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The Approximate Solutions and Stability Analysis for Blood Ethanol Concentration System

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Abstract: The purpose of this article is to present the nonlinear fractional order differential equations with Atangana-Baleanu operator using the natural variational iteration method, and by using the fixed-point theory to examine the solution's existence and uniqueness The system is solved numerically by using the suggested scheme and the resulting solutions are compared to the exact data. This combination will result in a better and more quickly convergent sequence, the equilibrium points and their stability are also studied.

Keywords: Fractional blood ethanol concentration system, Atangana-Baleanu fractional derivative operator, Natural transform, VIM, stability.

1 Introduction

Most of the applied specializations of the various sciences used differential equations to express them. also derivatives are widely used for modeling physical, chemical, and biological problems in science and technology .Recent studies have shown the suitability of this branch of mathematical analysis to accurately some physical systems Recently, researchers have been interested in fractional differential equations and finding appropriate solutions for them It describes nonlinear phenomena more accurately than classical equations We may face some difficulties in solving equations due to the complexity of fractional derivatives. With respect to the modeling of alcohol concentration in blood, the ability to model blood alcohol content as a function of time is an attractive concept to medical personnel as well. Besides modeling the blood concentration for the sole information, the techniques could perhaps be used to compare the metabolic ability of a given subject's liver to that which would be considered normal. most published research has concentrated on a global approach based on the fractional derivative introduced by Caputo, Caputo–Fabrizio and the Atangana–Baleanu.[1-26].

2 Basic Concepts

Definition 1: Let $\varepsilon_1, \varepsilon_2 \in H_1$ such that $\varepsilon_1 > \varepsilon_2$, we define the Atangana-Baleanu derivative (ABFD) as

$$({}_{AB})D_t^{\alpha} f(\tau) = \frac{m(\alpha)}{1-\alpha}\int_0^{\tau} E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(\tau-x)^{\alpha}\right) f'(x)\,dx, \tag{2.1}$$

where $0 < \alpha \leq 1$ and $m(\alpha)$ is a normalization function such that $m(1) = m(0) = 1$.

Definition 2: The natural transform of the function for $t \in \mathbb{R}$ is given by

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0,\ |f(t)| < Me^{|t|/\tau_j} \text{ if } t \in (-1)^j \times (0,\infty],\ j \in \mathbb{Z}^+ \right\},$$

and the natural transform is

$$N[f(t)] = R(s,u) = \int_0^{\infty} e^{-st} f(ut)\,dt, \quad s,u \in (-\infty,\infty). \tag{2.2}$$

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The equation (2.2) becomes the Laplace transformation if $u = 1$:

$$R(s,u) = \frac{1}{u}\int_0^{\infty} e^{-st/u} f(t)\,dt. \tag{2.3}$$

From [?] and Eq. (2.3), we obtain the relation:

$$N\left((_{AB})D_t^{\alpha} f(\tau)\right) = \frac{m(\alpha)}{1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}}}\left(R(s,u) - \frac{1}{s} f(0)\right). \tag{2.4}$$

The inverse natural transform of (2.4) becomes:

$$N^{-1}(R(s,u)) = f(t) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} e^{st/u} R(u,s)\,dt. \tag{2.5}$$

where s and u are natural transform variables, and p is a real constant.

3 Analysis of the Natural VIM

This section is focused on presenting the basis of the Natural VIM. For this, recall the general FDE:

$$(_{AB})D_t^{\alpha}\Phi(t) + R(\Phi,\Psi) + L(\Phi,\Psi) = g(t), \tag{3.1}$$

where $R(\Phi,\Psi)$, $N(\Phi,\Psi)$, and $g(t)$ are linear, nonlinear, and known functions, respectively. $g(t)$ is the non-homogeneous term.

Now, we apply the N-transform to Eq. (3.1) to get:

$$\frac{z(\alpha)}{1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}}}\left(N(\Phi(t)) - \frac{1}{s}\Phi(0)\right) = N\left(g(t) - R(\Phi,\Psi) - L(\Phi,\Psi)\right), \tag{3.2}$$

and

$$\bar{\Phi}(t) = \frac{1}{s}\Phi(0) - \left[\frac{1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}}}{z(\alpha)} N\left(g(t) - R(\Phi,\Psi) - L(\Phi,\Psi)\right)\right]. \tag{3.3}$$

Applying the variation iteration method:

$$\bar{\Phi}_{n+1}(t) = \bar{\Phi}_n(t) + \lambda\left[\Phi_n(t) - \frac{1}{s}\Phi(0) - \left(\frac{1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}}}{z(\alpha)} N\left(g(t) - R(\Phi,\Psi) - L(\Phi,\Psi)\right)\right)\right], \tag{3.4}$$

where $\lambda(s) = -1$ is the Lagrange coefficient. By using the inverse N-transform, we have:

$$\Phi_{n+1}(t) = \Phi_n(t) - N^{-1}\left[\left(1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}}\right) N\left(g(t) - R(\Phi,\Psi) - L(\Phi,\Psi)\right)\right]. \tag{3.5}$$

Under the initial condition $\Phi(0) = \Phi_0(0)$, we have:

$$\Phi(t) = \lim_{n\to\infty}\Phi_n(t).$$

4 Existence and uniqueness investigation

Consider the following system:

$$(_{AB})D_t^{\alpha}\Phi(t) = -\beta^{\alpha}\Phi(t) \tag{4.1}$$

$$(_{AB})D_t^{\alpha}\Psi(t) = \beta^{\alpha}\Phi(t) - \mu^{\alpha}\Psi(t) \tag{4.2}$$

with initial conditions:

$$\Phi(0) = \Phi_0, \quad \Psi(0) = 0.$$

The solutions for $\Phi(t)$ and $\Psi(t)$ are given by:

$$\Phi(t) = \Phi_0 + \frac{1-\alpha}{m(\alpha)} w(t,\Phi) + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w(t,\Phi)\, d\tau, \tag{4.3}$$

$$\Psi(t) = \Psi_0 + \frac{1-\alpha}{m(\alpha)} N(t,\Phi,\Psi) + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} N(t,\Phi,\Psi)\, d\tau. \tag{4.4}$$

where $N(t,\Phi,\Psi) = \beta^\alpha \Phi(t) - \mu^\alpha \Psi(t)$ and $w(t,\Phi) = -\beta^\alpha \Phi(t)$.

It is clear that N and w satisfy the Lipschitz conditions.

Let $\Phi(t), \Psi(t) \in H'$. Then, we define the following operators:

$$T = \Phi_0 + \frac{1-\alpha}{m(\alpha)} w + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w\, d\tau, \tag{4.5}$$

$$F = \Psi_0 + \frac{1-\alpha}{m(\alpha)} N + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} N\, d\tau. \tag{4.6}$$

These satisfy the Lipschitz condition.

Proof: Let $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ be assumed to be bounded, where $\Phi_1(0) = \Phi_2(0)$ and $\Psi_1(0) = \Psi_2(0)$.

We want to show that:

$$\|T(t,\Phi_1) - T(t,\Phi_2)\| = \frac{1-\alpha}{m(\alpha)} \|w(t,\Phi_1) - w(t,\Phi_2)\| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|w(t,\Phi_1) - w(t,\Phi_2)\|\, d\tau$$

$$\leq \frac{1-\alpha}{m(\alpha)} \|w(t,\Phi_1) - w(t,\Phi_2)\| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|w(t,\Phi_1) - w(t,\Phi_2)\|\, d\tau.$$

From the Lipschitz condition of $w(t,\Phi_1)$, we obtain:

$$\leq \left(\frac{1-\alpha}{m(\alpha)} + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau \right) L \|\Phi_1 - \Phi_2\|.$$

Thus,

$$= \gamma \|\Phi_1 - \Phi_2\|, \quad \text{where} \quad \gamma = \frac{1-\alpha}{m(\alpha)} + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau. \tag{4.7}$$

Similarly, for $F(t,\Psi_1)$ and $F(t,\Psi_2)$, we have:

$$\|F(t,\Psi_1) - F(t,\Psi_2)\| = \frac{1-\alpha}{m(\alpha)} \|N(t,\Psi_1) - N(t,\Psi_2)\| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|N(t,\Psi_1) - N(t,\Psi_2)\|\, d\tau.$$

$$\leq \frac{1-\alpha}{m(\alpha)} L \|\Psi_1 - \Psi_2\| + \frac{\alpha L}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\Psi_1 - \Psi_2\|\, d\tau.$$

Thus,

$$= \gamma \|\Psi_1 - \Psi_2\|, \quad \text{where} \quad \gamma = \frac{1-\alpha}{m(\alpha)} L + \frac{\alpha L}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau. \tag{4.8}$$

Theorem 2: Let $\Phi(t), \Psi(t) \in H'$. Then $\Phi(t)$ and $\Psi(t)$ are the solution for the system if and only if they satisfy the integral equations:

$$T = \Phi_0 + \frac{1-\alpha}{m(\alpha)} w + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w\, d\tau,$$

$$F = \Psi_0 + \frac{1-\alpha}{m(\alpha)} N + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} N\, d\tau.$$

Proof: We can prove that by applying the Atangana-Baleanu integral for the above system and simplifying, we will obtain the two operators T and F.

Theorem 3: Consider the above system. If the functions $w(t,\Phi)$ and $N(t,\Psi)$ satisfy the Lipschitz condition with the Lipschitz constant b such that:

$$b \leq \frac{m(\alpha)\Gamma(\alpha)}{(1-\alpha)\Gamma(\alpha)+1}, \tag{4.9}$$

then the system has a unique solution.

Proof: We will prove that T and F are contraction maps. Let $\Phi_1, \Phi_2 \in H'$, then:

$$\|T(t,\Phi_1)-T(t,\Phi_2)\| \leq \|\frac{1-\alpha}{m(\alpha)}[w(t,\Phi_1)-w(t,\Phi_2)]\| + \|\frac{\alpha}{m(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[w(t,\Phi_1)-w(t,\Phi_2)]\,d\tau\|.$$

$$\leq \frac{1-\alpha}{m(\alpha)}\|w(t,\Phi_1)-w(t,\Phi_2)\| + \frac{(1-\alpha)}{m(\alpha)}L\|\Phi_1-\Phi_2\| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau L\|\Phi_1-\Phi_2\|.$$

$$\leq \frac{(1-\alpha)\Gamma(\alpha)+1}{m(\alpha)\Gamma(\alpha)}L\|\Phi_1-\Phi_2\|. \tag{4.10}$$

Since $\frac{(1-\alpha)\Gamma(\alpha)+1}{m(\alpha)\Gamma(\alpha)}L \leq 1$, then T is a contraction map. By the Banach fixed-point theorem, T has a unique fixed point.

Similarly, for $\Psi_1, \Psi_2 \in H'$, we have:

$$\|F(t,\Psi_1)-F(t,\Psi_2)\| \leq \|\frac{1-\alpha}{m(\alpha)}[N(t,\Psi_1)-N(t,\Psi_2)]\| + \|\frac{\alpha}{m(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[N(t,\Psi_1)-N(t,\Psi_2)]\,d\tau\|.$$

$$\leq \frac{1-\alpha}{m(\alpha)}L\|\Psi_1-\Psi_2\| + \frac{\alpha L}{m(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau\|\Psi_1-\Psi_2\|.$$

$$\leq \frac{(1-\alpha)\Gamma(\alpha)+1}{m(\alpha)\Gamma(\alpha)}L\|\Psi_1-\Psi_2\|. \tag{4.11}$$

Since $\frac{(1-\alpha)\Gamma(\alpha)+1}{m(\alpha)\Gamma(\alpha)}L \leq 1$, then F is a contraction map. By the Banach fixed-point theorem, F has a unique fixed point.

5 Application

Consider the blood ethanol concentration system with initial value problem, applying the NVIM, suppose that $m(\alpha)=1$.

$$({}_{AB}D_t^{\alpha})\Phi(t) = -\beta^{\alpha}\Phi(t) \tag{5.1}$$

$$({}_{AB}D_t^{\alpha})\Psi(t) = \beta^{\alpha}\Phi(t) - \mu^{\alpha}\Psi(t) \tag{5.2}$$

Where we have the following descriptions for the included functions and parameters:

$\Phi(t)$: The alcohol's concentration in the stomach at time t (mg/l).

$\Psi(t)$: The alcohol's concentration in the blood at time t (mg/l).

β : The rate law constant 1 (min^{-1}).

μ : The rate law constant 2 (min^{-1}).

Applying the NVIM to equations (5.1) and (5.2) we get:

$$\Phi_{n+1}(t) = \Phi_n(t) - N^{-1}\left[(1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}})N(\beta^{\alpha}\Phi_n(t))\right] \tag{5.3}$$

$$\Psi_{n+1}(t) = \Psi_n(t) - N^{-1}\left[(1-\alpha+\frac{\alpha u^{\alpha}}{s^{\alpha}})N(\mu^{\alpha}\Psi_n(t)-\beta^{\alpha}\Phi_n(t))\right] \tag{5.4}$$

$$\Phi_0(t)=\Phi_0, \quad \Psi_0(t)=0$$

We find the approximate $\Phi(t)$ and $\Psi(t)$ respectively:

$$\Phi(t)=\lim_{n\to\infty}\Phi_n(t),\quad \Psi(t)=\lim_{n\to\infty}\Psi_n(t)$$

For $n=1$:

$$\Phi_1(t)=\Phi_0-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N(\beta^\alpha\Phi_0(t))\right]$$

$$=\Phi_0-\beta^\alpha\Phi_0N^{-1}\left[\left(\frac{1}{s}-\frac{\alpha}{s}+\frac{\alpha u^\alpha}{s^{\alpha+1}}\right)\right]$$

$$=\Phi_0-\beta^\alpha\Phi_0\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)$$

For $n=1$:

$$\Psi_1(t)=0-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N(0-\beta^\alpha\Phi_0(t))\right]$$

$$=\beta^\alpha\Phi_0N^{-1}\left[\left(\frac{1}{s}\right)\right]$$

$$=\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)$$

For $n=2$:

$$\Phi_2(t)=\Phi_0-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N(\beta^\alpha\Phi_1(t))\right]$$

$$=\Phi_0-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N\left(\beta^\alpha\left(\Phi_0-\beta^\alpha\Phi_0\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)\right)\right)\right]$$

$$=\Phi_0-\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)-\Phi_0\beta^2\alpha\left[(1-\alpha)^2-2(1-\alpha)\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}-\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$

For $n=2$:

$$\Psi_2(t)=\Psi_0(t)-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N(\mu^\alpha\Psi_1(t)-\beta^\alpha\Phi_1(t))\right]$$

$$=0-N^{-1}\left[(1-\alpha+\frac{\alpha u^\alpha}{s^\alpha})N\left(\mu^\alpha\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)-\beta^\alpha\left(\Phi_0-\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)\right)\right)\right]$$

$$=\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)-\Phi_0\beta^\alpha u^\alpha\left[(1-\alpha)^2+2(1-\alpha)\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$

$$+\Phi_0\beta^2\alpha\left[(1-\alpha)^2+2(1-\alpha)\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$

Therefore, the series solution $\Phi(t)$ and $\Psi(t)$ is given respectively by:

$$\Phi(t)=\Phi_0-\beta^\alpha\Phi_0\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)-\Phi_0\beta^2\alpha\left[(1-2\alpha+\alpha^2)-(2\alpha-2\alpha^2)\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+\cdots \quad (5.7)$$

$$\Psi(t)=\Phi_0\beta^\alpha\left(1-\alpha+\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right)+\Phi_0\beta^\alpha u^\alpha\left[(1-\alpha)^2-2(1-\alpha)\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}-\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$

$$-\Phi_0\beta^2\alpha\left[(1-\alpha)^2+2(1-\alpha)\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+\cdots \tag{5.8}$$

The approximate solution by considering the VIM with distinct values of α, $\beta=0.028$, $\mu=0.084$, and initial conditions $\Phi_0 = 4$ and $\Psi_0 = 0$ is shown in Figures 1. Figure 1 explains the exact solutions with distinct values of $\alpha = (0.9, 0.8, 0.7, 0.6, 1)$. As a result, the numerical solution is appropriate and dependent on α, β, and μ, which confirms that, in the case of fractional derivatives, the suggested approximation method is effectively used to solve the stated problem.

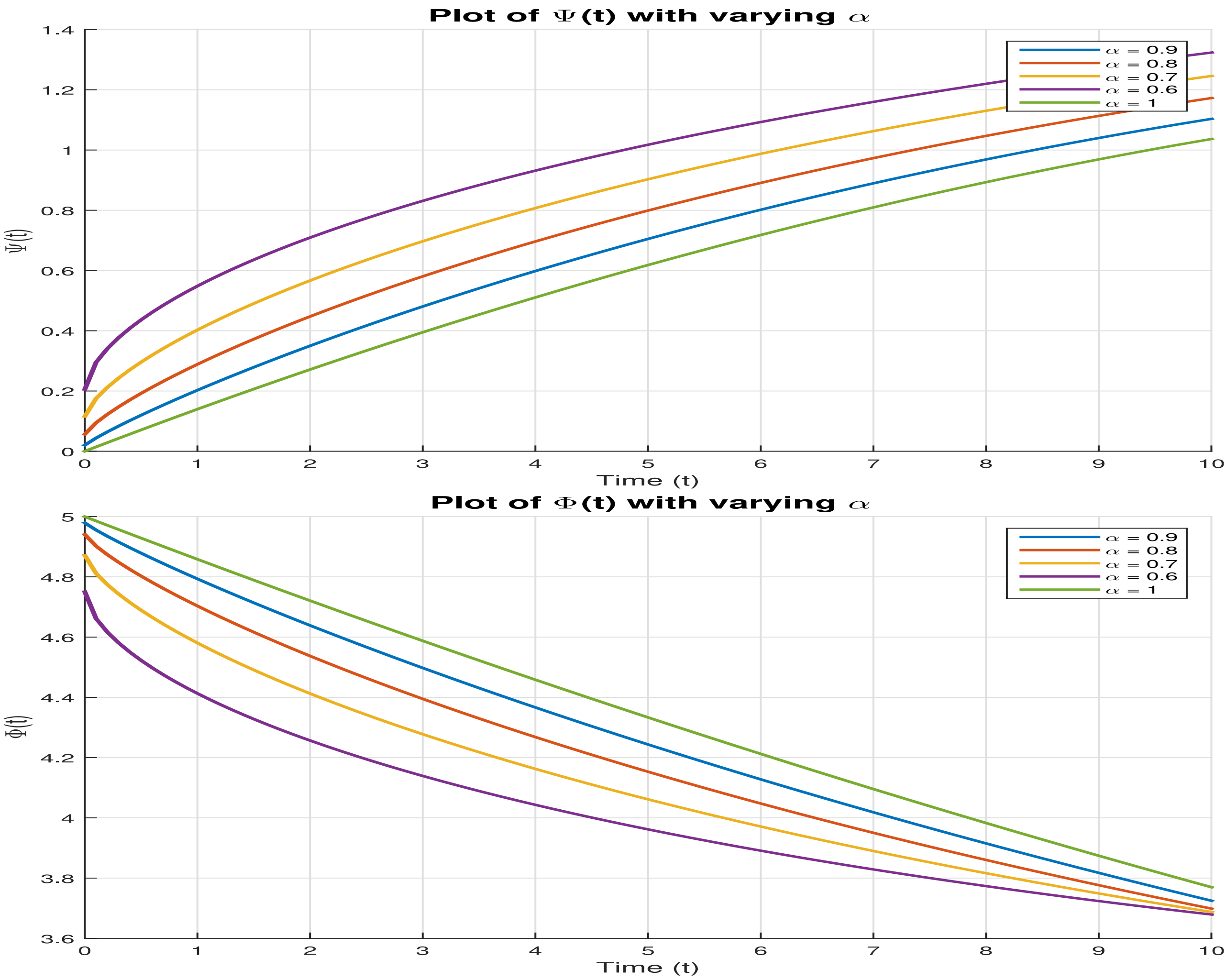

Fig. 1: The plots of the rough Result $u(\varkappa, \mathrm{t})$for a range of values of ⋉

6 Stability

In this section, the following proof is used to examine the stability of the system described by the Natural Atangana-Baleanu derivative [30].

$$({}_{AB}D_t^{\alpha})\Phi(t)=0 \tag{6.1}$$

$$({}_{AB}D_t^{\alpha})\Psi(t)=0 \tag{6.2}$$

$$-\beta^{\alpha}\Phi(t)=0, \quad \beta^{\alpha}\Phi(t)-\mu^{\alpha}\Psi(t)=0$$

The equilibrium point (EP) is $(0,0)$.
The Jacobian matrix for the system is given as:

$$J=\begin{pmatrix}-\beta^{\alpha} & 0\\ \beta^{\alpha} & -u^{\alpha}\end{pmatrix} \tag{6.3}$$

The stability of the fractional system is determined by the eigenvalues λ_i of the matrix A. The eigenvalues are found by solving:

$$\det(A-\lambda I)=0 \tag{6.4}$$

For this system, the characteristic equation is:

$$\begin{vmatrix}-\beta^{\alpha}-\lambda & 0\\ \beta^{\alpha} & -u^{\alpha}-\lambda\end{vmatrix}=0$$

This simplifies to:

$$(-\beta^{\alpha}-\lambda)(-u^{\alpha}-\lambda)=0$$

$$\lambda^2+(\beta^{\alpha}+u^{\alpha})\lambda+\beta^{\alpha}u^{\alpha}=0 \tag{6.5}$$

The eigenvalues are:

$$\lambda_1=-\beta^{\alpha}, \quad \lambda_2=-u^{\alpha} \tag{6.6}$$

For a fractional-order system $0<\alpha\leq 1$, the system is asymptotically stable if all eigenvalues satisfy $\mathrm{Re}(\lambda_i)<0$. In this case, both eigenvalues $-\beta^{\alpha}$ and $-u^{\alpha}$ are negative if $\beta^{\alpha}>0$ and $u^{\alpha}>0$.
The system is stable if the parameters $\beta^{\alpha}>0$ and $u^{\alpha}>0$. Stability improves as these parameters increase, as they ensure the system's states decay over time.

7 Conclusion

The Natural VIM is a good technique and approach that can deal with FDEs that are linear or nonlinear. We implemented this procedure solve the fractional blood ethanol concentration system. The suggested scheme is a more dependable method that converges more quickly the proposed method is clear, simple it may be extended to solve additional fractional-order.

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