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Generalized Alpha Power Transformed Truncated Lomax Distribution with Application

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Abstract: In this article, a new generalization of the truncated Lomax (TLo) distribution, which called the generalized alpha power transformed truncated Lomax (GAPTTLo) distribution is studied and discussed. The probability density function (PDF) of the GAPTTLo distribution is either declining, unimodal, or right skewed. The hazard rate function (HRF) of the GAPTTLo distribution, may be decreasing or bathtub shaped. Several statistical features such as, quantile function, median, moments, moment generating function, and order statistics. The three unknown parameters for the GAPTTLo distribution are estimated utilizing the maximum likelihood approach of estimation. An investigation into simulation is carried out in order to determine the most effective method of estimation. In addition, we study two real-world data sets using the GAPTTLo distribution, demonstrating that it displays superior performance in comparison to some of its competitors.

Keywords: Truncated Lomax distribution, generalized alpha power transformed , moments, estimation, simulation.

1 Introduction

In the past few years, many authors have come up with different ways to turn standard continuous distributions into flexible continuous distributions. Several analysts are interested in the related models because they can be used in many areas, such as physics, biology, health, finance, the economy, and engineering. Most of the time, these broader distributions are part of certain statistical groups based on modifying the parameters of a parent cumulative distribution function (CDF). The possible values of the new parameter(s) can then significantly improve the statistical abilities of the parent distribution. These values positively affect the asymmetry, kurtosis, weight on the tails, and center and dispersion parameters.
In subsequent years, several additional authors were motivated to develop families of models involving beta-G family by [1], Weibull-G family by [2], transformed-transformer-G (T-X) family by [3], Kavya-Manoharan-G family by [4], Kumaraswamy-G family by [5], logistic X-G family by [6], for more details see [[7]-[33]].
Recently [34] introduced the alpha power transformation (APT-G) method and it has the following CDF :

$$F(y;\alpha,\varpi)=\frac{\alpha^{G(y;\varpi)}-1}{\alpha-1} \quad , \alpha>0 \quad , \alpha\neq 1\ ,\ y\ \in R. \tag{1}$$

Several authors in literature have created novel probability distributions using the APT-G. For example, the APT Lindley by [35], APT inverse-Lindley by [36], APT power Lindley [37] APT extended exponential by [38], APT Pareto by [39], APT inverse-Weibull by [40], APT inverted exponential by [41], APT Weibull by [42], APT generalized exponential by [43] distributions.
Recently [43] added positive shape parameter β to the CDF 1 to introduc the generalized APT-G (GAPT-G) method to increase flexibility in APT-G family. The CDF and probability density function (PDF) of GAPT- G family is given by

$$F(y;\alpha,\beta,\varpi)=\frac{\alpha^{G(y;\varpi)^{\beta}}-1}{\alpha-1} \quad , \alpha,\beta>0 \quad , \alpha\neq 1\ ,\ y\ \in R, \tag{2}$$

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and

$$f(y;\alpha,\beta,\varpi)=\frac{\beta\log(\alpha)}{\alpha-1}\,g(y;\varpi)\,G(y;\varpi)^{\beta-1}\,\alpha^{G(y;\varpi)^{\beta}}\ ,\ \alpha,\beta>0\quad,\ \alpha\neq 1\ ,\ y\in R. \tag{3}$$

Recently, [44] proposed the truncated Lomax (TLo) distribution which has one shape parameter and it have the following PDF and CDF as follows:

$$g(y;\theta)=\frac{\theta\,(1+y)^{-\theta-1}}{1-2^{-\theta}}\ ;\ 0<y<1,\,\theta>0, \tag{4}$$

and

$$G(y;\theta)=\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\ ;\ 0<y<1,\,\theta>0, \tag{5}$$

Ref. [[45], [46]] used the CDF 4 and PDF 5 to give two generalizations of the TLo distribution called the Kumaraswamy TLo, and sine TLo distributions respectively.

The main goal of this research article is to introduce a new generalization of the truncated Lomax (TLo) distribution, which called the generalized alpha power transformed truncated Lomax (GAPTTLo) distribution as we discussed in Section 2. The rest of the paper is organized as follows: several statistical features of the GAPTTLo distribution are computed in Section 3. Section 4 presents the estimation of parameters using the maximum likelihood (ML) approach. Section 5 presents a simulation study that aims to estimate the model parameters of the distribution. Section 6 presents two actual data sets to demonstrate the effectiveness of the formulated model. Concluding remarks are stated in section 7.

2 Formulation of the GAPTTLo Distribution

In this section, we construct a new three parameter distribution called the generalized alpha power transformed truncated Lomax (GAPTTLo) distribution by inserting 4 and 5 into 2 and 3, we obtain the CDF and PDF of the GAPTTLo distribution as follows

$$F(y;\alpha,\beta,\theta)=\frac{\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}-1}{\alpha-1}\qquad,\ \alpha,\beta,\theta>0\quad,\ \alpha\neq 1\ ,\ 0<y\ <1. \tag{6}$$

and

$$f(y;\alpha,\beta,\theta)=\frac{\theta\beta\log(\alpha)}{(\alpha-1)\,(1-2^{-\theta})^{\beta}}\ (1+y)^{-\theta-1}\left(1-(1+y)^{-\theta}\right)^{\beta-1}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\ . \tag{7}$$

The reliability function (RF) and the hazard rate function (HRF), reversed HRF and cumulative HRF for the GAPTTLo distribution are given by

$$R(y;\alpha,\beta,\theta)=1-\frac{\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}-1}{\alpha-1}=\frac{\alpha-\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}}{\alpha-1},$$

$$h(y;\alpha,\beta,\theta)=\frac{\theta\beta\log(\alpha)\,(1+y)^{-\theta-1}\left(1-(1+y)^{-\theta}\right)^{\beta-1}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}}{\left(\alpha-\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\right)(1-2^{-\theta})^{\beta}},$$

$$\tau(y;\alpha,\beta,\theta)=\frac{\theta\beta\log(\alpha)\,(1+y)^{-\theta-1}\left(1-(1+y)^{-\theta}\right)^{\beta-1}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}}{\left(\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}-1\right)(1-2^{-\theta})^{\beta}},$$

and

$$H(y;\alpha,\beta,\theta)=-\log\left[\frac{\alpha-\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}}{\alpha-1}\right].$$

In Figure 1, the plots of the PDF and HRF for the GAPTTLo distribution are shown. From Figure 1, we can see that the probability density function (PDF) of the GAPTTLo distribution is either declining, unimodal, or right skewed. The HRF of the GAPTTLo distribution model, on the other hand, may be decreasing or bathtub shaped.

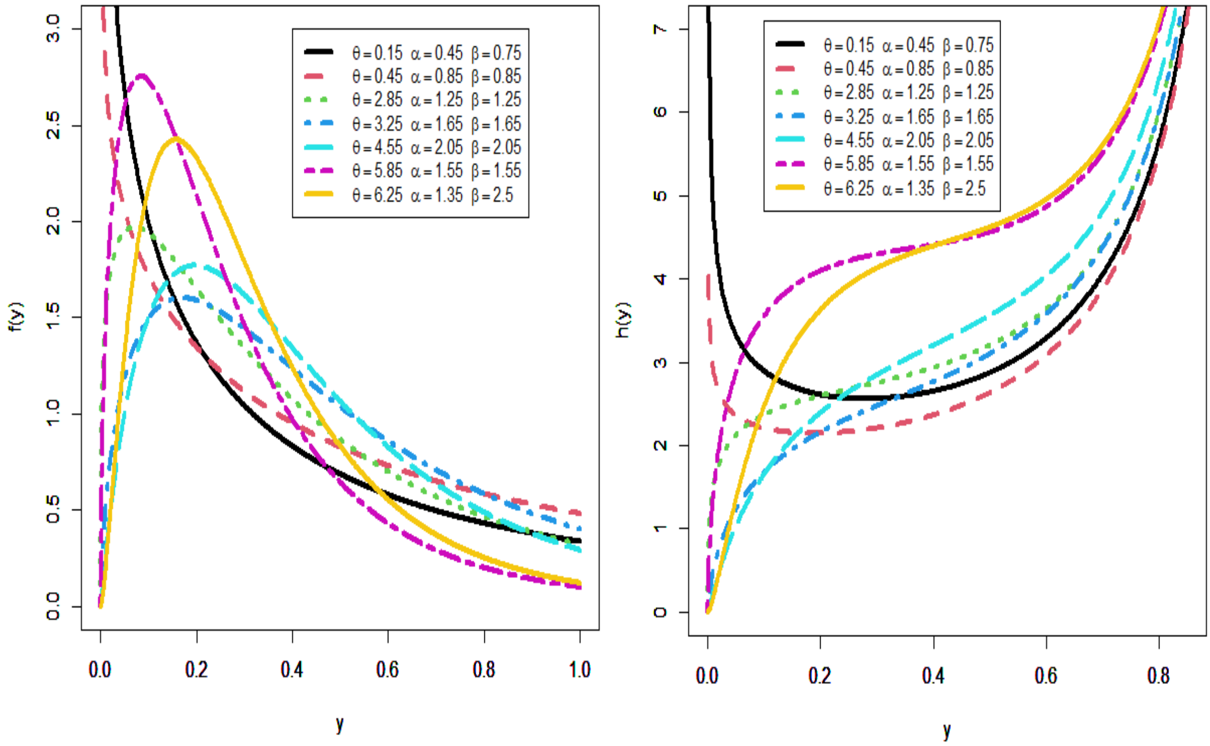

Fig. 1: Plots of PDF and HRF for the GAPTTLo distribution.

3 Statistical Properties

Some important statistical characteristics of the GAPTTLo distribution are covered in this section.

3.1 Important expansions

The PDF of the GAPTTLo distribution can be expressed explicitly using the following series form.

$$\alpha^{\varpi} = \sum_{i=0}^{\infty} \frac{(\log \alpha\,)^{i}}{i!}\,\varpi^{i}\,. \tag{8}$$

Employing 8 in 7, then we can rewrite the PDF 7 as follows:

$$f(y;\alpha,\beta,\theta) = \frac{\theta\beta}{(\alpha-1)} \sum_{i=0}^{\infty} \frac{(\log(\alpha))^{\;i+1}}{(1-2^{-\theta})^{\beta(i+1)}\,i!}\,(1+y)^{-\theta-1}\left(1-(1+y)^{-\theta}\right)^{\beta(i+1)-1}. \tag{9}$$

By using the binomial theory to the last term of Equation 9, then

$$\left(1-(1+y)^{-\theta}\right)^{\beta(i+1)-1} = \sum_{j=0}^{\infty}(-1)^{j}\binom{\beta(i+1)-1}{j}(1+y)^{-j\theta}\,. \tag{10}$$

By inserting 10 in 9 then the PDF of the GAPTTLo can be rewritten as

$$f(y;\alpha,\beta,\theta)=\sum_{i,j=0}^{\infty}\Xi_{i,j}\ (1+y)^{-\theta(j+1)-1}, \tag{11}$$

where $\Xi_{i,j}=\frac{(-1)^{j}\theta\beta(\log(\alpha))^{\ i+1}}{(\alpha-1)(1-2^{-\theta})^{\beta(i+1)}i!}\binom{\beta(i+1)-1}{j}$.

3.2 Quantile Function and Median:

If $p\sim(0,1)$ then the random variable $Y\sim$ GAPTTLo, the p-th quantile function of the GAPTTLo distribution is provided via

$$y_{p}=\left[1-\left(1-2^{-\theta}\right)\left(\frac{\log\left(1+p\left(\alpha-1\right)\right)}{\log\left(\alpha\right)}\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\theta}}-1. \tag{12}$$

The median of the GAPTTLo distribution can be computed by putting p =0.5 in Equation 12 as follows

$$y_{0.5}=\left[1-\left(1-2^{-\theta}\right)\left(\frac{\log\left(0.5\left(\alpha+1\right)\right)}{\log\left(\alpha\right)}\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\theta}}-1.$$

Some numerical values of Q1, Q2, Q3, skewness (S) and kurtosis (K) are provided in Table 1

Table 1: Some numerical values of Q1, Q2, Q3, S and K for the GAPTTLo distribution

α	β	θ	Q1	Q2	Q3	BSK	MKUR
1.5	0.5	4	0.0208	0.0871	0.2409	0.3980	1.4120
		4.2	0.0200	0.0837	0.2317	0.3986	1.4188
		4.4	0.0192	0.0804	0.2230	0.3992	1.4246
		4.6	0.0185	0.0774	0.2148	0.3996	1.4294
		4.8	0.0178	0.0746	0.2070	0.3999	1.4337
		5	0.0172	0.0719	0.1997	0.4000	1.4368
	1.2	4	0.1072	0.2349	0.4386	0.2297	1.2348
		4.2	0.1030	0.2259	0.4237	0.2336	1.2469
		4.4	0.0990	0.2174	0.4092	0.2370	1.2580
		4.6	0.0953	0.2094	0.3953	0.2397	1.2682
		4.8	0.0918	0.2018	0.3819	0.2420	1.2771
		5	0.0886	0.1946	0.3691	0.2441	1.2855
	1.8	4	0.1730	0.3222	0.5348	0.1752	1.1851
		4.2	0.1662	0.3103	0.5182	0.1814	1.1983
		4.4	0.1599	0.2990	0.5019	0.1869	1.2109
		4.6	0.1539	0.2882	0.4861	0.1917	1.2229
		4.8	0.1483	0.2780	0.4707	0.1959	1.2341
		5	0.1430	0.2682	0.4558	0.1995	1.2444
2.2	0.5	4	0.0278	0.1066	0.2748	0.3618	1.3552
		4.2	0.0267	0.1024	0.2644	0.3628	1.3634
		4.4	0.0257	0.0985	0.2545	0.3637	1.3701
		4.6	0.0247	0.0948	0.2452	0.3643	1.3759
		4.8	0.0238	0.0913	0.2364	0.3647	1.3812
		5	0.0230	0.0880	0.2281	0.3654	1.3851
	1.2	4	0.1246	0.2647	0.4769	0.2043	1.2050
		4.2	0.1197	0.2547	0.4612	0.2090	1.2177
		4.4	0.1151	0.2452	0.4458	0.2130	1.2297
		4.6	0.1108	0.2362	0.4311	0.2167	1.2408
		4.8	0.1067	0.2277	0.4168	0.2198	1.2508
		5	0.1029	0.2197	0.4031	0.2223	1.2600
	1.8	4	0.1945	0.3550	0.5720	0.1498	1.1603
		4.2	0.1869	0.3421	0.5551	0.1571	1.1735
		4.4	0.1798	0.3298	0.5385	0.1637	1.1864
		4.6	0.1731	0.3181	0.5221	0.1695	1.1987
		4.8	0.1668	0.3069	0.5062	0.1745	1.2105
		5	0.1609	0.2963	0.4906	0.1789	1.2216

3.3 Moments:

The s^{th} moments of GAPTTLo distribution can be computed from the next formula

$$\mu'_s = \int_0^{\infty} y^s f(y;\alpha,\beta,\theta)dy = \sum_{i,j=0}^{\infty} \Xi_{i,j} \ (1+y)^{-\theta(j+1)-1} .$$

By employing 11 in the previous equation then,

$$\mu'_s = \sum_{i,j=0}^{\infty} \Xi_{i,j} \ \int_0^{\infty} y^s \, (1+y)^{-\theta(j+1)-1} \, dy.$$

By using the prime beta function to the previous equation then,

$$\mu'_s = \sum_{i,j=0}^{\infty} \Xi_{i,j} B\left(s+1, \theta(j+1)-s\right) \ , \quad \theta(j+1) > s.$$

The moment generating function for GAPTTLo distribution is provided via

$$M_Y\left(t\right) = \sum_{s=0}^{\infty} \frac{t^s}{s!}\mu'_s = \sum_{s=0}^{\infty} \sum_{i,j=0}^{\infty} \Xi_{i,j} \frac{t^s}{s!} B\left(s+1, \theta(j+1)-s\right) \ , \quad \theta(j+1) > s.$$

Some numerical values of moments are provided in Table 2.

Table 2: Some numerical values of moments for the GAPTTLo distribution

α	β	θ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	σ^2	σ	S	K	CV
1.5	0.5	4	0.1698	0.0716	0.0419	0.0288	0.0428	0.2068	1.7158	5.5823	1.2179
		4.2	0.1645	0.0680	0.0393	0.0268	0.0409	0.2023	1.7656	5.8330	1.2299
		4.4	0.1593	0.0645	0.0368	0.0249	0.0391	0.1978	1.8149	6.0909	1.2414
		4.6	0.1544	0.0612	0.0345	0.0231	0.0374	0.1934	1.8636	6.3555	1.2524
		4.8	0.1496	0.0581	0.0323	0.0215	0.0357	0.1890	1.9117	6.6264	1.2629
		5	0.1451	0.0552	0.0302	0.0199	0.0341	0.1847	1.9592	6.9031	1.2729
	1.2	4	0.2997	0.1474	0.0917	0.0649	0.0576	0.2399	0.9410	3.0825	0.8005
		4.2	0.2909	0.1403	0.0862	0.0605	0.0557	0.2360	0.9887	3.2182	0.8112
		4.4	0.2824	0.1335	0.0810	0.0564	0.0538	0.2320	1.0359	3.3602	0.8214
		4.6	0.2741	0.1271	0.0761	0.0526	0.0519	0.2279	1.0825	3.5083	0.8313
		4.8	0.2661	0.1209	0.0715	0.0489	0.0501	0.2238	1.1286	3.6624	0.8408
		5	0.2584	0.1150	0.0671	0.0455	0.0482	0.2196	1.1741	3.8222	0.8499
	1.8	4	0.3724	0.1988	0.1285	0.0928	0.0602	0.2453	0.6531	2.5307	0.6586
		4.2	0.3620	0.1897	0.1211	0.0867	0.0586	0.2422	0.7015	2.6288	0.6690
		4.4	0.3518	0.1809	0.1140	0.0810	0.0571	0.2389	0.7495	2.7335	0.6790
		4.6	0.3420	0.1724	0.1073	0.0756	0.0555	0.2355	0.7969	2.8446	0.6887
		4.8	0.3324	0.1643	0.1010	0.0705	0.0538	0.2320	0.8437	2.9618	0.6981
		5	0.3231	0.1566	0.0949	0.0657	0.0522	0.2285	0.8900	3.0851	0.7071
2.2	0.5	4	0.1891	0.0824	0.0488	0.0338	0.0467	0.2160	1.5475	4.8984	1.1419
		4.2	0.1833	0.0783	0.0458	0.0314	0.0447	0.2114	1.5960	5.1199	1.1535
		4.4	0.1776	0.0743	0.0430	0.0292	0.0428	0.2069	1.6441	5.3482	1.1647
		4.6	0.1722	0.0706	0.0403	0.0272	0.0410	0.2024	1.6916	5.5828	1.1754
		4.8	0.1669	0.0670	0.0377	0.0253	0.0392	0.1979	1.7385	5.8235	1.1856
		5	0.1619	0.0637	0.0354	0.0235	0.0375	0.1935	1.7848	6.0698	1.1954
	1.2	4	0.3245	0.1657	0.1051	0.0752	0.0604	0.2458	0.8127	2.7976	0.7574
		4.2	0.3152	0.1579	0.0989	0.0702	0.0586	0.2420	0.8600	2.9157	0.7679
		4.4	0.3062	0.1505	0.0931	0.0655	0.0567	0.2382	0.9069	3.0400	0.7780
		4.6	0.2974	0.1433	0.0875	0.0611	0.0549	0.2343	0.9533	3.1704	0.7877
		4.8	0.2889	0.1365	0.0823	0.0569	0.0530	0.2303	0.9991	3.3067	0.7971
		5	0.2806	0.1299	0.0773	0.0530	0.0512	0.2262	1.0444	3.4486	0.8061
	1.8	4	0.3984	0.2208	0.1457	0.1065	0.0621	0.2491	0.5343	2.3504	0.6253
		4.2	0.3876	0.2109	0.1375	0.0997	0.0607	0.2463	0.5828	2.4329	0.6355
		4.4	0.3770	0.2014	0.1297	0.0933	0.0592	0.2433	0.6307	2.5220	0.6454
		4.6	0.3667	0.1922	0.1222	0.0871	0.0577	0.2402	0.6782	2.6175	0.6550
		4.8	0.3567	0.1833	0.1151	0.0814	0.0561	0.2369	0.7251	2.7191	0.6643
		5	0.3469	0.1749	0.1083	0.0759	0.0545	0.2336	0.7713	2.8268	0.6733

3.4 Order Statistics:

Assume that $Y_1,\ Y_2,\ \ldots,Y_n$ be random sample from the GAPTTLo distribution with order statistics $Y_{(1)},\ Y_{(2))},\ \ldots,Y_{(n)}$. The PDF of $Y_{(k)}$, is computed from the next formula

$$f_{Y_{(k)}}\left(y;\alpha,\beta,\theta\right)=\frac{n!}{\left(k-1\right)!\left(n-k\right)!}F^{k-1}\left(y;\alpha,\beta,\theta\right)f\left(y;\alpha,\beta,\theta\right)\left(1-F\left(y;\alpha,\beta,\theta\right)\right)^{n-k}.$$

The PDF of $Y_{(k)}$ can be calculated as

$$f_{Y_{(k)}}\left(y;\alpha,\beta,\theta\right)=\frac{n!\theta\beta\,\log(\alpha)(1+y)^{-\theta-1}\left(1-(1+y)^{-\theta}\right)^{\beta-1}}{(k-1)!(n-k)!(\alpha-1)^{n}\left(1-2^{-\theta}\right)^{\beta}}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\times\left(\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}-1\right)^{k-1}\left(\alpha-\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\right)^{n-k}.$$

The PDF of the smallest and biggest order statistics can be readily obtained from the preceding equation

$$f_{Y_{(1)}}\left(y;\alpha,\beta,\theta\right)=\frac{n\theta\beta\,\log\left(\alpha\right)\left(1+y\right)^{-\theta-1}\left(1-\left(1+y\right)^{-\theta}\right)^{\beta-1}}{\left(\alpha-1\right)^{n}\left(1-2^{-\theta}\right)^{\beta}}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\left(\alpha-\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\right)^{n-1},$$

and

$$f_{Y_{(n)}}\left(y;\alpha,\beta,\theta\right)=\frac{n\theta\beta\,\log\left(\alpha\right)\left(1+y\right)^{-\theta-1}\left(1-\left(1+y\right)^{-\theta}\right)^{\beta-1}}{\left(\alpha-1\right)^{n}\left(1-2^{-\theta}\right)^{\beta}}\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}\left(\alpha^{\left(\frac{1-(1+y)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}}-1\right)^{n-1}.$$

respectively.

4 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a method used to estimate the unknown parameters of a statistical distribution by maximizing the likelihood function. For a given random sample $y_1,...,y_n$ of size n from a distribution, the likelihood function is the product of the PDF evaluated at each data point. The log-likelihood function L_n is given by:

$$\begin{aligned}L_n=&n\log\left(\theta\right)+n\log\left(\beta\right)+n\log\left(\log\left(\alpha\right)\right)-n\log\left(\alpha-1\right)-n\beta\log\left(1-2^{-\theta}\right)\\&-(\theta+1)\sum_{i=1}^{n}\log\left(1+y_i\right)\\&+(\beta-1)\sum_{i=1}^{n}\log\left(1-\left(1+y_i\right)^{-\theta}\right)\\&+\sum_{i=1}^{n}\left(\frac{1-\left(1+y_i\right)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}\log\left(\alpha\right).\end{aligned}$$

The first partial derivative with respect to $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$

$$\frac{\partial L_n}{\partial\alpha}=\frac{n}{\alpha\log(\alpha)}-\frac{n}{(\alpha-1)}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{1-\left(1+y_i\right)^{-\theta}}{1-2^{-\theta}}\right)^{\beta},$$

$$\begin{aligned}\frac{\partial L_n}{\partial\beta}=&\frac{n}{\beta}-n\log\left(1-2^{-\theta}\right)+\sum_{i=1}^{n}\log\left(1-\left(1+y_i\right)^{-\theta}\right)\\&+\sum_{i=1}^{n}\left(\frac{1-\left(1+y_i\right)^{-\theta}}{1-2^{-\theta}}\right)^{\beta}\log\left(\frac{1-\left(1+y_i\right)^{-\theta}}{1-2^{-\theta}}\right)\log\left(\alpha\right),\end{aligned}$$

and

$$\frac{\partial L_n}{\partial \theta}=\frac{n}{\theta}-\frac{2^{-\theta}n\beta\log(2)}{1-2^{-\theta}}-\sum_{i=1}^{n}\log(1+y_i)+(\beta-1)\sum_{i=1}^{n}\frac{(1+y_i)^{-\theta}\log(1+y_i)}{1-(1+y_i)^{-\theta}}$$

$$+\sum_{i=1}^{n}\left[\beta\log(\alpha)\left(\frac{1-(1+y_i)^{-\theta}}{1-2^{-\theta}}\right)^{\beta-1}\left(\frac{(1+y_i)^{-\theta}\log(1+y_i)\left(1-2^{-\theta}\right)-2^{-\theta}\log(2)\left(1-(1+y_i)^{-\theta}\right)}{(1-2^{-\theta})^2}\right)\right].$$

To find the estimates of α, β, and θ, we need to solve the system of equations obtained by setting the partial derivatives to zero: $\frac{\partial L_n}{\partial \alpha}=0$, $\frac{\partial L_n}{\partial \beta}=0$, $\frac{\partial L_n}{\partial \theta}=0$. These equations can be solved numerically to obtain the MLEs of the parameters.

5 Simulation Study

We performed an extensive simulation study to assess the MLE method using R software (version 4.3.2). We generated 1000 samples from the GAPTTLo distribution with varying sample sizes of $\{10, 20, 30, 50, 100, 200, 700, 800, 900, 1000\}$, and the initial parameter values are provided as follows $(\theta,\ \alpha,\ \beta) =$ (1, 0.3, 1.3), (1.5, 0.3, 1.3), (2.5, 0.1, 2.0) and (3, 0.5, 2.5). We compute the estimated mean of parameters, mean square error (MSE), root mean square error (RMSE), average confidence interval length (AIL) and coverage probability(CP). From Tables 3 to 6 we can note that:

1.The mean estimates for the parameters θ, α, and β show a consistent trend of improving accuracy as the sample size n increases they become nearer to their initial values.
2.Both MSE and RMSE decrease as the sample size increases. This ensures the consistency of MLE.
3.AIL tends to decrease as the sample size increases, leading to narrower confidence intervals. This is an indication that the estimator is reliable.
4.Coverage probabilities across all tables 95% or higher, so the confidence intervals constructed for the parameter estimates are generally reliable, containing the true parameter values at the expected rate.

Table 3: Simulation results for the GAPTTLo distribution at $\theta = 1$, $\alpha = 0.3$, $\beta = 1.3$

n	Parameter	Mean	MSE	RMSE	AIL	CP
10	θ	0.8509	1.7337	1.3167	3.4163	99.8
	α	0.3707	0.0841	0.2900	0.9222	96.7
	β	1.2876	0.0216	0.1469	0.5740	97
20	θ	0.9160	1.5199	1.2328	3.3280	99.7
	α	0.3667	0.0632	0.2513	0.8419	96.5
	β	1.2938	0.0159	0.1261	0.4938	96.9
30	θ	0.9046	1.3856	1.1771	3.2052	99.7
	α	0.3618	0.0597	0.2442	0.8251	96.4
	β	1.2895	0.0141	0.1189	0.4644	96.8
50	θ	0.8929	1.3063	1.1430	3.1243	99.9
	α	0.3545	0.0557	0.2360	0.8047	96
	β	1.2894	0.0116	0.1079	0.4211	96.3
100	θ	0.9713	1.1764	1.0846	3.0973	99.8
	α	0.3656	0.0535	0.2312	0.8003	95.9
	β	1.2926	0.0095	0.0974	0.3807	97
200	θ	0.9466	1.0663	1.0326	2.9688	99.8
	α	0.3539	0.0463	0.2152	0.7625	95.9
	β	1.2915	0.0082	0.0904	0.3529	96.7
300	θ	0.9527	1.0660	1.0325	2.9752	99.8
	α	0.3578	0.0498	0.2232	0.7806	95.5
	β	1.2891	0.0083	0.0909	0.3540	97.6
500	θ	1.0517	0.9182	0.9582	2.9280	99.8
	α	0.3748	0.0510	0.2259	0.7927	95.9
	β	1.2954	0.0066	0.0812	0.3180	96.4

6 Applications

Here, we provide applications to two real data sets to illustrate the importance and potentiality of the GAPTTLo distribution. The goodness of-fit statistics for these distributions and other competitive distributions are compared and the MLEs of their parameters are provided.

Table 4: Simulation results for the GAPTTLo distribution at θ = 1.5, α = 0.3, β = 1.3

n	Parameter	Mean	MSE	RMSE	AIL	CP
10	θ	1.2762	1.7933	1.3391	3.8652	99.6
	α	0.3765	0.1586	0.3982	1.1428	96.8
	β	1.2834	0.0173	0.1316	0.5121	96.4
20	θ	1.2859	1.6083	1.2682	3.7370	99.9
	α	0.3699	0.1306	0.3613	1.0651	97
	β	1.2766	0.0137	0.1171	0.4498	96.8
30	θ	1.3648	1.3611	1.1667	3.6372	99.6
	α	0.3712	0.1371	0.3703	1.0837	97.5
	β	1.2867	0.0124	0.1115	0.4342	96.4
50	θ	1.3469	1.2685	1.1263	3.5350	99.6
	α	0.3531	0.0892	0.2986	0.9294	97.2
	β	1.2864	0.0100	0.0998	0.3879	96.8
100	θ	1.3182	1.1576	1.0759	3.3977	99.5
	α	0.3467	0.1036	0.3219	0.9712	97.5
	β	1.2813	0.0088	0.0938	0.3604	97.2
200	θ	1.4007	1.0414	1.0205	3.3924	99.6
	α	0.3453	0.0508	0.2254	0.7782	97
	β	1.2886	0.0076	0.0875	0.3400	96.9
300	θ	1.3852	1.0034	1.0017	3.3365	99.8
	α	0.3421	0.0498	0.2231	0.7717	96.1
	β	1.2880	0.0065	0.0803	0.3115	97.4
500	θ	1.3870	0.9606	0.9801	3.2961	99.5
	α	0.3372	0.0434	0.2083	0.7391	95.6
	β	1.2888	0.0059	0.0768	0.2980	97.2

Table 5: Simulation results for the GAPTTLo distribution at θ = 2.5, α = 0.1, β = 2

n	Parameter	Mean	MSE	RMSE	AIL	CP
10	θ	2.6550	2.5286	1.5902	5.7584	98.7
	α	0.1934	0.0520	0.2281	0.6014	92.7
	β	1.9988	0.0636	0.2523	0.9894	96.5
20	θ	2.8610	2.3060	1.5186	5.7535	98.6
	α	0.2025	0.0515	0.2270	0.5997	93.2
	β	2.0344	0.0523	0.2286	0.8864	96.8
30	θ	2.8222	2.0408	1.4286	5.4583	98.3
	α	0.1943	0.0468	0.2163	0.5761	92.9
	β	2.0238	0.0444	0.2106	0.8207	96.7
50	θ	2.7738	1.7694	1.3302	5.1052	97.7
	α	0.1804	0.0376	0.1939	0.5265	92.8
	β	2.0221	0.0375	0.1936	0.7543	96.5
100	θ	2.8301	1.8074	1.3444	5.1111	97.4
	α	0.1864	0.0419	0.2046	0.5501	92.6
	β	2.0284	0.0335	0.1829	0.7088	97.4
200	θ	2.7850	1.5864	1.2595	4.8116	95.9
	α	0.1793	0.0402	0.2005	0.5405	92
	β	2.0245	0.0296	0.1721	0.6681	96.5
300	θ	2.7703	1.5606	1.2492	4.7832	96.6
	α	0.1710	0.0319	0.1785	0.4922	92.2
	β	2.0235	0.0304	0.1744	0.6778	97.4
500	θ	2.8287	1.5950	1.2629	4.7824	97
	α	0.1815	0.0393	0.1981	0.5356	92.1
	β	2.0240	0.0278	0.1666	0.6466	97.3

Table 6: Simulation results for the GAPTTLo distribution at $\theta = 3$, $\alpha = 0.5$, $\beta = 2.5$

n	Parameter	Mean	MSE	RMSE	AIL	CP
10	θ	3.2039	1.6422	1.2815	4.9618	99.8
	α	0.9898	1.6454	1.2827	3.3146	95.0
	β	2.4741	0.1073	0.3275	1.2805	97.2
20	θ	3.1406	1.4874	1.2196	4.7512	99.7
	α	0.9630	1.5565	1.2476	3.2347	94.8
	β	2.4583	0.0802	0.2833	1.0989	96.3
30	θ	3.1483	1.4094	1.1872	4.6196	99.4
	α	0.9156	1.2460	1.1163	2.9472	95.5
	β	2.4565	0.0677	0.2603	1.0063	97.3
50	θ	3.1083	1.2250	1.1068	4.3200	99.5
	α	0.8202	0.7331	0.8562	2.3773	93.9
	β	2.4629	0.0561	0.2369	0.9176	97.2
100	θ	3.1442	1.1586	1.0764	4.1835	99.3
	α	0.8650	1.0668	1.0329	2.7597	95.1
	β	2.4663	0.0459	0.2142	0.8294	97.4
200	θ	3.1392	1.0985	1.0481	4.0741	99.5
	α	0.7958	0.6828	0.8263	2.3087	95.2
	β	2.4815	0.0454	0.2132	0.8328	97.4
300	θ	3.1357	1.1148	1.0559	4.1066	99.4
	α	0.8082	0.6945	0.8333	2.3265	95
	β	2.4723	0.0427	0.2066	0.8030	96.6
500	θ	3.1642	1.0180	1.0090	3.9043	99.6
	α	0.8017	0.6214	0.7883	2.2298	94.7
	β	2.4724	0.0352	0.1875	0.7275	96.8

The first real data set we look at statistics on the number of months it takes for renal dialysis patients to get infected, as stated in [47]. The "times of infection" data set is: $\{2.5, 2.5, 3.5, 3.5, 3.5, 4.5, 5.5, 6.5, 6.5, 7.5, 7.5, 7.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 12.5, 13.5, 14.5, 14.5, 21.5, 21.5, 22.5, 22.5, 25.5, 27.5\}$. We now perform a normalization procedure by dividing this data by thirty, resulting in values between 0 and 1. The data set has been changed to: $\{0.08333, 0.08333, 0.116667, 0.850000, 0.116667, 0.116667, 0.15000, 0.18333, 0.216667, 0.916667, 0.216667, 0.25000, 0.25000, 0.25000, 0.25000, 0.28333, 0.316667, 0.35000, 0.38333, 0.416667, 0.416667, 0.750000, 0.450000, 0.483333, 0.483333, 0.716667, 0.716667, 0.750000\}$.
The second data set is obtained from [48] with respect to the flood data for 20 observations "0.265, 0.392, 0.297, 0.3235, 0.402, 0.269, 0.315, 0.654, 0.338, 0.379, 0.418, 0.423, 0.379, 0.412, 0.416, 0.449, 0.484, 0.494, 0.613, 0.74".
These real data sets are utilized to assess the goodness of fit of the GAPTTLo distribution. The suggested model is compared with TLo, xlindley (XL) [49], power xlindley (PXL) [50], Kumaraswamy (Kw) [51], beta (B) [52], inverse Lindley (IL) [53], exponential pareto (EP) [54], unit exponential Pareto (UEP) [55], kumaraswamy Kumaraswamy (KwKw) [56], exponentiated Kumaraswamy (EKw) [57], and Marshall-Olkin Kumaraswamy (MOKw) [58] models.
Some descriptive statistics of the both data sets are provided in Table 7, and the relevant boxplot are shown in Figure 2.

Table 7: Descriptive statistics for data set

	N	Mean	Median	Mode	Var	S	K
DATA1	28	0.377	0.300	0.250	0.061	0.809	-0.447
DATA2	20	0.423	0.407	0.379	0.016	1.156	1.153

The total time test (TTT) plot is an important approach for verify whether the data can be applied to a specific model. The TTT plots of the real data sets are shown in Figure 3.

The maximum likelihood estimators (MLEs) and standard errors (SEs) of the model parameters are computed. In order to assess the distribution models, various criteria are taken into account, including the Akaike information criterion (AIC), correct AIC (CAIC), Bayesian IC (BIC), Hannan-Quinn IC (HQIC), Kolmogorov-Smirnov (KS) test, p-value (PV) test, Cramér–Von Mises (W^{*}) and Anderson–Darling (A^{*}). In contrast, the broader dissemination is associated with reduced

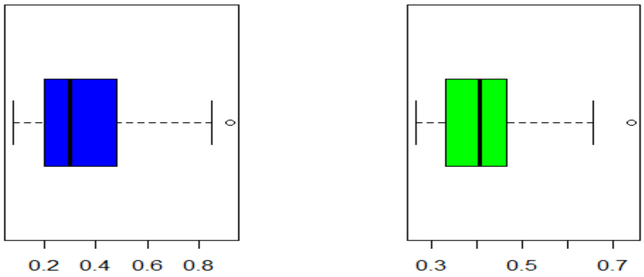

Fig. 2: Boxplots for the both data sets.

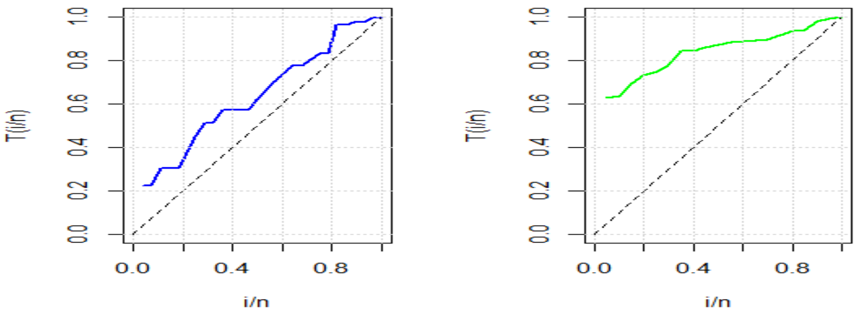

Fig. 3: TTT plots for the both data sets.

values of AIC, CAIC, BIC, HQIC, KS, W^*, A^* and the highest magnitude of PV. The maximum likelihood estimators (MLEs) of the competitive models, along with their standard errors (SEs) and values of AIC, CAIC, BIC, HQIC, PV, W^*, A^*, and KS for the suggested data sets, are displayed in Tables 8 – 11.
It has been observed that the GAPTTLo distribution, characterized by three parameters, exhibits superior goodness of fit compared to alternative models. This distribution exhibits the lowest values of AIC, CAIC, BIC, HQIC, W*, A* and KS, and the highest value of PV among the distributions under consideration in this analysis. Furthermore, Figures 4-7 exhibit the graphical representations of the estimated pdf, cdf, ccdf, and probability-probability (PP) plots for the competitive model applied to the given data sets.

7 Concluding Remarks

The present study examines and discusses a novel extension of the truncated Lomax (TLo) distribution, namely the generalised alpha power transformed truncated Lomax (GAPTTLo) distribution. The probability density function (PDF) of the GAPTTLo distribution exhibits either a decreasing trend, a single mode, or extreme right skewness. The hazard rate function (HRF) obtained from the GAPTTLo distribution may exhibit a declining or bathtub-shaped pattern. Quantile function, median, moments, moment generating function, and order statistics are among the statistical aspects. We estimate the three unknown parameters for the GAPTTLo distribution using the maximum likelihood method of estimation. Exploration of simulation is conducted to ascertain the optimal approach for estimate. Furthermore, we analyse two practical data sets using the GAPTTLo distribution, elucidating its exceptional performance relative to some

Table 8: MLEs and SEs for the first data set

Model	Estimates (standard errors)			
	θ	α	β	λ
GAPTTLo	4.077 (3.14)	0.444 (0.914)	2.798 (1.067)	
TLo	1.033 (0.957)			
XL	2.827 (0.487)			
PXL	1.637 (0.241)	4.224 (0.943)		
Kw	1.265 (0.25)	2.080 (0.571)		
B	1.357 (0.333)	2.106 (0.550)		
IL	0.406 (0.055)			
EP	1.638 (0.241)	0.326 (8.047)	0.650 (26.292)	
UEP	0.753 (0.101)	0.066 (0.511)	0.123 (0.725)	
KwKw	0.935 (0.003)	14.253 (0.003)	5.499 (0.007)	0.133 (0.025)
EKw	0.015 (0.013)	1.716 (0.264)	868.646 (1354.074)	
MOKw	0.127 (0.167)	1.988 (0.458)	1.080 (0.921)	

Table 9: Measures of fitting for the first data set

Model	AIC	BIC	CAIC	HQIC	KS	PV	W^*	A^*
GAPTTLo	-5.084	-1.087	-4.084	-3.862	0.115	0.856	0.040	0.306
TLo	-2.595	-1.026	-2.441	-2.187	0.155	0.510	0.079	0.515
XL	3.352	4.685	3.506	3.760	0.197	0.226	0.045	0.369
PXL	-3.554	-0.890	-3.075	-2.740	0.121	0.808	0.063	0.465
Kw	-3.325	-0.661	-2.845	-2.510	0.138	0.663	0.114	0.705
B	-3.555	-0.891	-3.075	-2.741	0.141	0.632	0.110	0.686
IL	-0.334	0.998	-0.181	0.073	0.155	0.508	0.089	0.637
EP	-1.555	2.442	-0.555	-0.333	0.121	0.809	0.063	0.465
UEP	71.916	75.913	72.916	73.138	0.187	0.282	0.215	1.227
KwKw	-3.652	1.676	-1.913	-2.023	0.147	0.582	0.058	0.382
EKw	-4.039	-0.043	-3.039	-2.817	0.125	0.771	0.067	0.448
MOKw	-4.546	-0.550	-3.546	-3.324	0.118	0.831	0.045	0.338

Table 10: MLEs and SEs for the second data set

Model	Estimates (standard errors)			
	θ	α	β	λ
GAPTTLo	13.779 (10.546)	0.422 (2.222)	93.306 (186.037)	
TLo	0.001 (1.111)			
XL	2.545 (0.513)			
PXL	3.526 (0.565)	14.513 (5.943)		
Kw	3.363 (0.603)	11.787 (5.358)		
B	6.758 (2.095)	9.113 (2.852)		
IL	0.634 (0.104)			
EP	3.526 (0.565)	0.288 (1.760)	0.180 (3.869)	
UEP	1.640 (0.252)	0.143 (0.548)	0.044 (0.274)	
KwKw	1.599 (0.013)	34.571 (0.012)	130.734 (0.142)	0.154 (0.035)
EKw	0.095 (0.071)	4.022 (0.841)	16265.970 (29021.920)	
MOKw	0.016 (0.037)	6.397 (1.260)	5.384 (11.143)	

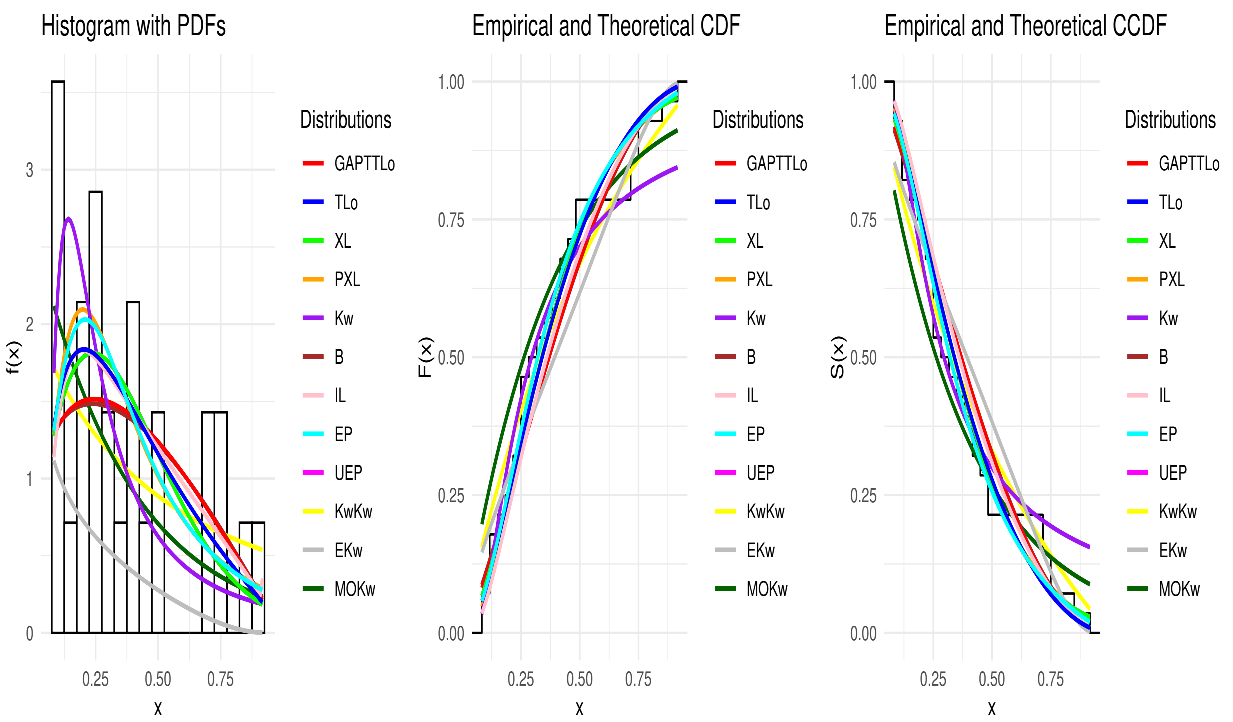

Fig. 4: Estimated pdf, cdf and ccdf plots of the competitive models for the first data set.

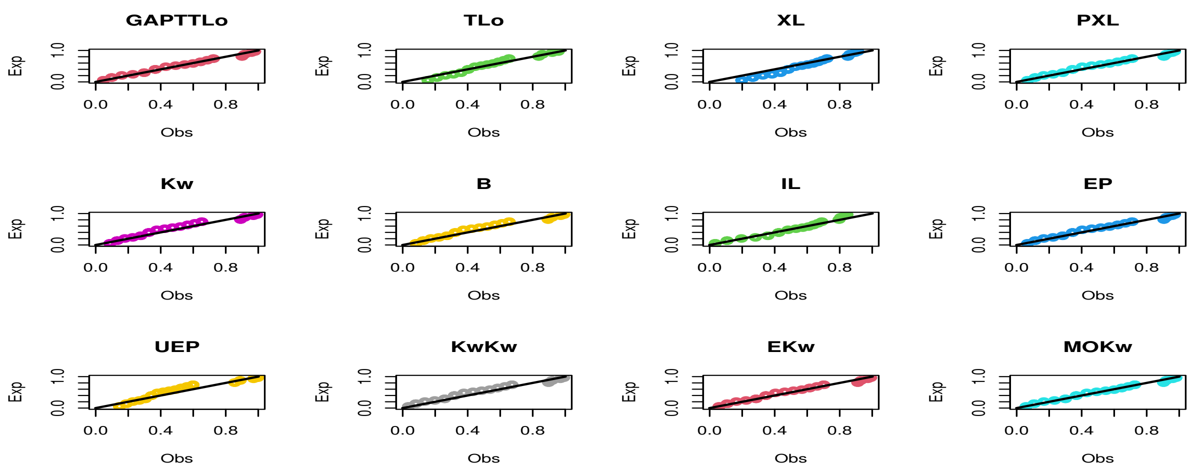

Fig. 5: The PP plots of the fitted models for the first data set.

competitors models. For future works many authors can use the suggested model to estimate its parameters using different censored schemes.

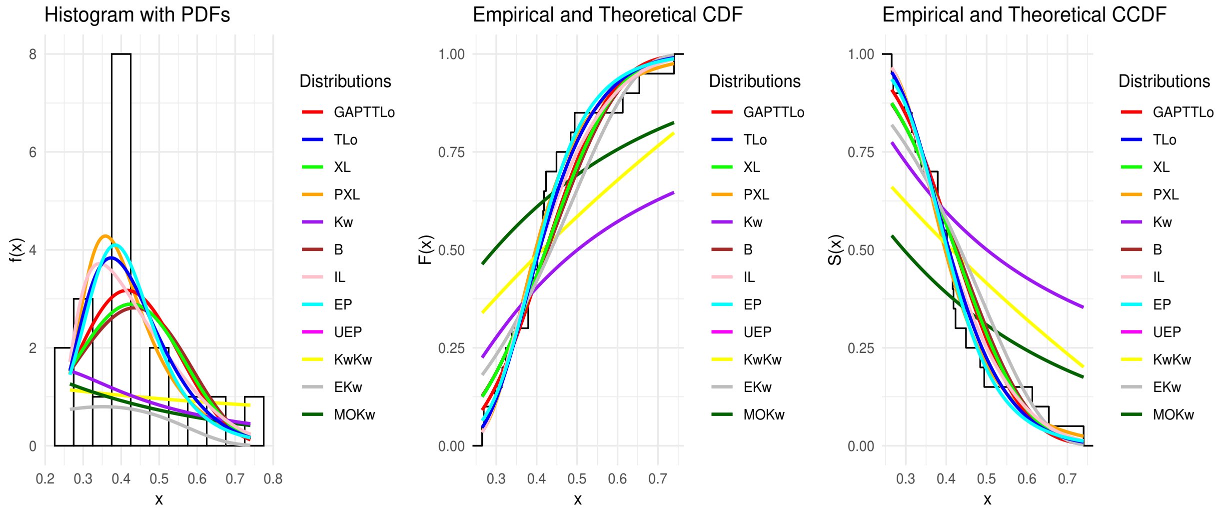

Fig. 6: Estimated pdf, cdf and ccdf plots of the competitive models for the second data set.

Table 11: Measures of fitting for the second data set

Model	AIC	BIC	CAIC	HQIC	KS	PV	W^*	A^*
GAPTTLo	-26.685	-23.698	-25.185	-26.102	0.120	0.936	0.044	0.264
TLo	1.314	2.309	1.536	1.508	0.239	0.201	0.116	0.699
XL	7.477	8.473	7.699	7.671	0.206	0.375	0.074	0.464
PXL	-22.528	-20.536	-21.822	-22.139	0.199	0.408	0.146	0.868
Kw	-21.732	-19.741	-21.027	-21.344	0.211	0.336	0.166	0.972
B	-24.124	-22.133	-23.419	-23.736	0.199	0.408	0.126	0.752
IL	0.827	1.822	1.049	1.021	0.356	0.013	0.046	0.281
EP	-20.528	-17.541	-19.028	-19.945	0.199	0.408	0.146	0.868
UEP	32.006	34.993	33.506	32.589	0.228	0.247	0.263	1.474
KwKw	-24.723	-20.740	-22.056	-23.945	0.148	0.771	0.062	0.350
EKw	-26.028	-23.041	-24.528	-25.445	0.143	0.810	0.059	0.356
MOKw	-25.564	-22.577	-24.064	-24.981	0.127	0.906	0.051	0.334

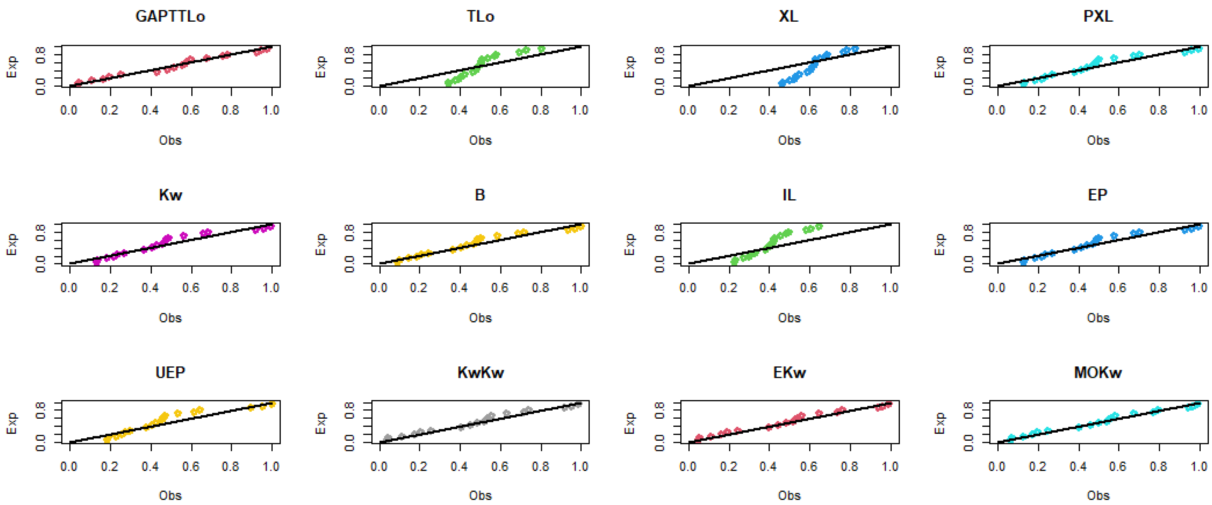

Fig. 7: The PP plots of the fitted models for the second data set.

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