using equations (19), (18), we have kink wave solution

$$u_1 = \frac{se^{\Gamma(\beta+1)(sx^\alpha - v_1t^\alpha + c_1)/\alpha}}{1 + e^{\Gamma(\beta+1)(sx^\alpha - v_1t^\alpha + c_1)/\alpha}}, \\ v_1 = \frac{s[s(\lambda+1) + 2B + 2\lambda q]}{2}, \quad (20)$$

with the condition $k = k_1 = \frac{\tau[\lambda(2q+s)+2B+s]^2 - 2(\lambda+1)}{4}$.

secondly, the case when $a - -q$ we have

$$\varphi_1 = e^{\Gamma(\beta+1)(-qx^\alpha - v_2t^\alpha + c_1)/\alpha}, \quad (21)$$

$$v_2 = \frac{-q[2B - q(\lambda+1) - 2\lambda s]}{2}, \quad (22)$$

Using equations (21), (22), we have kink wave solution

$$u_2 = \frac{-qe^{\Gamma(\beta+1)(-qx^\alpha - v_2t^\alpha + c_2)/\alpha}}{1 + e^{\Gamma(\beta+1)(-qx^\alpha - v_2t^\alpha + c_2)/\alpha}}, \\ v_2 = \frac{-q[2B - q(\lambda+1) - 2\lambda s]}{2}, \quad (23)$$

with the condition $k = k_2 = \frac{\tau[\lambda(2q+s)+q-2B]^2 - 2(\lambda+1)}{4}$.

The dynamical behavior of the one kink soliton solution are introduced here via 2D and 3D graphics in figures 1 and 2. The parameter selections for figure 1, we have $\alpha = 0.8, \beta = 1, s = 0.4, q = 0.5, B = 3, c_1 = 10, \lambda = 2$. But for figure 2, we take $\alpha = 0.9, \beta = 1, s = -0.6, q = -0.7, B = 3, c_2 = 20, \lambda = 4$.

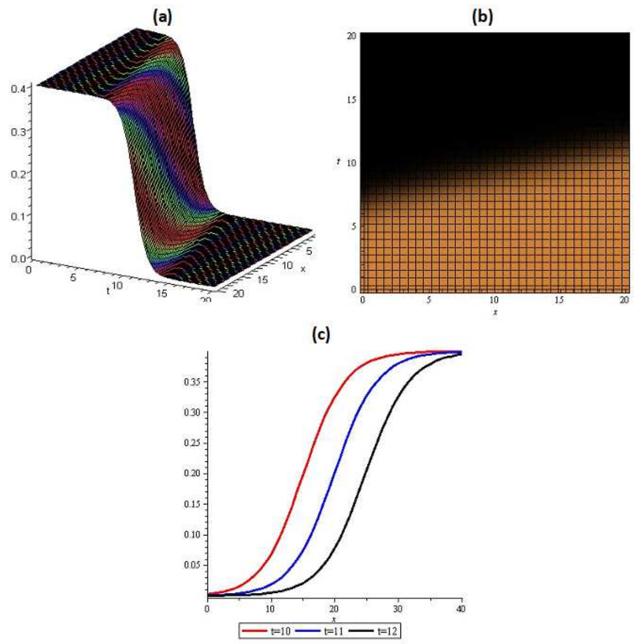

Fig. 1: 3-D plot, density plot, and 2-D plot of one kink soliton solution given by u_1 with the selection $\alpha = 0.9, \beta = 1, s = 0.4, q = 0.5, B = 3, c_1 = 10, \lambda = 2$.

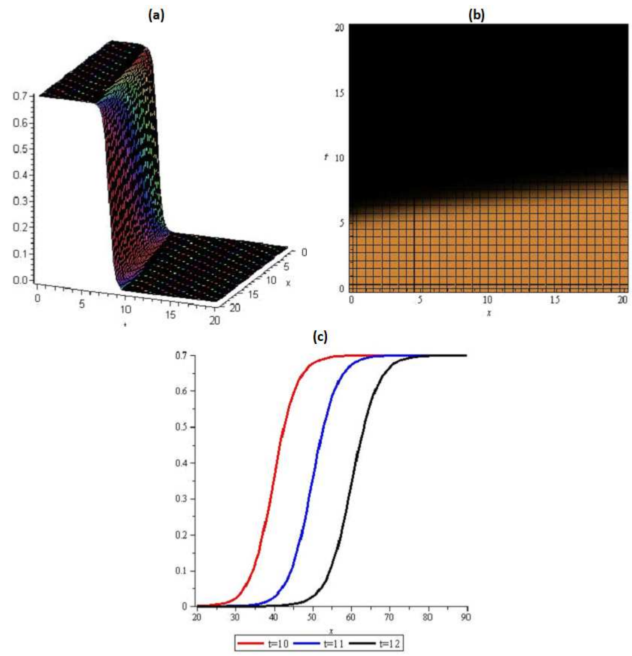

Fig. 2: 3-D plot, density plot, and 2-D plot of one kink soliton solution given by u_2 with the selection $\alpha = 0.8, \beta = 1, s = -0.6, q = -0.7, B = 3, c_1 = 20, \lambda = 4$.

3 Double soliton of the model

We seek for the solution as

$$u(x^\alpha, t^\alpha) = f(\xi_1, \xi_2),$$
$$\xi_1 = \Gamma(\beta+1)(sx^\alpha - v_1 t^\alpha + c_1)/\alpha,$$
$$\xi_2 = \Gamma(\beta+1)(-qx^\alpha - v_2 t^\alpha + c_2)/\alpha, \quad (24)$$

equation (24) represented the interactions of many travelling wave when the two travelling waves ξ_1 and ξ_2 do not have a proportional relationship.

In our attempts to investigate the double soliton wave using Hirota's procedure, we encounter an extremely complex system of nonlinear algebraic relations. To make this system solvable, we need to impose certain hypotheses on the parameters. Given that the solution we seek is asymptotically represented by one or two traveling wave fronts, depending on the type of interaction, we choose the parameters and according to formulas (19) and (22). For a simpler and more straightforward investigation of the double soliton wave, it is helpful to assume that the function f is a combination of functions representing traveling waves in the form

$$f = 1 + \varepsilon(e^{\xi_1} + e^{\xi_2}) + R\varepsilon^2 e^{\xi_1+\xi_2}, \quad (25)$$

Substituting (25) with equation (11) into hyperbolic GFBM (10), we obtain six order series in the parameter ε. Equating each coefficient of ε by zero we have six nonlinear algebraic equations (denoted by $E_1, E_2, ..., E_6$) in the determined constants and the products of $e^{m\xi_1}, e^{n\xi_2}$. In the next, we use the abbreviation $X^m = e^{m\xi_1}$ and $Y^n = e^{n\xi_2}$. Therefore, the system of nonlinear algebraic equations E_k expressed by the abbreviation X^m and Y^n in the form

$$\sum_{n=0}^{k} a_n^k X^n Y^{k-n}, \quad k = 1, 2, ..., 6. \quad (26)$$

with a_n^k containing the determined constants $\tau, B, \kappa, \lambda, s, q$ and R.

Since we will not take into account the case in which the waves ξ_1 and ξ_2 are proportional, therefore we treat the variables X and Y as independent and unrelated variables. Thus system (20) have been satisfied iff $a_n^k = 0$. Based on the symbolic software "Maple" or "Mathematica", we solve the algebraic system $a_n^k = 0, 0 \le n \le k \le 6$. Note that we do not consider solutions given by $s = 0, q = 0, q = -s$ and $\lambda = -1/3$.

Now we begin to explain the procedure which enabling us to obtain the double soliton-based ansatz (11). Firstly, since $a_0^1 = a_1^1 = 0$, we have

$$\begin{aligned} &2\lambda[2qs\tau - 1 + s^2\tau + 2B\tau(2q+s)] - 2 \\ &-4\kappa + 4B^2\tau + 4Bs\tau + (2q+s)^2\lambda^2\tau = 0, \end{aligned} \quad (27)$$

$$\begin{aligned}\lambda[2q^2\tau-2+4qs\tau-4B\tau(2s+q)]-2\\-4\kappa+4B^2\tau-4Bq\tau+q^2\tau+(q+2s)^2\lambda^2\tau=0,\end{aligned} \tag{28}$$

By subtracting (28) from (27), we have

$$\tau[(s-q)(1-\lambda)+4B](1+3\lambda)(q+s)=0. \tag{29}$$

According to what we agreed upon previously, we reject the cases when $q=-s$ and $\lambda=-1/3$, so equation (29) satisfied only when $\tau=0$ or the null value of expression in the brackets. It is clear that the first choice cannot be neglected in some steps, so we choose it from the very beginning. Using this choice in expression (27), we get this formula

$$\kappa=-\frac{1+\lambda}{2}, \tag{30}$$

Secondly, we consider $a_3^6=0$, so

$$(s-q)qsR^3=0,$$

which gives either $R=0$ or $s=q$. Obviously, we are forced into the first selection, since when using the second selection, we get $R=0$ in the calculations. Cases $R=0,\tau=0$ and $\kappa=-(1+\lambda)/2$ verifies that the system of algebraic equations $E_1,E_2,...,E_6$ take the zero value. The direct analysis, which can be considered somewhat extensive analysis of some other prospects investigate the algebraic equations $a_n^k=0$ displays that the introduced cases are being the generalized ones and many other non-trivial double soliton solutions, which can be gained from resolving the algebraic equations in different techniques, that could be everything given from this one, while non-trivial solutions of this kind correspond to non-zero $\tau=0$ is not present.

Through this procedure, we can conclude that obtaining a double-soliton solution that satisfies equation (10) using the Hirota method is not feasible. The result is as follows

$$\begin{aligned}u_{11}&=\frac{se^{\Gamma(\beta+1)(sx^\alpha-v_1t^\alpha+c_1)/\alpha}-qe^{\Gamma(\beta+1)(-qx^\alpha-v_2t^\alpha+c_2)/\alpha}}{1+e^{\Gamma(\beta+1)(sx^\alpha-v_1t^\alpha+c_1)/\alpha}+e^{\Gamma(\beta+1)(-qx^\alpha-v_2t^\alpha+c_2)/\alpha}},\\v_1&=\frac{s[s(\lambda+1)+2B+2\lambda q]}{2},\\v_2&=\frac{-q[2B-q(\lambda+1)-2\lambda s]}{2},\end{aligned} \tag{31}$$

since the reminder of parameters are optional.

The dynamical behavior of the double-kink soliton solution of is presented via 2D and 3D graphics in figures 3. The parameter selections are given by $\alpha=0.9,\beta=1,s=9,q=2,B=0.3,c_1=1,c_2=5,\lambda=0.25$.

4 The technique realized by using Hirota's modified procedure

We can succeed in obtaining the double-soliton solutions of fractional model (10) by adding a simple modification to relation (11). Assuming that

$$u=\frac{g}{f}, \tag{32}$$

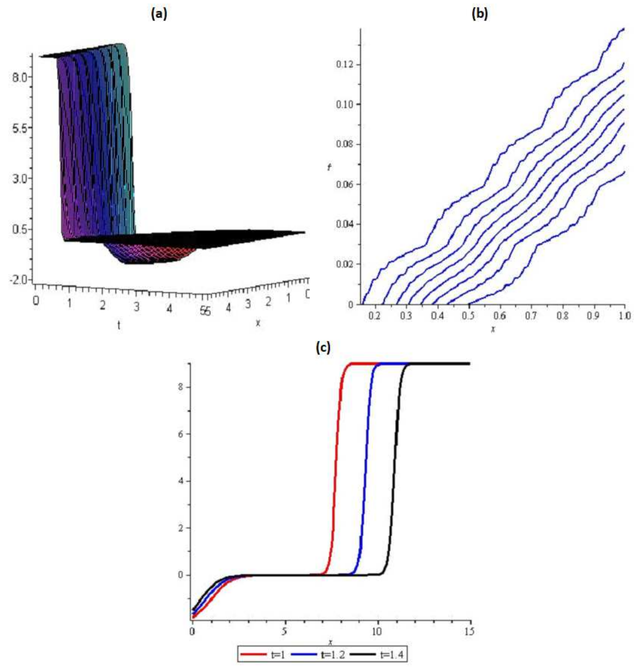

Fig. 3: 3-D plot, contour plot, and 2-D plot of double-soliton kink soliton given by u_{11} with the selection $\alpha=0.9,\beta=1,s=9,q=2,B=0.3,c_1=1,c_2=5,\lambda=0.25$.

that is we have

$$f=1+\varepsilon(e^{\xi_1}+e^{\xi_2})+R\varepsilon^2e^{\xi_1+\xi_2}, \tag{33}$$

$$g=\varepsilon(r_1e^{\xi_1}+r_2e^{\xi_2})+A\varepsilon^2e^{\xi_1+\xi_2}, \tag{34}$$

it is clear from the hypothesis of the two functions f and g functions that we do not present a relationship between these two functions from the beginning which is in contrast to what followed in the procedure of Hirota's.

By making direct substitution using the relations (32)-(34) into hyperbolic GFBM (10), we obtain a series of degree six in ε. Equating each coefficient of ε by zero we have six nonlinear algebraic equations (denoted by $E_1,E_2,...,E_6$) in the determined constants and the products of $e^{m\xi_1},e^{n\xi_2}$. Using the same abbreviations above we can obtain fifteen nonlinear algebraic equations when $a_n^k=0, 0\le n\le k\le 6$. Based on the symbolic software "Maple" or "Mathematica", we can solve this system $a_n^k=0$. As we presented in the above section, we turn a blind eye to the cases of $s=0,q=0,\lambda=-1/3$ and $q=-s$. Moreover, we will not take into account such cases $r_2=0,R=1$ and $A=r_1$ because those choices lead to solutions that describe single travelling waves. Obviously, we can notice the transformations

$$r_1\to r_2,r_2\to r_1,q\to -s,s\to -q, \tag{35}$$

be an equivalence relationship in the group of double soliton, so we have one investigation for every pair.

At first, we select the status when $r_1 \neq 0, r_2 \neq 0$ from $E_1 \neq 0$ we have

$$B\frac{(s-q)(\lambda-1)}{4} \tag{36}$$

$$\kappa = \frac{\tau(1+3\lambda)^2(q+s)^2 - 8(1+\lambda)}{16} \tag{37}$$

using (36) and (37) into $a_0^2 = a_2^2 = a_0^3 = a_3^3 = 0$, we obtain

$$\begin{aligned} &r_1(s-r_1)[s+\lambda(s-q)] = 0, \\ &r_2(q-r_2)[q+\lambda(q-s)] = 0, \\ &r_1\lambda(s-r_1)(q+r_1) = 0, \\ &r_2\lambda(s-r_2)(q+r_2) = 0, \end{aligned} \tag{38}$$

When solving the previous set of algebraic equations, we find that they are achieved if we give $r_1 = r_2 = s$ and

$$\lambda = \frac{q}{s-q}, \tag{39}$$

from $a_3^6 = 0$, we have

$$A(A+qR)(A-sR) = 0, \tag{40}$$

that is achieved for example when $A = -qR$, using this selection in $a_2^5 = 0$, we get

$$R^2(q+s)^3 = 0, \tag{41}$$

therefore, we must take $R = 0$ in this equation. The reminder equation $a_1^2 = 0$, so we have

$$s^2(q+s)^2[2(q-s) + \tau q(2q+s)^2] = 0, \tag{42}$$

that is gives

$$\tau = -\frac{2(q-s)}{q(2q+s)^2}, \tag{43}$$

Hence, the hyperbolic GFBM (10) have the double soliton in the form

$$\begin{aligned} u_{22} &= \frac{s(e^{\Gamma(\beta+1)(sx^\alpha - v_1 t^\alpha + c_1)/\alpha} + e^{\Gamma(\beta+1)(-qx^\alpha - v_2 t^\alpha + c_2)/\alpha})}{1 + e^{\Gamma(\beta+1)(sx^\alpha - v_1 t^\alpha + c_1)/\alpha} + e^{\Gamma(\beta+1)(-qx^\alpha - v_2 t^\alpha + c_2)/\alpha}}, \\ v_1 &= \frac{s(2q^2+3qs+s^2)}{4(s-q)}, \\ v_2 &= \frac{q(2q^2+3qs+s^2)}{4(s-q)}, \end{aligned} \tag{44}$$

where $\tau = -\frac{2(q-s)}{q(2q+s)^2}, \lambda = \frac{q}{s-q}, B = -\frac{s-2q}{4}$ and $\lambda = \frac{s-q}{8q}$.

Currently, we suggest that $r_1 \neq 0$ and $r_2 = 0$, through this hypothesis the constant continuous can still be obtained from the expression $a_1^1 = 0$. Also, we can get from the expressions $a_2^2 = a_3^3 = a_3^6 = 0$, that $r_1 = s$ and $A = Rs$. From $a_2^3 = 0$ we have

$$\tau sq(R-1)(1-3\lambda)(q+s)^2[q(\lambda-1)+4B+2s(\lambda+1)] = 0, \tag{45}$$

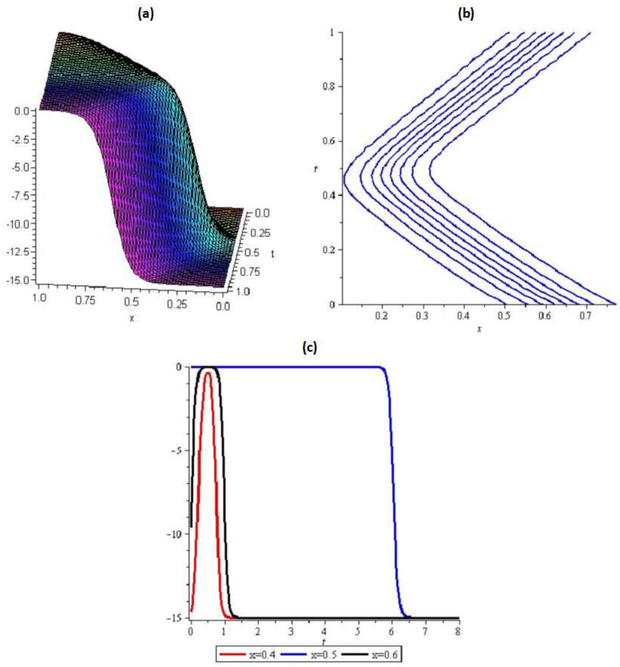

Fig. 4: 3-D plot, contour plot, and 2-D plot of double-soliton kink soliton given by u_{22} with the selection $\alpha = 0.9, \beta = 1, s = -15, q = 20, c_1 = 10, c_2 = -5$.

when $R = 1$, we do not have the double soliton solution but we have one soliton solution. Therefore, we must have

$$B = \frac{q(1-\lambda) - 2s(\lambda+1)}{4} \tag{46}$$

using (46), we can write $a_1^3 = 0$ as

$$sq(R-1)[q+s+\lambda(q+2s)] \tag{47}$$

which gives

$$\lambda = -\frac{q+s}{q+2s}, \tag{48}$$

The reminder two equations $a_1^2 = 0$ and $a_2^4 = 0$, having the same factor

$$2q+4s+\tau(q+s)(2q+s)^2,$$

that is compute the parameter τ as

$$\tau = -\frac{2(q+2s)}{(q+s)(2q+s)^2}, \tag{49}$$

Thus, the hyperbolic GFBM (10) have the double soliton in the form

$$
\begin{aligned}
u &= \frac{s e^{\Gamma(\beta+1)(sx^{\alpha}-v_1 t^{\alpha}+c_1)/\alpha}\left(1+R e^{\Gamma(\beta+1)(-qx^{\alpha}-v_2 t^{\alpha}+c_2)/\alpha}\right)}{1+e^{\Gamma(\beta+1)(sx^{\alpha}-v_1 t^{\alpha}+c_1)/\alpha}+e^{\Gamma(\beta+1)(-qx^{\alpha}-v_2 t^{\alpha}+c_2)/\alpha}+\Theta_1},\\
\Theta_1 &= R e^{\Gamma(\beta+1)(sx^{\alpha}-v_1 t^{\alpha}+c_1)/\alpha+\Gamma(\beta+1)(-qx^{\alpha}-v_2 t^{\alpha}+c_2)/\alpha}\\
v_1 &= -\frac{qs(s+2q)}{4(2s+q)},\\
v_2 &= -\frac{q(s+2q)}{4}
\end{aligned}
\tag{50}
$$

with $B = (2q-s)/4, \lambda = -(s+q)/(2s+q), \tau = -2(2s+q)/q+s)(2q+s)^2, \kappa = -(2s+q)/8(s+q)$ and s, q, R, c_1, c_2 are constants. According to the conditions, in application of the physical meaning of the constants τ, κ as well as the parameters s, q in above must investigate the following inequality $(s+q)(2s+q) < 0$. We should note that for the random selection of the parameter R involves that the solution (50) designates both regular and singular solutions when $R \geq 0$ and $R < 0$ respectively. When $\alpha = \beta = 1$ all the results of Vladimirov and Maczka in 2007 are recovered [26].

5 Spatial geometric demonstration

In exploring the geometric interpretations of the solutions to the hyperbolic GFBM described by equation (10), we express the solutions in terms of $U = U(x^{\alpha}, t^{\alpha})$, which represents 2-dimensional space in a fractional surface defined by the non-integer variables x^{α} and t^{α}. This approach enables us to derive the M-truncated fractional Monge (FM) space, expelling the measurement of key M-truncated fractional differential geometry amounts, such as the fractional Gaussian (FG) curvature K_{α} and the fractional mean (FM) curvature H_{α} .

The FM space is a concept that generalizes Monge space by incorporating FC into its framework. Monge space typically deals with the study of surfaces and their properties in terms of Monge's equations, which describe surfaces in a simplified form. In FM space, fractional derivatives are used instead of traditional integer-order derivatives. This approach allows for the analysis of surfaces with more complex geometrical and topological features that are not well-described by classical methods. It is useful in studying surfaces and shapes in fractional-dimensional spaces and can provide deeper insights into phenomena where traditional calculus might be inadequate.

The FG curvature extends the concept of Gaussian curvature to fractional calculus. It measures curvature in spaces where traditional integer-order derivatives are replaced with fractional ones, providing insights into the geometric properties of surfaces in a more generalized framework. FG curvature has several applications across various fields. Examples include, in complex geometry which provides a way to analyze surfaces and shapes in geometries that are not easily described by traditional integer-order calculus. This can be particularly useful in studying complex surfaces in higher-dimensional spaces. In the study of materials with intricate microstructures, FG curvature can help describe and predict the behavior of these materials under various conditions, such as stress or deformation. In areas such as general relativity and quantum field theory, FG curvature can offer new perspectives on the geometry of spacetime and the nature of gravitational fields. It can be applied to model and understand the complex shapes of biological structures, such as the surfaces of cell membranes or the forms of certain proteins. In graphics and visualization, FG curvature can assist in rendering and simulating complex surfaces and textures, contributing to more realistic and detailed visualizations. These applications leverage the fractional approach to gain deeper insights into curvature and geometry in various contexts where traditional methods might fall short.

The FM curvature is a generalization of the classical mean curvature, incorporating fractional calculus into its definition. In classical differential geometry, mean curvature is a measure of how a surface bends in space. For a surface, it is defined as the average of the principal curvatures at a given point. It provides insight into how the surface locally curves and how it might behave under deformations. FM curvature extends this concept by replacing the traditional integer-order derivatives used in calculating mean curvature with fractional-order derivatives. This generalization allows for the analysis of surfaces and shapes in fractional spaces, where the traditional calculus may not apply or may be insufficient. FM curvature has several applications across various fields. Examples include, in complex geometry analysis which helps in studying surfaces in fractional-dimensional spaces or those exhibiting complex, non-integer-dimensional features. In analyzing and modeling materials with intricate microstructures or surfaces with fractional properties. Provides new ways to explore and model physical phenomena in fractional space-time geometries. It assists in rendering and simulating surfaces with fractional characteristics, leading to more detailed and realistic visual representations. Overall, FM curvature allows for a more nuanced understanding of curvature and surface behavior in contexts where traditional methods might not be applicable.

The steps to calculate the M-truncated fractional Gaussian and mean curvatures are as follows:

$$
\begin{aligned}
K_{\alpha} &= \frac{\ell_{11}\ell_{22}-\ell_{12}^2}{G_{11}G_{22}-G_{12}^2},\\
H_{\alpha} &= \frac{\ell_{11}G_{22}+\ell_{22}G_{11}-2\ell_{12}G_{12}}{2(G_{11}G_{22}-G_{12}^2)}, \quad G_{11}G_{22}-G_{12}^2 \neq 0,
\end{aligned}
\tag{51}
$$

where $G_{11} = D_{M,x}^{\alpha,\beta}\Phi . D_{M,x}^{\alpha,\beta}\Phi$, $G_{22} = D_{M,t}^{\alpha,\beta}\Phi . D_{M,t}^{\alpha,\beta}\Phi$, $G_{12} = D_{M,x}^{\alpha,\beta}\Phi . D_{M,t}^{\alpha,\beta}\Phi$, $\ell_{11} = D_{M,x}^{\alpha\alpha,\beta}\Phi . N$, $\ell_{22} = D_{M,t}^{\alpha\alpha,\beta}\Phi . N$, $\ell_{12} = D_{M,x}^{\alpha,\beta}(D_{M,t}^{\alpha,\beta}\Phi) . N$, and $N = (D_{M,x}^{\alpha,\beta}\Phi \wedge D_{M,t}^{\alpha,\beta}\Phi)/\|D_{M,x}^{\alpha,\beta}\Phi \wedge D_{M,t}^{\alpha,\beta}\Phi\|$. Here

$G_{11} > 0, G_{22} > 0$ represent the squares of the spatial fractional velocities for the x and t parameter curves of Φ, while G_{12} measures the M-truncated fractional coordinate angle between $D_{M,x}^{\alpha,\beta}\Phi$ and $D_{M,t}^{\alpha,\beta}\Phi$ (the tangent to the fractional coordinate curves). The function Φ takes the form $\Phi = \Phi(x^{\alpha}, t^{\alpha}, \varphi(x^{\alpha}, t^{\alpha}))$. On the other hand, given the M-truncated FG curvature K_{α} and M-truncated FM curvature H_{α}, we can readily determine the principal curvatures $k_{1_{\alpha}}$ and $k_{2_{\alpha}}$, given by the solutions of the following equation

$$k_{\alpha}^2 - H_{\alpha}k_{\alpha} + K_{\alpha} = 0, \tag{52}$$

this gives $H_{\alpha} \pm \sqrt{H_{\alpha}^2 - K_{\alpha}}$ with $k_{1_{\alpha}}$ and $k_{2_{\alpha}}$ be the M-truncated fractional principal curvatures of the fractional space patch $\varphi(u)$ so the FG curvature K_{α} and FM curvature VH_{α} of φ are

$$K_{\alpha} = k_{1_{\alpha}}k_{2_{\alpha}}, \;\; H_{\alpha} = \frac{1}{2}(k_{1_{\alpha}} + k_{2_{\alpha}}). \tag{53}$$

Additionally, we can calculate the M-truncated FG curvature K_{α} and M-truncated FM curvature H_{α} from the first and second forms of the M-truncated fractional space,

$$E_{\alpha}du_{\alpha}^2 + 2F_{\alpha}du_{\alpha}dv_{\alpha} + G_{\alpha}dv_{\alpha}^2, \tag{54}$$

and

$$L_{\alpha}du_{\alpha}^2 + 2M_{\alpha}du_{\alpha}dv_{\alpha} + N_{\alpha}dv_{\alpha}^2, \tag{55}$$

Thus, we use the following two matrices:

$$F_1 = \begin{pmatrix} E_{\alpha} & F_{\alpha} \\ F_{\alpha} & G_{\alpha} \end{pmatrix}, \;\; F_2 = \begin{pmatrix} L_{\alpha} & M_{\alpha} \\ M_{\alpha} & N_{\alpha} \end{pmatrix}, \tag{56}$$

The eigenvalues of $F_1^{-1}F_2$ represent the M-truncated fractional principal curvatures, and $k_{1_{\alpha}}k_{2_{\alpha}}$ denotes the matrix determinant, therefore:

$$K_{\alpha} = det(F_1^{-1}F_2) = det(F_1^{-1})det(F_2) = \frac{L_{\alpha}N_{\alpha} - M_{\alpha}^2}{E_{\alpha}G_{\alpha} - F_{\alpha}^2}. \tag{57}$$

The trace of a matrix is equal to the sum of its eigenvalues, which equals double the M-truncated FM curvature. Therefore, we obtain

$$H_{\alpha} = \frac{1}{2}trace(F_1^{-1}F_2) = \frac{1}{2}\frac{L_{\alpha}G_{\alpha} - 2M_{\alpha}F_{\alpha} + N_{\alpha}E_{\alpha}}{E_{\alpha}G_{\alpha} - F_{\alpha}^2}. \tag{58}$$

Another method to get K_{α} and H_{α} involves using the fact that the principal curvatures are similarly the roots of the equation

$$\begin{aligned}(E_{\alpha}G_{\alpha} - F_{\alpha}^2)k_{\alpha}^2 - (L_{\alpha}G_{\alpha} - 2M_{\alpha}F_{\alpha} + N_{\alpha}E_{\alpha})k_{\alpha} \\ +(L_{\alpha}N_{\alpha} - M_{\alpha}^2) = 0.\end{aligned} \tag{59}$$

The creation K_{α} and the amount $2H_{\alpha}$ of the two roots can be directly computed from the coefficients. These outcomes are consistent with those obtained previously. When $\alpha = 1$, the expressions simplify to the standard forms used in differential geometry.

Example: For the M-truncated FM patch $\psi = \psi(x^{\alpha}, y^{\alpha})$, that is designated by $\varphi = (x^{\alpha}, y^{\alpha}, \psi(x^{\alpha}, y^{\alpha}))$. Initially, we calculate the single and double M-truncated fractional differentials as

$$\begin{aligned}
\varphi_{M,x}^{\alpha,\beta}\varphi &= (\alpha/\Gamma(\beta+1), 0, \psi_{M,x}^{\alpha,\beta}),\\
\varphi_{M,y}^{\alpha,\beta}\varphi &= (0, \alpha/\Gamma(\beta+1), \psi_{M,y}^{\alpha,\beta}),\\
\varphi_{M,x}^{\alpha\alpha,\beta}\varphi &= (0, 0, \psi_{M,x}^{\alpha\alpha,\beta}),\\
\varphi_{M,xy}^{\alpha\alpha,\beta}\varphi &= (0, 0, \psi_{M,xy}^{\alpha\alpha,\beta}),\\
\varphi_{M,y}^{\alpha\alpha,\beta}\varphi &= (0, 0, \psi_{M,y}^{\alpha\alpha,\beta}),
\end{aligned} \tag{60}$$

since $D_{M,x}^{\alpha,\beta}\varphi = \varphi_{M,x}^{\alpha,\beta}$, $D_{M,y}^{\alpha,\beta}\varphi = \varphi_{M,y}^{\alpha,\beta}$, $D_{M,x}^{\alpha\alpha,\beta}\varphi = \varphi_{M,x}^{\alpha\alpha,\beta}$, $D_{M,y}^{\alpha\alpha,\beta}\varphi = \varphi_{M,y}^{\alpha\alpha,\beta}$, and $D_{M,x}^{\alpha,\beta}(D_{M,y}^{\alpha,\beta}\varphi) = \varphi_{M,xy}^{\alpha\alpha,\beta}$.

Therefore, the coefficients for the M-truncated fractional first fundamental form are calculated as follows:

$$\begin{aligned}
E_{\alpha} &= \varphi_{M,x}^{\alpha,\beta}.\varphi_{M,x}^{\alpha,\beta} = \left(\tfrac{\alpha}{\Gamma(\beta+1)}\right)^2 + (\psi_{M,x}^{\alpha,\beta})^2,\\
F_{\alpha} &= \varphi_{M,x}^{\alpha,\beta}, \qquad \varphi_{M,y}^{\alpha,\beta} = \psi_{M,x}^{\alpha,\beta}.\psi_{M,y}^{\alpha,\beta},\\
G_{\alpha} &= \varphi_{M,y}^{\alpha,\beta}.\varphi_{M,y}^{\alpha,\beta} = \left(\tfrac{\alpha}{\Gamma(\beta+1)}\right)^2 + (\psi_{M,y}^{\alpha,\beta})^2.
\end{aligned} \tag{61}$$

Therefore, the M-truncated fractional unit normal to the fractional patch is

$$\begin{aligned}
N &= \frac{\varphi_{M,x}^{\alpha,\beta} \wedge \varphi_{M,y}^{\alpha,\beta}}{\|\varphi_{M,x}^{\alpha,\beta} \wedge \varphi_{M,y}^{\alpha,\beta}\|}\\
&= \frac{(-\psi_{M,x}^{\alpha,\beta}, -\psi_{M,y}^{\alpha,\beta} + \alpha/\Gamma(\beta+1))}{\sqrt{(\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2}},
\end{aligned} \tag{62}$$

The coefficients of the M-truncated fractional second fundamental form are calculated as follows:

$$\begin{aligned}
L_{\alpha} &= \varphi_{M,x}^{\alpha\alpha,\beta}, N = \frac{\alpha\psi_{M,x}^{\alpha\alpha,\beta}/\Gamma(\beta+1)}{\sqrt{(\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2}},\\
M_{\alpha} &= \varphi_{M,xy}^{\alpha\alpha,\beta}, N = \frac{\alpha\psi_{M,xy}^{\alpha\alpha,\beta}/\Gamma(\beta+1)}{\sqrt{(\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2}},\\
N_{\alpha} &= \varphi_{M,y}^{\alpha\alpha,\beta}, N = \frac{\alpha\psi_{M,y}^{\alpha\alpha,\beta}/\Gamma(\beta+1)}{\sqrt{(\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2}}.
\end{aligned} \tag{63}$$

Therefore,

$$K_{\alpha} = \frac{\psi_{M,x}^{\alpha\alpha,\beta}\psi_{M,y}^{\alpha,\beta} - (\psi_{M,xy}^{\alpha,\beta})^2}{((\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2)^2}, \tag{64}$$

$$\begin{aligned}
H_{\alpha} = &\frac{1}{\sqrt{((\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2 + (\psi_{M,x}^{\alpha,\beta})^2)^3}}\\
&\times\Big\{((\alpha/\Gamma(\beta+1))^2 + (\psi_{M,y}^{\alpha,\beta})^2)\psi_{M,x}^{\alpha\alpha,\beta}\\
&+(\alpha/\Gamma(\beta+1))^2 + (\psi_{M,x}^{\alpha,\beta})^2)\psi_{M,y}^{\alpha\alpha,\beta}\\
&+ 2\psi_{M,xy}^{\alpha\alpha,\beta}\psi_{M,y}^{\alpha,\beta}\psi_{M,x}^{\alpha,\beta}\Big\}
\end{aligned} \tag{65}$$

6 Concluding Remarks

In this manuscript, we apply the fractional Hirota bilinear technique to derive analytical solutions for the hyperbolic generalized space-time fractional Burgers model. We develop both single and double soliton waves for the fractional differential model under investigation. These computations are conducted using symbolic computation tools like Maple, which has become increasingly popular among researchers, demonstrating that the Hirota bilinear method is a promising and straightforward approach for addressing nonlinear differential models of both integer and fractional orders. Our results confirm that this method is easy to apply, effective, and well-suited for a wide range of engineering and physics problems. The ability to select arbitrary fractional orders allows us to construct more complex structures, and variations in the soliton solutions based on fractional order provide broader applications in the applied sciences. The geometric analysis enables us to understand the properties of the solutions, offering new physical insights into the problem. For a fractional order of one, our results align with those of Vladimirov and Maczka [26]. We suggest that this method could be applied to further nonlinear fractional systems. In upcoming research, we aim to investigate the geometric properties of 3-dimensional surfaces in M-truncated fractional space and apply this technique to other nonlinear space-time fractional differential models.

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