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Fitting Statistical Parent Distributions to Quantify Financial Risk in the South African Financial Index (J580): an Extended Analysis

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Abstract: Statistical distribution fitting and risk estimation are an important exercise in financial data analysis because the fitted distribution properties tend to provide more theoretically established information about the data. Consequently, in this paper, we correct and extend the analysis of the South African Financial Index, also known as J580 (which is an important index, with a significant market capitalization and it is listed in the Johannesburg Stock Exchange). A paper published in this journal, fitted four distributions (gamma, Weibull, exponential and Burr) to the J580 returns using two goodness-of-fit tests (Akaike information criterion (AIC) and Bayesian information criterion (BIC)) and two risk measures (value-at-risk and expected shortfall). This latter paper had incorrect descriptives and the analysis is of little depth; thus, in this paper, we conduct a more thorough analysis by extending this work by using six goodness-of-fit (AIC, BIC, Kolmogorov-Smirnov, Anderson-Darling, Cramer von Mises, negative log-likelihood) and the same two risk measures to instead nineteen statistical distributions (i.e. we re-evaluate the four distributions in full and conduct model fitting of fifteen additional ones). To strike a balance between goodness-of-fit and risk measures performance, a ranking method is implemented, and it is observed that the Burr distribution (also identified as the best distribution by the paper published in this journal) is consistently competitive in modelling losses (especially with an excellent goodness-of-fit and partially good with risk measures). The other strong competitors that can be implemented instead of the Burr are the generalized Pareto and transformed beta distributions. However, the exponential distribution which was identified by the paper published in this journal is not even in the top 5 of best distributions under gains; instead, the most ideal distributions are the transformed gamma, transformed beta and Weibull (in that order). From the findings as well as since the kurtosis and skewness of the losses are much greater than that of the gains, it is advisable to hold a short position in the long run; however, in the short run, investors need to closely monitor the candlestick trends so that they can make rational decisions based on the underlying patterns observed.

Keywords: Expected shortfall, Goodness-of-fit, Heavy-tailed, Light-tailed, Financial index, Value-at-risk (VaR), Risk measures.

1 Introduction

Investment portfolios and risk management strategies play a pivotal role in financial decision-making, particularly in assessing potential losses and gains [1]. Banks and investment entities need to be extremely cautious of different types of losses as they tend to cause bankruptcy in many financial institutions. Financial data returns often exhibit heavy tails, that are highly leptokurtic and have a positive skewness [2]. Model uncertainty arises as traditional models may inadequately capture extreme events which affects risk predictions. A heavy-tailed (light-tailed) data distribution is characterized by an increased (decreased) probability of extreme events, respectively, deviating from the characteristics of a normal distribution. Heavy-tailed distributions exhibit a slower decay in tail probabilities compared to normal distributions which makes extreme events more probable. In exploring heavy-tailed (light-tailed) distributions, we are exposed to scenarios where the likelihood of extreme events is heightened (lowered), respectively.

Heavy- and light-tailed distributions can be used to assess several risks which are mentioned in [2]. Incorporating heavy-tailed (light-tailed) distributions into risk models provides a more realistic and robust framework for managing and

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mitigating financial risks in situations where extreme outcomes are more (less) probable, respectively. Consequently, it is important to know whether gain or loss returns of a specific financial data exhibit heavy or light tail characteristics.

While overall there are many basic statistical distributions, similar to [3] in this research work we intend to limit our discussion to the following 19 non-Gaussian statistical distributions for light-tailed to heavy-tailed data (these can be classified into three categories, each containing either 1, 2, 3 or 4 parameter(s) [4]:

1. *The transformed beta family (9 in total)*
   - 4-parameter: Transformed beta distribution.
   - 3-parameter: Generalized Pareto, Burr, and inverse Burr distributions.
   - 2-parameter: Pareto, inverse Pareto, loglogistic, paralogistic, and inverse paralogistic distributions.
2. *The transformed gamma family (8 in total)*
   - 3-parameter: Transformed gamma and inverse transformed gamma distributions
   - 2-parameter: Gamma, inverse gamma, Weibull and inverse Weibull distributions.
   - 1-parameter: Exponential and inverse exponential distributions.
3. *Other distributions (2 in total)*
   - 2-parameter: Lognormal and inverse Gaussian distributions.

Many academic studies have been conducted on statistical loss distributions that are outlined above using different types of datasets, some of these are summarized in Table 1. As we document these publications, we take note of other light-tailed and heavy-tailed distributions that are not listed on Table 1 (this is denoted with tick on the sub-column labelled 'Other distributions considered' within the main column 'Statistical distributions' in Table 1). Note that characteristics such as parameter estimation, goodness-of-fit and risk measures are not thoroughly discussed in the next two paragraphs of the literature review because these are comparatively summarized in Table 1.

Table 1: Summary of different publications that discuss on literature based on 19 statistical distributions in the context of light-tailed and heavy-tailed data

Publication	Illustrative Data	Statistical Distributions																				Parameter Estimation	Goodness-of-fit	Risk Metric
		Burr	Exponential	Gamma	Generalized Pareto	Inverse Burr	Inverse Exponential	Inverse Gamma	Inverse Gaussian	Inverse Paralogistic	Inverse Pareto	Inverse Transformed Gamma	Inverse Weibull	Loglogistic	Lognormal	Paralogistic	Pareto	Transformed Beta	Transformed Gamma	Weibull	Other distributions considered			
[2]	Six examples	✓	✓	✓											✓		✓			✓		MLE	KS, AD	-
[3]	SA taxi claims Danish fire claims	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		MLE	KS, AD, CvM, NLL, AIC, BIC	VaR, TVaR
[4]	Theory and basic examples	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	Bayes, MoM, PE, MLE	LR, χ^2 KS, AD	VaR, TVaR
[5]	SA taxi claims	✓	✓	✓											✓		✓			✓		MLE	KS, AD	VaR
[6]	SA Financial Index - J580	✓	✓	✓																✓		MLE	AIC, BIC	VaR, ES
[7]	SA Financial Index - J580	✓	✓	✓																✓		MLE	KS, AD, CvM, NLL, AIC, BIC	VaR, ES
[8]	Vehicle Insurance loss																			✓		MLE	AIC, BIC, HAQUI, CAIC	-
[9]	US property claim indices	✓		✓											✓		✓					MLE	KS, CvM, AD, χ^2	-
[10]	Insurance and economic							✓	·													MWL	AIC, BIC, dAIC, TIC	-
[11]	Third-party liability claims			✓											✓		✓			✓		MLE, MoM	DT, χ^2 KS, AD	-
[12]	US indemnity losses Danish fire claims																				✓		KS, AIC, LL	VaR, TVaR
[13]	Simulated data																				✓	-	-	VaR, TVaR, WP
[14]	8 Norwegian stocks								✓												✓	Bootstrap	-	VaR
[15]	Unemployment insurance Simulated data		✓	✓																✓	✓	MLE, OLS, WLS, AD, CvM, PE	AIC, CAIC, BIC, HAQUI. AD,	VaR, TVaR, TV, TVP
[16]	Operational loss data			✓											✓		✓			✓		MLE	KS, AD	-
[17]	LDCE operational loss	✓	✓	✓										✓	✓		✓			✓	✓	MLE	χ^2	-
[18]	LDCE operational loss		✓	✓											✓		✓				✓	MLE	KS, AD	-
[19]	Legal loss data		✓														✓			✓		MLE	KS, AD	-
[20]	Legal liability loss		✓																	✓	✓	MLE	-	-
[21]	Simulated																				✓	-	-	-
[22]	Simulated																✓					-	-	-
[23]	SA Industrial Index - J520	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	MLE	KS, AD, CvM, NLL, AIC, BIC	VaR, TVaR

Acronym: , AD – Anderson-Darling, AIC – Akaike Information Criterion, BIC – Bayesian Information Criterion, CAIC – consistent Akaike information criterion, CvM – Cramer von Misses, dAIC – difference AIC, DT – Density Trace, ES – Expected Shortfall, HAQUI - Hannan-Quinn information, KS – Kolmogorov-Smirnov, LL – log-likelihood, LR – Likelihood Ratio, MLE – Maximum Likelihood Estimation, MoM – Method of Moments, NLL – Negative Log Likelihood, OLS - ordinary least-squares, PE – percentile estimation, χ^2- Chi-Squared, TIC - Takeuchi Information Criterion , TVaR – Tail Value-at-Risk, TV - tail variance, TVP – tail variance premium, VaR – Value-at-Risk, WLS - weighted least-square, WP - weighted premium

More recently, [3] fitted the 19 distributions to two datasets, i.e., the South African taxi claims data and the Danish fire claims data. Next, [5] only considered six distributions and fitted these to the South African taxi claims data. Using the KS and AD tests to assess the goodness-of-fit, the lognormal distribution was shown to have a better fit. [6] used the South African Financial Index (J580) to fit four loss distributions using only the AIC and BIC, the Burr and exponential distributions had the best goodness-of-fit for losses and gains, respectively.

Thereafter, [7] reevaluated the results in [6] and found some errors. [8] used the vehicle insurance loss data for which they only considered the Weibull distribution, and the proposed model is compared with other heavy tailed distributions, not considered in this study. [9] illustrated their findings using the United Stated (US) property claim services indices data, of which they analysed 4 of the 19 distributions considered in this study, and the lognormal distribution provided a better fit in comparison to the Pareto distribution. [10] investigated the inverse Gaussian distribution with application to insurance data about bodily injury claims and economic data about incomes of Italian households. [11] only considered 4 distributions to model the claim amounts in motor third-party liability insurance.[12] used two datasets for their research study (the US indemnity losses and Danish fire losses) by fitting two distributions, i.e. skew-normal and skew-logistic. [13] derived the theoretical properties of the skew-elliptical distribution without using a practical dataset; however, a simulated multivariate skew-normal dataset with 5 different returns was used. [14] fitted the symmetrical Gaussian, non-symmetrical normal inverse Gaussian distribution, and non-parametric model on eight Norwegian stocks (1997-1999) data. [15] used a number of distributions (not considered in this study) as well as the gamma and Weibull to model the monthly unemployment insurance (July 2008 - April 2013).

[16] modelled the operational loss data (1950-2002) which was obtained from worldwide institutions compiled by the 'IC2 Operational Loss F1RST Database' and fitted five distributions considered in this research work. [17] also analysed the LDCE operational loss data where they analysed the data on bank-by-bank basis instead of amalgamated as in [18] . Note that [18] analysed the LDCE operational loss data using the extremes distributions and four distributions considered in this research work. [19] studied the legal loss data using three distributions while [20] analysed the legal liability loss data where it was realized that it is highly leptokurtic. Based on the lowest value of the AD for the goodness-of-fit, the Weibull distribution was shown to have a better fit in [20]. Note that [21] and [22] used simulated data instead of real-life data and this is mostly caused by lack of availability of data (subject to strict privacy laws). Finally, [23] fitted 22 distributions to the South African Industrial Index (J520), where six goodness-of-fit tests and two risk measures are used to find the most appropriate distribution to model the index's returns.

In [6], the authors considered only 4 of the 19 distributions in Table 1 (i.e., Burr, exponential, gamma and Weibull) to separately analyze the goodness-of-fit for losses and gains of the J580 and concluded that the best distributions to fit were the Burr and exponential distributions, respectively. Thereafter, they computed the corresponding risk metrics, i.e. VaR and ES using the Burr for losses and exponential for gains. Given the critical discussion of [7] about the accuracy of the paper by [6], we believe that considering a larger class of distributions as done by [3,23] would be of benefit to check whether there might be a better distribution that fits the loss and gain returns rather than the considered fewer four distributions. Such a larger class of distributions would bring in a more in-depth analysis of the J580 returns and given the importance of this index, it is important to thoroughly analyse the goodness-of-fit and risk measures so that investors can be provided with valuable insights for decision-making. According to [24], most investors are often driven by loss aversion (i.e. prioritizing the avoidance of losses over the pursuit of gains), thus given the importance of the J580 index, investors may have a very keen interest in fully understanding the loss and gain returns characteristics, underlying statistical distribution's properties and tail properties in order to understand potential behavior of losses/gains.

The portion of the J580 dataset we are interested in is for the period June 1995 to January 2018 (this is the period used by [6]), and it consists of 272 observations. The J580 index is one of the three main sub-indices of the South African All Share Index (ALSI) with one of the highest market capitalizations in the Johannesburg Stock Exchange (JSE) and it consist of the banking, insurance and securities industries. The total economic growth relies heavily on the financial sector as it has a significant majority of the tertiary/college educated work force in South Africa and sponsors financially many community-based projects as well as sporting activities, see [7]. Note that the daily J580 index is also available publicly online on Yahoo Finance. The main objective is to extend the number of fitted distributions to the J580 returns from 4 distributions to 19 distributions to better capture the intricate features of the losses and gains separately. This information will provide historical information about the index on how it tends to perform so that investors can make rational decisions. This research work intends to provide answers to the following research questions (RQ) about the J580 returns:

– **RQ1**: By considering the additional 15 distribution, does the Burr distribution still fit the loss returns best?
– **RQ2**: By considering the additional 15 distribution, does the Weibull distribution still fit the gains returns best?
–**RQ3**: What are the real-life implications of the identified best distribution with respect to riskiness of investing in the index?

The rest of the paper is structured as follows: In Section 2, the theoretical background and methodology properties of the study are discussed. The extended empirical analysis is conducted in Section 3, with the additional discussion in Section 4. Finally, the corresponding concluding remarks are done in Section 4.

2 Statistical distributions and methodology

2.1 Statistical distributions

The 19 statistical distributions to be considered in this study are summarized in Table 2 (sorted alphabetically using the first column). A theoretical description and properties of these 19 statistical distributions are provided in the Appendix of [4]. Extending from the summary table compiled by [3], the last column was added to indicate which statistical distributions are a special case of each type of family distributions. Understanding the properties of statistical distributions in Table 2 is crucial for analyzing investment indices as well as claims data in actuarial science since investors and risk analysts can use these properties to better fit the statistical distributions to refine risk assessments in investment portfolios [2], particularly when tail characteristics play a crucial role in anticipating and managing extreme outcomes. The careful selection of distributions and consideration of their specific parameter values are key in aligning models with empirical patterns observed in investment indices.

2.2 Goodness-of-fit and risk measures

Three goodness-of-fit statistical measures are considered in this research work, those are the Kolmogorov-Smirnov (KS), Cramer-von-Mises (CvM), and Anderson-Darling (AD) tests which are defined as follows by [2]:

$$KS : \max_x |F_n(x) - F(x)|, \tag{1}$$

$$CvM : n\int (F_n(x) - F(x))^2 f(x)dx, \tag{2}$$

$$AD : n\int \frac{(F_n(x) - F(x))^2}{F(x)(1-F(x))} f(x)dx, \tag{3}$$

where n is the number of observations, $F_n(x)$ is the empirical cdf, $F(x)$ is the theoretical (fitted) cdf and $f(x)$ is the corresponding pdf, for all returns data point x. Goodness-of-fit statistics are used to compare models which are not nested, and which do not have a direct relationship. The KS statistic computes the maximum absolute vertical differences between the empirical cdf and the theoretical cdf, see [2] and [5]. The KS statistic captures the differences between the middle of the data and the proposed model. The CvM statistic considers the integral of the squared differences between the empirical cdf and the theoretical cdf rather than just considering differences between points. The AD statistic emphasizes where $F(x) or 1 - F(x)$ are small. The values for which $F(x) or 1 - F(x)$ are small are the tail areas of the distribution. Therefore, the AD statistic places emphasis on the tails of the distribution.

Three information criteria are considered in this research work, those are the negative log-likelihood (NLL), Akaike information criterion (AIC) and the Bayesian information criterion (BIC) [4]. For all three criteria, a lower value implies that the theoretical model is simpler (has less parameters) or provides a better fit to the data. The NLL is appropriate only when comparing models with the same number of parameters; however, the AIC and BIC are more appropriate for comparing models with different number of parameters. Let $l(\theta)$ denote the maximized log-likelihood function of a model, then the NLL is defined as

$$NLL = -l(\theta) \tag{4}$$

The AIC is defined as

$$AIC = 2NLL + 2p \tag{5}$$

where p is the number of free parameters or degrees of freedom. The BIC is defined as

$$BIC = 2NLL + p\log n \tag{6}$$

where n is the number of observations or sample size.

Value-at-Risk (VaR)is the capital desirable to ascertain with a high level of confidence p, that a business is not rendered technically insolvent [4]. The VaR is also interpreted as the lower bound of the capital required for the company to remain

Table 2: List of 19 statistical distributions and their corresponding properties.

Distribution	**Parameters**	**PDF**	**CDF**	$\mathbb{E}[X^k]$	**Special Cases**
Burr	$\alpha>0,\gamma>0,$ $\theta>0$	$\frac{\alpha\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^{\alpha+1}}$	$1-u^{\alpha},\quad u=\frac{1}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$	$\frac{\theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\alpha)}$, for $-\gamma<k<\alpha\gamma$	Transformed beta when $\tau=1$
Exponential	$\theta>0$	$\frac{e^{-\frac{x}{\theta}}}{\theta}$	$1-e^{-\frac{x}{\theta}}$	$\begin{cases}\theta^k\Gamma(k+1), \text{if } k>-1\\ \theta^k k!, \text{if } k \text{ is a } \mathbb{Z}^+\end{cases}$	Transformed gamma when $\alpha=\tau=1$
Gamma	$\alpha>0,\theta>0$	$\frac{\left(\frac{x}{\theta}\right)^{\alpha}e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)}$	$\Gamma\left(\alpha;\frac{x}{\theta}\right)$	$\begin{cases}\frac{\theta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}, \text{ if } k>-\alpha\\ \theta^k(\alpha+k-1)\cdots\alpha, \text{if } k \text{ is a } \mathbb{Z}^+\end{cases}$	Transformed gamma when $\tau=1$
Generalized Pareto	$\alpha>0,\tau>0,$ $\theta>0$	$\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\frac{\theta^{\alpha}x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}$	$\beta(\tau,\alpha;u),\quad u=\frac{x}{x+\theta}$	$\begin{cases}\frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, & -\tau<k<\alpha\\ \frac{\theta^k\tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)}, \text{if } k \text{ is a } \mathbb{Z}^+\end{cases}$	Transformed beta when $\gamma=1$
Inverse Burr	$\tau>0,\gamma>0,$ $\theta>0$	$\frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^{\tau+1}}$	$u^{\tau},\quad u=\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$	$\frac{\theta^k\Gamma(\tau+\frac{k}{\gamma})\Gamma(1-\frac{k}{\gamma})}{\Gamma(\tau)}$, if $-\tau\gamma<k<\gamma$	Transformed beta when $\alpha=1$
Inverse Exponential	$\theta>0$	$\frac{\theta e^{-\frac{\theta}{x}}}{x^2}$	$e^{-\frac{\theta}{x}}$	$\theta^k\Gamma(1-\mathrm{k}), \mathrm{k}<1$	Inverse transformed gamma when $\alpha=\tau=1$
Inverse Gamma	$\alpha>0,\theta>0$	$\frac{\left(\frac{\theta}{x}\right)^{\alpha}e^{-\frac{\theta}{x}}}{x\Gamma(\alpha)}$	$1-\Gamma\left(\alpha;\frac{\theta}{x}\right)$	$\begin{cases}\frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}, \text{ if } k<\alpha\\ \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, \text{if } k \text{ is a } \mathbb{Z}^+\end{cases}$	Inverse transformed gamma when $\tau=1$
Inverse Gaussian	$\mu>0,\theta>0$	$\left(\frac{\theta}{2\pi x^3}\right)^{\frac{1}{2}}e^{-\frac{\theta z^2}{2x}}$, $z=\frac{x-\mu}{\mu}$	$\Phi\left[z\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right]+e^{\frac{2\theta}{\mu}}\Phi\left[-y\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right]$ $z=\frac{x-\mu}{\mu}$	$\sum_{n=0}^{k-1}\frac{(k+n-1)!}{(k-n-1)!\,n!}\frac{\mu^{n+k}}{(2\theta)^n}, k=1,2,\ldots,$	
Inverse Paralogistic	$\tau>0,\theta>0$	$\frac{\tau^2\left(\frac{x}{\theta}\right)^{\tau^2}}{x\left[1+\left(\frac{x}{\theta}\right)^{\tau}\right]^{\tau+1}}$	$u^{\tau},\quad u=\frac{\left(\frac{x}{\theta}\right)^{\tau}}{1+\left(\frac{x}{\theta}\right)^{\tau}}$	$\frac{\theta^k\Gamma\left(\tau+\frac{k}{\tau}\right)\Gamma\left(1-\frac{k}{\tau}\right)}{\Gamma(\tau)}$, if $-\tau^2<k<\tau$	Transformed beta when $\tau=\gamma,\alpha=1$
Inverse Pareto	$\tau>0,\theta>0$	$\frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$	$\left(\frac{x}{x+\theta}\right)^{\tau}$	$\begin{cases}\frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \text{if } -\tau<k<1\\ \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \text{if } k \text{ is a } \mathbb{Z}^-\end{cases}$	Transformed beta when $\gamma=\alpha=1$
Inverse Transformed Gamma	$\alpha>0,\theta>0,$ $\tau>0$	$\frac{\tau u^{\alpha}e^{-u}}{x\Gamma(\alpha)}, u=\left(\frac{\theta}{x}\right)^{\tau}$	$1-\Gamma(\alpha;u)$	$\frac{\theta^k\Gamma(\alpha-\frac{k}{\tau})}{\Gamma(\alpha)}$, if $k<\alpha\tau$	
Inverse Weibull	$\tau>0,\theta>0$	$\frac{\tau\left(\frac{\theta}{x}\right)^{\tau}e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x}$	$e^{-\left(\frac{\theta}{x}\right)^{\tau}}$	$\theta^k\Gamma\left(1-\frac{k}{\tau}\right)$, if $k<\tau$	Inverse transformed gamma when $\alpha=1$
Loglogistic	$\gamma>0,\theta>0$	$\frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^2}$	$\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$	$\theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$, if $-\gamma<k<\gamma$	Transformed beta when $\alpha=\tau=1$
Lognormal	$\mu>0,\sigma>0$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{z^2}{2}}=\frac{\phi(z)}{\sigma x}$ $z=\frac{\ln x-\mu}{\sigma}$	$\Phi(z)$	$e^{k\mu+\frac{1}{2}k^2\sigma^2}$	
Paralogistic	$\alpha>0,\theta>0$	$\frac{\alpha^2\left(\frac{x}{\theta}\right)^{\alpha}}{x[1+\left(\frac{x}{\theta}\right)^{\alpha}]^{\alpha+1}}$	$1-u^{\alpha},\quad u=\frac{1}{1+\left(\frac{x}{\theta}\right)^{\alpha}}$	$\frac{\theta^k\Gamma(1+\frac{k}{\alpha})\Gamma(\alpha-\frac{k}{\alpha})}{\Gamma(\alpha)}, -\alpha<k<\alpha^2$	Transformed beta when $\alpha=\gamma,\tau=1$
Pareto	$\alpha>0,\theta>0$	$\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$	$1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$	$\begin{cases}\frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \text{if } 1<k<\alpha\\ \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \text{if } k \text{ is a positive } \mathbb{Z}\end{cases}$	Transformed beta when $\gamma=\tau=1$
Transformed Beta	$\alpha>0,\theta>0,$ $\gamma>0,\tau>0$	$\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\alpha+\tau}}$	$\beta(\tau,\alpha;u), u=\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$	$\frac{\theta^k\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\alpha)\Gamma(\tau)}, -\tau\gamma<k<\alpha\gamma$	
Transformed Gamma	$\alpha>0,\tau>0,$ $\theta>0$	$\frac{\tau u^{\alpha}e^{-u}}{x\Gamma(\alpha)},\quad u=\left(\frac{x}{\theta}\right)^{\tau}$	$\Gamma(\alpha;u)$	-	
Weibull	$\tau>0,\theta>0$	$\frac{\tau\left(\frac{x}{\theta}\right)^{\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}$	$1-e^{-\left(\frac{x}{\theta}\right)^{\tau}}$	$\theta^k\Gamma\left(1+\frac{k}{\tau}\right)$, for $k>-\tau$	Transformed gamma when $\alpha=1$

insolvent. If the insurer has that amount on hand, then it can absorb $100p\%$ of potential outcomes. Only $100(1-p)\%$ of the possible outcomes may potentially result in the insurer declaring bankruptcy [4]. Let X denote the loss random variable, then the $VaR$of Xat a $100p\%$ security level, denoted π_p or $VaR_p(X)$, is the $100p^{th}$ percentile of the distribution of X and

it is known as a quantile risk measure given by

$$VaR_p(X) = \underset{X\geq 0}{inf}[x|F_x(x) \leq p], 0 < p < 1. \tag{7}$$

The empirical estimate for VaR of $X$is given by

$$VaR_p(X) = \hat{F}^{-1}(p), \tag{8}$$

where the empirical cdf is denoted by $F(.)$. Note that VaR is a lower limit for large losses, that is, VaR simply provides the lower bound for large losses and does not provide any information of the tail beyond the cutoff (threshold) [2].

The tail VaR (TVaR) is also known as expected shortfall (ES), average VaR, conditional-tail-expectation, tail conditional expectation, and conditional value at risk. The use of the term ES is popular in Europe, while CTE and TCE are more popular in North America, see [4] for more details. TVaR is defined as the expected loss if the loss exceeds VaR or the average of all VaR values exceeding security level, p, where$0 < p < 1$. The TVaR of X at the $100p\%$ security level, denoted $TVaR_p(X)$, is defined as

$$TVaR_p(X) = \frac{1}{1-p}\int_p^1 VaR_u(X)du. \tag{9}$$

If X is continuous at $VaR_p(X)$, then $F_X[VaR_p(X)] = p$ and this yields,

$$TVaR_p(X) = E[X|X > VaR_p(X)] = E[X|X > \pi_p] = \frac{\int_{\pi_p}^{\infty} xf(x)dx}{1-F_X(\pi_p)} \tag{10}$$

Stated differently, it can be shown that,

$$TVaR_p(X) = E[X|X > VaR_p(X)] = \pi_p + e(\pi_p) \tag{11}$$

where $e(\pi_p)$ denotes the mean excess loss function evaluated at the $100p^{th}$ percentile. In other words, the TVaR can be interpreted as the corresponding VaR plus the average excess of all losses that exceed VaR [4]. The general advantages of TVaR is that, unlike VaR, it describes the tail behavior beyond the cutoff. The empirical estimate for the TVaR is given by,

$$TVaR_p(X) = 1/(1-p)\int_p^1 \hat{F}^{-1}(s)ds \tag{12}$$

where the empirical cdf is denoted by $F(.)$.

3 Analysis and Discussion

3.1 Descriptives

It is important that we analyse the descriptive statistics as shown in Table 3 for losses and gains separately. These descriptive statistics are different from those that are incorrect provided in [6]. These statistics indicate whether gains or losses have been more prevalent during the index's lifetime, specifically examining the outliers and extreme values identified by the leptokurtic and skewness measures. Furthermore, the skewness of losses is at most six times large than the skewness of gains, and the kurtosis of losses and gains is 51.2 and 6.4, respectively. Additionally, from the descriptive values the maximum of losses and gains is 0.51195 and 0.21652, respectively. These findings indicate that losses tend to exhibit more extreme values and outliers, suggesting that the index is more likely to experience significant losses (rather significant gains) on a month-to-month basis. From Table 3, both the losses and gains are asymmetric with the losses being much more leptokurtic than the gains. So given the leptokurtic nature and skewness of the J580 returns, working with a large class of distributions to fit the losses and gains allows for a better and broader evaluation of the data through several distributions from different families (outlined in Table 2) with different number of parameters (more flexibility on the tails). Such wide options of distributions could not be obtained using only 4 distributions (gamma, Weibull, exponential and Burr) that were considered in [6] or [7], where Burr is the only distribution with more than two parameters.

Table 3: Descriptives of the J580 returns

Descriptive	Losses	Gains
Mean	0.04123	0.04223
Median	0.02867	0.03188
Maximum	0.51195	0.21652
Minimum	0.00104	0.00028
Variance	0.00299	0.00128
Std deviation	0.05468	0.03577
Skewness	6.05210	1.50399
Kurtosis	51.28085	6.44453
Count	110	161

3.2 Parameter Estimates

When analyzing the J580 dataset, obtaining accurate parameter estimates from various statistical distributions, as indicated in Table 3 is required for evaluating model fit and assessing risks associated with losses and gains, respectively. The standard errors are provided in the parenthesis, and the parameters were estimated using maximum likelihood estimation technique. Certain values are denoted as NA (not any), these occur when the variance-covariance matrix is divergent and thus, it is not possible to obtain an exact value of the standard error. These parameters such as shape (α), scale (θ), define the characteristics of each distribution and how well it captures behaviour of the data. For example, the Burr distribution's parameters for losses (as seen in Table 4): $\alpha = 2.2679$, $\gamma = 1.4487$, $\theta = 0.0573$ and for gains: $\alpha = 9145.3783$, $\gamma = 1.1384$, $\theta = 133.2060$. For the Burr distribution, it is observed that that the parameters for losses and gains are different, which affects how the model fits each data subset and how different behaviours (i.e., location and shape) and spread are exhibited or different in each fitted Burr distribution. In essence, each distribution reflects different aspects of the data, the Burr distribution for example, with its flexibility in modelling various shapes, can capture heavy tails or skewness that might be present in losses but not in gains. By accurately estimating these parameters, we ensure that the chosen distribution reflects the actual data patterns, leading to more reliable risk assessments and predictions.

3.3 Goodness-of-fit

In our extended analysis of the J580 dataset in Tables 5 and 6, the goodness of fit of 19 different distributions are analyzed, that is, there are 15 additional distributions added compared to the 4 distributions fitted in [6] and [7]. The goodness-of-fit analysis for the losses in the J580 dataset uses various metrics, including the KS, AD, CvM, NLL, AIC and BIC are shown in Table 5. Our evaluation of each distribution reveals that for each of the goodness-of-fit statistic under the losses, the best fit under KS is Burr, the best under AD, CvM and NLL is transformed beta, and the best fit for AIC and BIC is paralogistic (see the yellow highlighted values). Next, the extended evaluation for gains returns using the 19 fitted distributions in Table 6 reveals that overall, the best fit under KS, AD and CvM is transformed beta, the best performer under NLL is inverse exponential, the best fit for AIC is Weibull and under BIC is the exponential (see the yellow highlighted values). If we had to choose based on this information alone, the most rational option would be the transformed beta distribution under both losses and gains.

To evaluate the performance of the distributions under each goodness-of-fit statistic, we ranked them from 1st (best) to 19th (worst), these values are given in square brackets in Tables 5 and 6. Thereafter, on the last column we compute the average rank (Avg. Rank) of all six goodness-of-fit statistics for each distribution. Based on the average rank, the top 5 best fitting distribution across all six goodness-of-fit statistics for the losses are Burr, inverse Burr, transformed beta, paralogistic and generalized Pareto, respectively. However, for the gains, the top 5 best distributions are transformed gamma, transformed beta, Weibull, inverse Burr and gamma, respectively.

3.4 Risk Measures

In evaluating the performance of various distributions for risk measures on the J580 dataset for losses and gains on the Tables 7 and 8, we compared each distribution's VaR and ES metrics against the empirical values. The fit of the theoretical model is also assessed by comparing the empirical risk estimates to the theoretical risk estimates. The cdf of the empirical distribution is computed by $F_n(x) = 1/n \# i : x_i \leq x$, where # denotes the number of observations $\leq x$, and

Table 4: Parameter estimates of the 19 distributions fitted on the J580 dataset

Distribution	No. of parameters	Gains	Losses
Burr	3	$\alpha = 9145.3783$ (NA) $\gamma = 1.1384$ (0.0711) $\theta = 133.2060$ (39.8368)	$\alpha = 2.2679$ (1.0469) $\gamma = 1.4487$ (0.1713) $\theta = 0.0573$ (0.0263)
Exponential	1	$\theta = 23.6812$ (1.8663)	$\theta = 24.2560$ (2.3127)
Gamma	2	$\alpha = 1.1780$ (0.1168) $\theta = 0.0358$ (0.0044)	$\alpha = 1.1951$ (0.1435) $\theta = 0.0345$ (0.0051)
Weibull	2	$\tau = 1.1385$ (0.0714) $\theta = 0.0441$ (0.0032)	$\tau = 1.0318$ (0.0686) $\theta = 0.0418$ (0.0041)
Generalized Pareto	3	$\alpha = 16293.04$ (NA) $\tau = 1.1780$ (0.1173) $\theta = 584.10$ (NA)	$\alpha = 4.5464$ (1.9945) $\tau = 1.7044$ (0.3224) $\theta = 0.0843$ (0.0554)
Inverse Burr	3	$\tau = 0.2130$ (0.0600) $\gamma = 3.9779$ (0.0844) $\theta = 0.0788$ (0.0095)	$\tau = 0.4999$ (0.1614) $\gamma = 0.8703$ (0.4529) $\theta = 0.0455$ (0.0099)
Inverse Exponential	1	$\theta = 0.0084$ (0.0007)	$\theta = 0.0137$ (0.0013)
Inverse Gamma	2	$\alpha = 0.5520$ (0.0500) $\theta = 0.0046$ (0.0006)	$\alpha = 0.9287$ (0.1089) $\theta = 0.0127$ (0.0019)
Inverse Gaussian	2	$\mu = 0.0422$ (0.0067) $\theta = 0.0105$ (0.0012)	$\mu = 0.0412$ (0.0055) $\theta = 0.0206$ (0.0028)
Inverse Paralogistic	2	$\tau = 1.3026$ (0.0609) $\theta = 0.0223$ (0.0023)	$\tau = 1.4714$ (0.0857) $\theta = 0.0186$ (0.0020)
Inverse Pareto	2	$\tau = 1.5741$ (0.2594) $\theta = 0.0162$ (0.0038)	$\tau = 3.2233$ (0.8129) $\theta = 0.0064$ (0.0020)
Inverse Transformed Gamma	3	$\alpha = 4.1988$ (NA) $\theta = 0.9686$ (NA) $\tau = 0.3570$ (NA)	$\alpha = 4.9574$ (NA) $\theta = 0.4219$ (NA) $\tau = 1.1330$ (NA)
Inverse Weibull	2	$\tau = 0.6587$ (0.0340) $\theta = 0.0134$ (0.0017)	$\tau = 0.8703$ (0.0561) $\theta = 0.0151$ (0.0017)
Loglogistic	2	$\gamma = 1.5724$ (0.1047) $\theta = 0.0300$ (0.0026)	$\gamma = 1.7568$ (0.1396) $\theta = 0.0271$ (0.0026)
Paralogistic	2	$\alpha = 1.5081$ (0.0809) $\theta = 0.0439$ (0.0039)	$\alpha = 1.5794$ (0.0987) $\theta = 0.0406$ (0.0041)
Lognormal	2	$\mu = -3.6458$ (0.0946) $\sigma = 1.2001$ (0.0669)	$\mu = -3.6622$ (0.0974) $\sigma = 1.0219$ (0.0689)
Pareto	2	$\alpha = 12152375.7$ (NA) $\alpha = 10.0242$ (7.3021)	$\theta = 513145.3$ (7958.1) $\theta = 0.3688$ (0.2929)
Transformed Beta	4	$\alpha = 3.0419$ (0.5514) $\gamma = 2.3828$ (0.2239) $\alpha = 1.6021$ (0.6077) $\gamma = 1.8086$ (0.0623)	$\theta = 0.1218$ (0.2785) $\tau = 0.3778$ (0.0963) $\theta = 0.0519$ (0.1248) $\tau = 0.7277$ (0.0217)
Transformed Gamma	3	$\alpha = 0.6260$ (NA) $\theta = 1.5362$ (NA) $\tau = 0.0670$ (NA)	$\alpha = 6.4761$ (NA) $\theta = 0.0003$ (NA) $\tau = 0.4007$ (NA)

n is the total number observations in the sample. Here, the empirical distribution is used as a reference 'model'. The percentage deviation of the theoretical VaR and the empirical VaR is computed by:

$$\%deviationVaR = \frac{VaR_{Theoretical} - VaR_{Empirical})}{VaR_{Empirical}} \quad (13)$$

Table 5: Goodness-of-fit risk metrics of the loss returns for the J580 dataset

Distribution	KS	AD	CvM	NLL	AIC	BIC	Avg. Rank
Burr	0.0377 [1]	0.1943 [2]	0.0241 [2]	-247.2 [2]	-488.5 [2]	-480.4 [3]	[2.0]
Exponential	0.0720 [11]	1.0285 [12]	0.1252 [12]	-240.8 [13]	-479.5 [12]	-476.8 [8]	[11.3]
Gamma	0.0616 [8]	0.5299 [7]	0.0697 [8]	-241.8 [11]	-479.6 [11]	-474.2 [12]	[9.5]
Weibull	0.0638 [9]	0.8362 [10]	0.0993 [9]	-240.9 [12]	-477.7 [13]	-472.3 [13]	[11.0]
Generalized Pareto	0.0429 [3]	0.2304 [4]	0.0305 [4]	-246.9 [4]	-487.9 [4]	-479.8 [5]	[4.0]
Inverse Burr	0.0448 [4]	0.2004 [3]	0.0241 [3]	-247.2 [3]	-488.4 [3]	-480.3 [4]	[3.2]
Inverse exponential	0.1650 [19]	4.6714 [19]	0.9059 [19]	-223.9 [19]	-445.7 [18]	-443.0 [18]	[18.7]
Inverse gamma	0.1538 [18]	4.3553 [18]	0.8044 [18]	-224.1 [18]	-444.1 [19]	-438.7 [19]	[18.3]
Inverse Gaussian	0.1420 [17]	3.1646 [16]	0.6392 [17]	-234.5 [16]	-465.0 [16]	-459.6 [16]	[16.3]
Inverse paralogistic	0.0686 [10]	0.8993 [11]	0.1107 [10]	-242.9 [9]	-481.7 [9]	-476.3 [9]	[9.7]
Inverse Pareto	0.1382 [16]	2.7992 [15]	0.3984 [15]	-234.6 [15]	-465.2 [15]	-459.8 [15]	[15.2]
Inverse transformed gamma	0.1082 [14]	1.8750 [14]	0.3306 [14]	-238.3 [14]	-470.6 [14]	-462.5 [14]	[14.0]
Inverse Weibull	0.1288 [15]	3.7274 [17]	0.5929 [16]	-226.4 [17]	-448.9 [17]	-443.5 [17]	[16.5]
Loglogistic	0.0541 [7]	0.5508 [8]	0.0649 [7]	-245.2 [7]	-486.5 [6]	-481.0 [2]	[6.2]
Paralogistic	0.0466 [5]	0.3012 [6]	0.0375 [5]	-246.9 [5]	-489.8 [1]	-484.4 [1]	[3.8]
Lognormal	0.0735 [12]	0.7349 [9]	0.1173 [11]	-244.4 [8]	-484.7 [8]	-479.3 [6]	[9.0]
Pareto	0.0841 [13]	1.4608 [13]	0.2081 [13]	-242.3 [10]	-480.6 [10]	-475.2 [11]	[11.7]
Transformed beta	0.0383 [2]	0.1819 [1]	0.0218 [1]	-247.3 [1]	-486.6 [5]	-475.8 [10]	[3.3]
Transformed gamma	0.0469 [6]	0.2989 [5]	0.0394 [6]	-246.1 [6]	-486.1 [7]	-478.0 [7]	[6.2]

Table 6: Goodness-of-fit risk metrics of the gain returns for the J580 dataset

Distribution	KS	AD	CvM	NLL	AIC	BIC	Avg. Rank
Burr	0.0610 [8]	1.1541 [7]	0.1581 [8]	-341.9 [9]	-677.8 [9]	-668.6 [10]	[8.5]
Exponential	0.0810 [9]	1.3320 [8]	0.2442 [10]	-348.5 [8]	-695.0 [6]	-691.9 [1]	[7.0]
Gamma	0.0498 [5]	0.6122 [6]	0.0860 [6]	-349.8 [5]	-695.6 [4]	-689.5 [3]	[4.8]
Weibull	0.0375 [3]	0.4368 [4]	0.0499 [4]	-350.5 [4]	-697.0 [1]	-690.9 [2]	[3.0]
Generalized Pareto	0.0498 [6]	0.6119 [5]	0.0859 [5]	-349.8 [6]	-693.6 [7]	-684.4 [7]	[6.0]
Inverse Burr	0.0442 [4]	0.2726 [3]	0.0428 [3]	-351.3 [3]	-696.5 [3]	-687.3 [5]	[3.5]
Inverse Exponential	0.3601 [19]	38.0830 [19]	7.7528 [19]	-243.5 [19]	-485.1 [19]	-482.0 [19]	[19.0]
Inverse Gamma	0.2429 [17]	16.6458 [17]	3.2650 [17]	-267.9 [18]	-531.7 [18]	-525.6 [18]	[17.5]
Inverse Gaussian	0.2910 [18]	22.5381 [18]	4.7898 [18]	-285.2 [17]	-566.4 [17]	-560.2 [17]	[17.5]
Inverse Paralogistic	0.1101 [12]	3.1785 [12]	0.3562 [12]	-329.7 [12]	-655.6 [12]	-649.2 [12]	[12.0]
Inverse Pareto	0.1622 [15]	5.7625 [14]	0.8368 [14]	-319.9 [14]	-635.7 [14]	-629.6 [14]	[14.2]
Inverse Transformed Gamma	0.1606 [14]	7.5326 [15]	1.2876 [15]	-310.7 [15]	-615.4 [15]	-606.1 [15]	[14.8]
Inverse Weibull	0.1894 [16]	10.9924 [16]	1.8705 [16]	-288.4 [16]	-572.9 [16]	-566.7 [16]	[16.0]
Loglogistic	0.0829 [11]	2.2516 [11]	0.2077 [9]	-335.3 [11]	-666.7 [11]	-660.5 [11]	[10.7]
Paralogistic	0.0578 [7]	1.6372 [10]	0.1408 [7]	-340.8 [10]	-677.6 [10]	-671.4 [9]	[8.8]
Lognormal	0.1169 [13]	3.9711 [13]	0.6155 [13]	-329.2 [13]	-654.3 [13]	-648.2 [13]	[13.0]
Pareto	0.0810 [10]	1.3320 [9]	0.2443 [11]	-348.5 [7]	-693.0 [8]	-686.9 [6]	[8.5]
Transformed Beta	0.0319 [1]	0.2102 [1]	0.0267 [1]	-351.7 [1]	-695.4 [5]	-683.1 [8]	[2.8]
Transformed Gamma	0.0326 [2]	0.2419 [2]	0.0282 [2]	-351.4 [2]	-696.8 [2]	-687.5 [4]	[2.3]

where theoretical implies anyone of the distributions listed in Table 2. Similarly, the percentage deviation of the theoretical ES and the empirical ES is computed by:

$$\%deviationES = \frac{ES_{Theoretical} - ES_{Empirical})}{ES_{Empirical}} \tag{14}$$

This is important because underestimating the risk measures may result in under-reserving, which may lead to insolvency (i.e., not enough capital to cover future claims). Overestimating the risk measures may result in over-reserving, which may affect the profitability of the insurer due to fewer funds available for investment purpose.

Firstly, the green shaded distribution must not be used as a candidate best distribution because at least one of the confidence level values of VaR or ES is equal to or greater than one. These distributions result in VaR or ES values that are irrationally high value(s); thus, must be ignored as it is both impractical and impossible to keep more than a 100% of cash on reserves. Stated differently, maintaining reserves exceeding 100% of cash is both impractical and impossible;

Table 7: Risk measures for gain returns on the J580 dataset showing VaR and ES

Distribution	VaR0.95	VaR0.99	VaR0.995	ES0.95	ES0.99	ES0.995	Avg. Rank
Empirical	0.108	0.155	0.318	0.190	0.334	0.512	
Exponential	0.124 (14.8%)	0.190 (22.9%)	0.218 (-31.3%)	0.165 (-13.1%)	0.231 (-30.8%)	0.260 (-49.3%)	[4.0]
Gamma	0.116 (7.8%)	0.174 (12.5%)	0.199 (-37.6%)	0.152 (-19.8%)	0.209 (-37.4%)	0.234 (-54.3%)	[1.5]
Weibull	0.121 (12.6%)	0.184 (19.0%)	0.211 (-33.8%)	0.160 (-15.6%)	0.222 (-33.5%)	0.249 (-51.4%)	[2.7]
Burr	0.115 (7.0%)	0.211 (36.8%)	0.268 (-15.7%)	0.181 (-4.4%)	0.316 (-5.5%)	0.396 (-22.6%)	[6.0]
Generalized Pareto	0.118 (9.8%)	0.211 (36.6%)	0.262 (-17.7%)	0.180 (-5.2%)	0.298 (-10.8%)	0.363 (-29.0%)	[5.5]
Inverse Burr	0.114 (5.9%)	0.227 (46.6%)	0.303 (-4.9%)	0.198 (4.3%)	0.387 (15.8%)	0.515 (0.7%)	[8.2]
Inverse Exponential	0.267 (148.5%)	1.365 (783.4%)	2.736 (760.2%)	-	-	-	-
Inverse Gamma	0.324 (201.0%)	1.863 (1106%)	3.938 (1137.8%)	-	-	-	-
Inverse Gaussian	0.147 (36.2%)	0.291 (88.5%)	0.364 (14.3%)	0.238 (25.6%)	0.401 (20.0%)	0.479 (6.5%)	[11.3]
Inverse Paralogistic	0.180 (67.3%)	0.550 (256.2%)	0.884 (177.9%)	0.575 (203.2%)	1.725 (416.3%)	2.765 (440.1%)	-
Inverse Pareto	0.399 (270.7%)	2.049 (1226.1%)	4.111 (1192.5%)	-	-	-	-
ITG	0.235 (118.0%)	0.657 (325.2%)	1.048 (229.5%)	0.585 (208.8%)	1.476 (341.8%)	2.159 (321.8%)	-
Inverse Weibull	0.457 (324.7%)	2.974 (1824.9%)	6.614 (1979.3%)	-	-	-	-
Loglogistic	0.145 (34.8%)	0.371 (140.1%)	0.552 (73.6%)	0.344 (81.3%)	0.865 (158.8%)	1.284 (150.9%)	-
Paralogistic	0.122 (13.1%)	0.248 (60.6%)	0.332 (4.3%)	0.215 (13.2%)	0.422 (26.3%)	0.461 (-10.0%)	[9.8]
Lognormal	0.138 (28.2%)	0.277 (79.1%)	0.347 (9.1%)	0.231 (21.8%)	0.416 (24.4%)	0.520 (1.6%)	[11.2]
Pareto	0.129 (19.4%)	0.215 (39.2%)	0.257 (-19.2%)	0.184 (-3.1%)	0.280 (-16.3%)	0.326 (-36.3%)	[6.8]
Transformed Beta	0.114 (5.5%)	0.212 (37.3%)	0.273 (-14.1%)	0.183 (-3.4%)	0.330 (-1.2%)	0.422 (-17.6%)	[6.5]
Transformed Gamma	0.121 (12.8%)	0.207 (34%)	0.249 (-21.7%)	0.176 (-7.2%)	0.272 (-18.5%)	0.319 (-37.7%)	[4.5]

Table 8: Risk measures for gain returns on the J580 dataset showing VaR and ES

Distribution	VaR0.95	VaR0.99	VaR0.995	ES0.95	ES0.99	ES0.995	Avg. Rank
Empirical	0.102	0.157	0.183	0.139	0.195	0.217	
Exponential	0.127 (24.0%)	0.195 (23.7%)	0.224 (22.4%)	0.169 (21.8%)	0.237 (21.1%)	0.266 (22.8%)	[7.5]
Gamma	0.122 (19.5%)	0.183 (16.4%)	0.209 (14.4%)	0.160 (15.4%)	0.221 (12.8%)	0.247 (13.9%)	[6.3]
Weibull	0.116 (13.4%)	0.169 (7.4%)	0.191 (4.5%)	0.149 (7.3%)	0.200 (2.5%)	0.222 (2.5%)	[3.2]
Burr	0.116 (13.4%)	0.169 (7.4%)	0.191 (4.5%)	0.149 (7.3%)	0.201 (2.6%)	0.222 (2.6%)	[3.7]
Generalized Pareto	0.120 (17.3%)	0.179 (14.1%)	0.205 (12.2%)	0.157 (13.1%)	0.242 (23.6%)	0.242 (11.6%)	[6.2]
Inverse Burr	0.109 (7.3%)	0.169 (7.4%)	0.202 (10.4%)	0.149 (7.7%)	0.226 (15.9%)	0.270 (24.7%)	[4.8]
Inverse Exponential	0.164 (60.8%)	0.836 (432.0%)	1.677 (817.8%)	-	-	-	-
Inverse Gamma	1.303 (1178.9%)	24.11 (15237%)	84.64 (46227%)	-	-	0.263 (21.4%)	-
Inverse Gaussian	0.175 (71.4%)	0.419 (166.8%)	0.551 (201.7%)	0.331 (139.0%)	0.624 (219.2%)	0.771 (256.1%)	[10.8]
Inverse Paralogistic	0.263 (158.0%)	0.930 (497.8%)	1.589 (769.9%)	1.164 (740.7%)	4.027 (1960.7%)	6.860 (3068.5%)	-
Inverse Pareto	0.489 (379.8%)	2.528 (1508.1%)	5.077 (2678.8%)	-	-	-	-
ITG	0.321 (215.4%)	1.265 (704.6%)	1.265 (592.3%)	1.417 (923.2%)	4.752 (2331.8%)	7.870 (3535.0%)	-
Inverse Weibull	1.222 (1099.0%)	14.51 (9130.9%)	41.72 (22736%)	-	-	-	-
Loglogistic	0.196 (91.9%)	0.558 (255.2%)	0.871 (376.5%)	0.550 (297.1%)	1.541 (688.8%)	2.397 (1007.3%)	-
Paralogistic	0.149 (45.9%)	0.322 (105%)	0.442 (142.1%)	0.280 (102.4%)	0.585 (199.5%)	0.796 (267.5%)	[10.2]
Lognormal	0.188 (84.4%)	0.426 (170.8%)	0.574 (214.3%)	0.352 (154.2%)	0.697 (256.9%)	0.906 (318.3%)	[12.0]
Pareto	0.127 (24.1%)	0.195 (23.7%)	0.224 (22.4%)	0.169 (21.8%)	0.237 (21.1%)	0.266 (22.8%)	[7.5]
Transformed Beta	0.110 (8.0%)	0.159 (0.8%)	0.181 (-1.1%)	0.141 (1.7%)	0.193 (-1.4%)	0.217 (0.2%)	[2.0]
Transformed Gamma	0.113 (10.5%)	0.157 (-0.4%)	0.174 (-4.9%)	0.140 (0.8%)	0.181 (-7.6%)	0.197 (-9.0%)	[1.3]

therefore, any VaR or ES greater than 1 must be disregarded. Secondly, to interpret the VaR and ES simultaneously for transformed beta distribution under losses at a 95% level is as follows: the loss returns are not expected to go beyond 11.4% (VaR=0.114) at a 95% confidence level which is 5.5% higher than the empirical distribution. However, if it goes beyond 11.4%, it will average 18.3% (ES=0.183) at a 95% confidence level which is 3.4% lower than the empirical distribution. The other values in Tables 7 and 8 are interpreted in a similar manner. Thirdly, for the computed confidence levels under losses, i.e. 95%, 99% and 99.5%, the distributions that lead to a VaR that is closest to the corresponding empirical values are transformed beta, gamma and paralogistic, respectively (see yellow shaded values in Table 7).

However, for the ES, the best distributions under 95%, 99% and 99.5% confidence levels are Pareto, transformed beta and inverse Burr, respectively. For the gains in Table 8, the distributions that lead to a VaR that is closest to the corresponding empirical values are inverse Burr, transformed gamma and transformed beta under the 95%, 99% and 99.5% confidence levels, respectively. However, for the corresponding ES, the best distributions are transformed gamma (for 95%) and transformed beta (for both 99% and 99.5%). Finally, following a similar ranking method used in Section 3.3, the top 5 distributions with respect to risk measures done in Table 7 for losses are gamma, Weibull, exponential, transformed gamma and generalised Pareto, respectively. However, for the gains in Table 8, the top 5 distributions are transformed gamma, transformed beta, Weibull, Burr and inverse Burr, respectively.

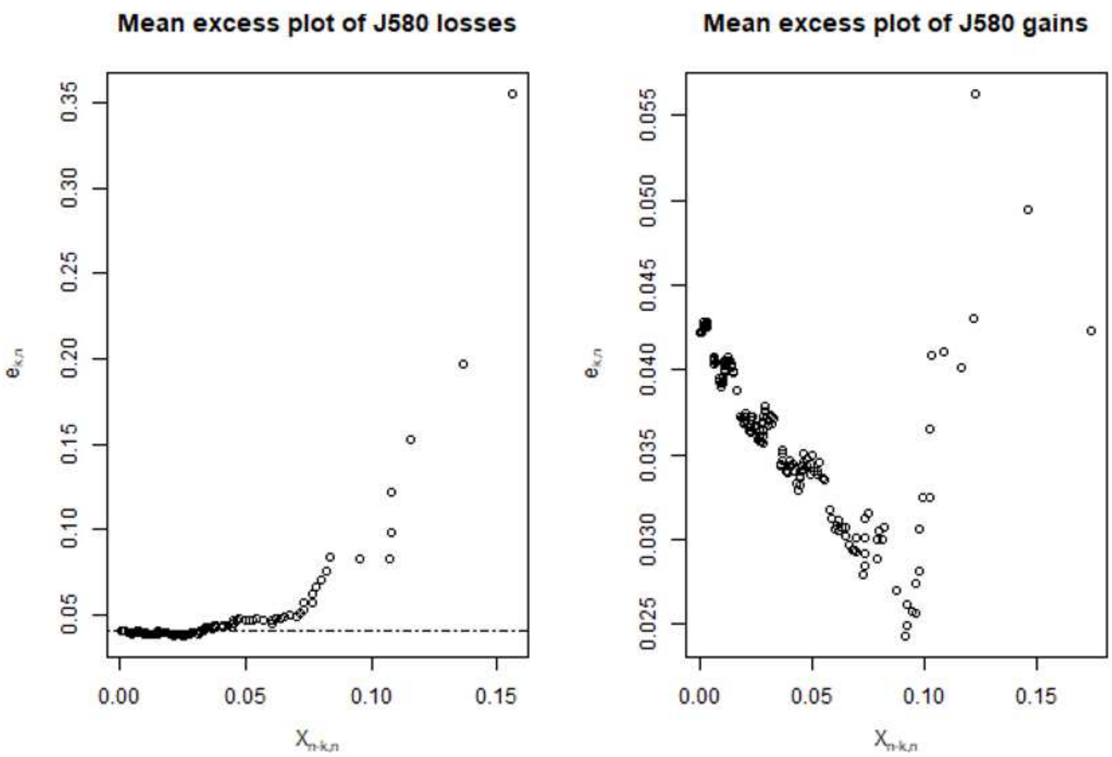

Fig. 1: Mean excess plots of the J580 absolute losses (on the left) and the gains (on the right)

4 Discussion

In checking which distribution is the most appropriate to fit for losses and separately gains, the degree of asymmetry, kurtosis, best fit on the tail, collectively the goodness-of-fit tests and risk measures are used to justify the final choice. The latter shall assist in answering the research questions (RQ1, RQ2 and RQ3) posed at the end of Section 1.

Note that mean excess plots are generally used in model building to identify the tails the data depict (i.e. light- or heavy-tailed); that is, visually examining mean excess plots can provide a possible shortlist of distributions to be tested. Consequently, the mean excess plots in Figure 1 suggests that a combination of two distributions (called composite distribution, see [25,26]) that are connected by a threshold, could be better than fitting a single distribution. For example, in the case of losses, the exponential distribution can be tested as the head component while the lognormal / Pareto can be tested as the tail component. However, the mean excess plot of gains suggest that the head (before the threshold) could be the uniform distribution while the tail (after the threshold) visually could be either the Pareto, lognormal or Weibull (tau ¡ 1) distribution. This trial-and-error procedure could be conducted for a number of distributions, see for instance [25] and [26], where separately 256 composite distributions were fitted on the Danish fire claims and South African taxi claims datasets, respectively. Next, keeping in mind that from Tables 7 and 8 (shaded in green), it is recommended that the following seven distributions must not be candidate best because they result in irrationally high values of VaR and ES: inverse exponential, inverse gamma, inverse paralogistic, inverse Pareto, inverse transformed gamma, inverse Weibull and loglogistic. Thus, excluding these latter distributions, it follows that:

- For loss returns, combing the last columns of Tables 5 and 7 to compute the average of ‘average ranks’ leads to the following top 5 distributions (numbers 1 to 5): Burr, generalized Pareto, transformed beta, transformed gamma and gamma. Although the transformed beta seems to have the mode in terms of most goodness-of-fit statistics (see Table 5), however, in terms of risk measures closeness to the empirical values, it was not the best performer (see Table 7). Thus, to strike balance between the goodness-of-fit tests and the risk measures, we recommend that the J580 loss returns be modelled using anyone of the top 5 distribution with the most recommended one being the Burr distribution. Therefore, to answer RQ1 posed at the end of Section 1, it follows that even when considering the additional 15 distributions, the

Burr (which was identified as the best in [6,7] when four distributions were considered) seems to be very competitive especially in terms of the different goodness-of-fit tests and partially in terms of risk measures.

– For gain returns, similarly, combining the last columns of Tables 6 and 8 to compute the average of 'average ranks' leads to the following top 5 distributions (numbers 1 to 5): transformed gamma, transformed beta, Weibull, inverse Burr and gamma. Although the transformed beta seems to have the mode in terms of most goodness-of-fit statistics (see Table 6), however, in terms of risk measures closeness to the empirical values, it is the second best (after the transformed gamma, see Table 7). Thus, to strike balance between the goodness-of-fit tests and the risk measures, we recommend that the J580 loss returns be modelled either using anyone of the top 5 distribution with the most recommended two being the transformed gamma or transformed beta distribution. Therefore, to answer RQ2 posed at the end of Section 1, it follows that when considering the additional 15 distributions, the transformed gamma and transformed beta seems to be very competitive in terms of both the different goodness-of-fit tests and risk measures than the Weibull distribution; however, the Weibull that was identified in [7] as best to fit when four distributions are considered, is consistently the third best distribution out of the 19. Note that the exponential distribution selected by [6] as the best to fit when four distributions are considered, is mostly the ninth best distribution out of the 19. To answer RQ3 in Section 1, the following points explain what the real-life implications of the chosen distributions are:

– The Burr and transformed gamma distributions, which best fit the loss and gain returns of the J580 index, respectively, reveal significant insights into its risk profile. The Burr distribution's coupled by the significant values for skewness and kurtosis of loss returns (see Table 3), heavy tail indicates a high likelihood of extreme losses, suggesting substantial downside risk. This means that investing in the J580 index exposes investors to sudden market downturns, which can be intensified by economic shocks or liquidity issues. Furthermore, the index's price volatility implies a higher probability of incurring significant losses rather than generating large profits. Heavy-tailed distributions, while useful for capturing extreme losses, can be sensitive to outliers and market anomalies, potentially leading to overestimated risk. Note that the transformed gamma distribution which is best for gains has a skewness and kurtosis that is much lower than that of the losses; hence, it means that it is not too prone to extreme values that losses are subjected to.

– Heavy-tailed distributions, such as the Burr distribution, indicate a higher likelihood of extreme losses, enabling investors to more accurately model downside risk. By utilizing these distributions, investors can calculate VaR to gain insights into potential losses at specified confidence levels, informing decisions on position sizing and risk tolerance. Furthermore, understanding the tail properties of these distributions can guide the development of risk management strategies, including tighter stop-loss thresholds and frequent portfolio rebalancing. This approach can also improve hedge ratios, allowing for better risk coverage. Given the potential for extreme downside risk, investors may want to exercise caution when holding long positions in the J580 index, considering strategic shorting or hedging instead. Additionally, by fitting known distributions to historical data, traders may be able to predict future price movements and identify early warning signs for exiting or reversing positions.

– Heavy-tailed distributions are closely tied to fundamental market risk factors, including macroeconomic volatility, industry demand fluctuations, and sector-specific risks inherent in the diverse industries represented in the J580 index. Our analysis reveals that these distributions effectively capture extreme loss events, highlighting the need for investors, particularly those holding long positions, to exercise heightened caution. The tail properties observed suggest a market environment prone to systematic risks and large-scale losses, underscoring the importance of strategies that mitigate extreme downside risk for portfolio managers. Due to the losses having a higher probability of being extreme in the J580 returns, a short position is advised in the long-run rather than a long position. However, it might not be a good strategy for the short-run as there is some volatility. To mitigate the risk of extreme losses in the short-run, one can use candlestick patterns which depict different patterns that can help an investor track the performance of the index using the 4 important index values over a specific time period: minimum, maximum, closing, and opening; see for instance [27]. Note that the time period to keep track of the 4 index values could be 1 minute, 1 hour, daily, weekly, monthly, etc. There are in total 35 patterns which can be categorised into 3 main trends [28], namely continuation (no overall change or reversal, either increase, decrease or constant will persist), bearish reversal (the ongoing uptrend is going to reverse to a downtrend) and bullish reversal (the ongoing downtrend is going to reverse to an uptrend). These patterns help investors make informed investment decisions to manage the risks they are exposed to in real time. Another way of reducing the risk of extreme losses would be to diversify the portfolio across different markets that have little to no correlation so that the portfolio is not exposed to the same kinds of risks. Hedging strategies like buying put options, selling call options, and selling futures can also be used to offset potential extreme losses in the short-term.

5 Conclusion

While [6,7] fitted 4 standard distributions on the J580 returns, [7] found that there are mistakes in [6] and corrected them. The mistakes inspired the extended analysis conducted in this study, which is a model fitting analysis implemented to a larger class of 19 standard distributions that were separately fitted to loss and gain returns. Some of these distributions, especially the flexible 3 or 4 parameters (not the less flexible 1 or 2 parameters) were observed to be able to capture the tail / extreme occurrences of the index over the long-term, to provide valuable insight investors could use to make rational decisions. We considered the large number of extreme occurrences of losses and concluded to short the stock over the long-term would be the most profitable course of action. The large class of distributions provided more insight in the flexible nature of the empirical and financial data since we noticed that 3- to 4-parameter distributions tend to model the data closely, given their flexible nature. Simpler models often fail to capture the complexity of heavy-tailed losses. For loss returns, the three parameter Burr distribution seems to be the best at capturing expected losses, so that the reserve needed can be efficiently estimated. On the other hand, for gains return, the transformed gamma distribution with three parameters outperformed others with four parameters, making it the preferred choice due to its simplicity. Taking into account the latter deductions as well as the skewness and kurtosis of the J580 returns, the recommended strategy, given the high probability of extreme losses, would be to take a short position in the J580 index. For future studies, using mixed or composite models [25,26], which combine multiple distributions to provide a more thorough and better understanding of the J580 index. These models can better capture the stochastic periodicity of both usual and extreme markets, leading to enhanced risk management strategies. The mean excess plot suggests that losses and separately gains exhibit characteristics of multiple distributions, supporting the use of composite and mixture models.

Data Availability Statement:

All the data are publicly available on Yahoo Finance and can be requested from the authors. In addition, the R software code used for analysis here can be requested from the authors.

Conflicts of Interest Statement:

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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