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A Novel Extension of the Inverse Rayleigh Distribution: Theory, Simulation, and Real-World Application

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Abstract: Generalized distributions benefit applied statisticians, and there are several approaches to expanding common distributions. This article presents a new sub-model, the Odd Burr XII-inverse Rayleigh (OBXII-IR) distribution, as well as a new family of generalized distributions, the Odd Burr-G family. To generate the OBXII-G family, we combine the T-X family with the Burr XII distribution. Examine a few OBXII-IR distribution statistics. These statistics include moments, incomplete moments, quantile function, order statistics, harmonic mean, and Rényi entropy. The methods used for parameter estimation include maximum likelihood, least squares, weighted least squares, Anderson Darling, and right-tailed Anderson Darling. This research evaluates the performance of these different estimators using a Monte Carlo simulation. The distribution's applicability to two real data sets proves that the OBXII-IR distribution is really useful and practical.

Keywords: T-X family, Inverse Rayleigh, Moments, Simulation, Estimation methods

1 Introduction

An important part of statistics is using probability distributions to create models of real-world occurrences. Probability distributions are used to represent the risk and uncertain occurrences seen in the natural world. Given the wide variety of natural occurrences, it becomes necessary to develop a multitude of probability distributions. However, for certain natural occurrences, the well-established probability distributions fail to characterize the data adequately. This has led to the advancement of generalized probability distributions, which have undergone expansion and modification in response to the widespread use of additional factors. By including certain parameters, we were able to more accurately define the distribution tail shape and make the known probability distributions more consistent with data from natural events.

Over the last several years, many expansions of the inverse Rayleigh (IR) distribution have been created using diverse mathematical methods, often derived from overarching distribution families. The following distribution has been identified: the beta IR distribution [1], the transmuted IR distribution [2], the weighted IR distribution [3], the odd Fréchet IR distribution [4], the type II Topp-Leone IR distribution [5], the exponentiated IR distribution [6], the half-logistic IR distribution [7], the alpha-power exponentiated IR distribution [8], extended odd Weibull IR distribution [9], and a new IR distribution [10].

Definition 1.1. Consider X a random variable with an Inverse Rayleigh (IR) distribution and a scaling parameter $\delta > 0$. The cumulative distribution function (cdf) and probability density function (pdf) are as follows:

$$G(x)_{IR} = e^{-\frac{\delta}{x^2}}, \tag{1}$$

and,

$$g(x)_{IR} = \frac{2\delta}{x^3} e^{-\frac{\delta}{x^2}}, x, \delta > 0. \tag{2}$$

Definition 1.2. [11] Let r(t) be the pdf of a random variable, say T, where $T \in [p,q]$ $for -\infty < p < q < \infty$, and let $W[G(x;\xi)]$ be a function of cdf of a random variable X, which satisfies the following conditions:

i.$W(G(x;\xi)) \in [p,q], \ -\infty < p < q < \infty.$

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ii. $W(G(x;\xi))$ is differentiable and monotonically non-decreasing.
iii. $W(G(x;\xi)) \to p, as\ x \to -\infty,\ and\ W(G(x;\xi)) \to q, as\ x \to \infty,\ i.e\ W(G(x;\xi)) \to 0, as\ x \to 0,\ and\ W(G(x;\xi)) \to 1, as\ x \to \infty.$

The T-X distribution family cdf is

$$K(X) = \int_{p}^{W(G(x;\xi))} r(t)dt, \quad (3)$$

where $W(G(x;\xi))$ satisfies the conditions i, ii, and iii. The pdf of the T-X distribution family is

$$k(x) = \left[\frac{\partial}{\partial x} W(G(x;\xi))\right] r[W(G(x;\xi))], \quad x \in R.$$

Using the T-X family principle, various new distribution classes have been developed in the literature. Table 1 lists $W(G(x;\xi))$ function for T-X family members.

Now, we introduce the proposed family. Let T$\sim$ Burr XII distribution, then its cdf is:

$$R(t) = 1 - \left[1 + t^{\lambda}\right]^{-\theta}. \quad (4)$$

And, the pdf of Burr XII distribution is

$$r(t) = \theta\lambda t^{\lambda-1}\left[1+t^{\lambda}\right]^{-(\theta+1)};\ t \geq 0, \theta, \lambda > 0. \quad (5)$$

When $r(t)$ follows (5), and setting $W[G(x;\xi)] = \frac{[G(x;\xi)]^2}{1-G(x;\xi)}$ in (3), we define the cdf of the Odd Burr XII-G (OBXII-G) family by:

$$K(x;\xi)_{OBXII-G} = 1 - \left[1 + \left[\frac{[G(x;\xi)]^2}{1-G(x;\xi)}\right]^{\lambda}\right]^{-\theta}. \quad (6)$$

The PDF of the OBXII-G family is

$$k(x;\xi)_{OBXII-G} = \theta\lambda G(x;\xi)\, g(x;\xi)\,(2-G(x;\xi))\,(1-G(x;\xi))^{-2}\left[\frac{[G(x;\xi)]^2}{1-G(x;\xi)}\right]^{\lambda-1}\left[1+\left[\frac{[G(x;\xi)]^2}{1-G(x;\xi)}\right]^{\lambda}\right]^{-(\theta+1)}. \quad (7)$$

Where ξ represents the parameters of the underlying distribution, $x > 0, \theta, \lambda > 0$ and $G(x;\xi)$ denotes the baseline cumulative distribution function.

This article aims to provide a more flexible and better Odd Burr XII-G family and sub-model Odd Burr XII IR (OBXII-IR) distribution. The OBXII-IR distribution is used to make the IR distribution more flexible through OBXII-G. It is used to present a modified IR distribution in the form of a liner combination.

Section 2 defines the cdf, pdf, and hazard function of the OBXII-IR model, which are using to frame the results presented in this paper. Section 4-9 studies several statistical aspects. Section 10 describes five different approaches to estimating the model's parameters. Section 11 provides and examines the simulation results. Section 12 describes a practical investigation using two real datasets. Finally, Section 13 discusses conclusions.

2 Sub-Model description

Here we present the Odd Burr XII Inverse Rayleigh distribution, a sub-model of the proposed family.

Definition 2.1. A continuous random variable X has the Odd Burr XII Inverse Rayleigh (OBXII-IR) distribution. Substitute the cdf and pdf of the Inverse Rayleigh distribution into Eq. (6) and (7). Then the cdf and pdf are as follows:

$$K(x)_{OBXII-IR} = 1 - \left[1 + \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}, \quad (8)$$

$$k(x;\xi)_{OBXII-IR} = \frac{2\theta\lambda\delta}{x^3} e^{-\frac{2\delta}{x^2}}\left(2-e^{-\frac{\delta}{x^2}}\right)\left(1-e^{-\frac{\delta}{x^2}}\right)^{-2}\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda-1}\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-(\theta+1)} \quad (9)$$

Table 1: Includes T-X family members

$W(G(x;\xi))$	Range of x	Members of the T-X family
$G(x;\xi)$	[0,1]	Beta-G [12]
$G(x;\xi)$	[0,1]	Kumaraswamy-G [13]
$G(x;\xi)$	[0,1]	Generalized Marshal-Olkin-G [14]
$G(x;\xi)$	[0,1]	exponentiated Marshal-Olkin-G [15]
$G(x;\xi)$	[0,1]	Proportional reversed hazard rate-G [16]
$-log[1-G(x;\xi)]$	(0, ∞)	Gamma-G Type-2 [17]
$-log[G(x;\xi)]$	(0, ∞)	Gamma-G Type-1 [18]
$\frac{G(x;\xi)}{1-G(x;\xi)}$	(0, ∞)	Gamma-G Type-3 [19]
$log\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]$	(-∞, ∞)	Logistic-G [20]
$\frac{[G(x;\xi)]^2}{1-G(x;\xi)}$	[0, ∞)	Generalized Rayleigh -G [21]

where $x > 0, \theta, \lambda,\ \delta > 0$.

and, the hazard function for the OBXII-IR is

$$h(x) = \frac{2\delta\theta\lambda}{x^3}\, e^{-\frac{2\delta}{x^2}} \left(2 - e^{-\frac{\delta}{x^2}}\right)\left(1 - e^{-\frac{\delta}{x^2}}\right)^{-2} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda-1}\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-1}. \quad (10)$$

Figure 1 shows the OBXII-IR distribution pdf, $h(x)$, cdf, and $s(x)$ graphs. Figure 1 displays that the OBXII-IR distribuiton can be unimodel, right-skewed, and decreasing-shaped. The $h(x)$ may have aincresing, decreasing and inverted J-shaped increase.

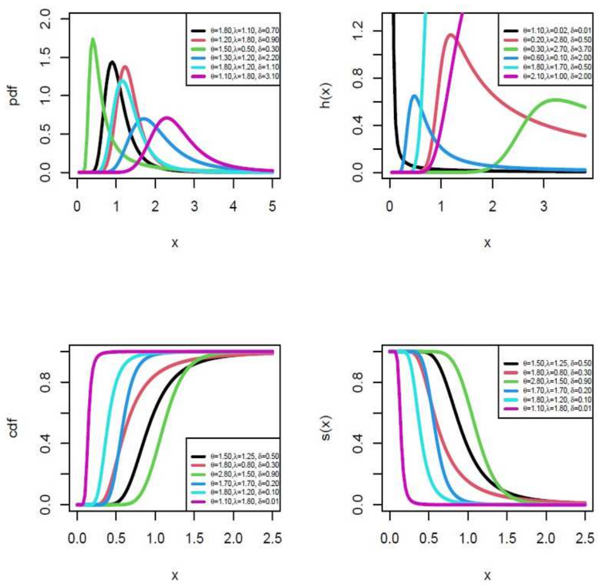

Fig. 1: pdf, h(x), cdf, and s(x) plots for the OBXII-IR distribution

3 Linear representation

Theorem 3.1. The pdf of the OBXII-IR distribution may be represented linearly as follows

$$k(x)_{OBXII-IR} = \sum_{j=0}^{\infty} \psi_{m,v,r,j} \frac{4\delta}{x^3} e^{-\frac{\delta(j+1)}{x^2}} - \sum_{j=0}^{\infty} \psi_{m,v,r,j} \frac{2\delta}{x^3} e^{-\frac{\delta(j+2)}{x^2}}, \quad (11)$$

where

$$\psi_{m,v,r,j} = \sum_{m=v=r=0}^{\infty} \frac{(-1)^{m+r+j}\Gamma(\lambda m+\lambda+1+v)\theta\lambda}{v!\Gamma(\lambda m+\lambda+1)}\binom{\theta+1}{m}\binom{v+2\lambda m+2\lambda-1}{r}\binom{r}{j}.$$

Proof. Utilizing the subsequent generic series expansions as given in [22,23,24], we have

$$[1-u]^s = \sum_{r=0}^{\infty}(-1)^r\binom{s}{r}u^r,\ [1-u]^{-s} = \sum_{m=0}^{\infty}\frac{\Gamma(s+j)}{m!\Gamma(p)}u^m\ :\ |u|<1,\ s>0,$$

and exponential expansion

$$e^{-a} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}a^j.$$

Equation (7) reflects our family density, which binomial series expansion may extend. Then

$$\begin{aligned}
k(x;\xi)_{OBXII-G} &= \theta\lambda\ G(x;\xi)\, g(x;\xi)\,(2-G(x;\xi))\,(1-G(x;\xi))^{-2}\left[\frac{[G(x;\xi)]^2}{1-G(x;\xi)}\right]^{\lambda-1}\left[1+\left[\frac{[G(x;\xi)]^2}{1-G(x;\xi)}\right]^{\lambda}\right]^{-(\theta+1)} \\
&= \sum_{m=0}^{\infty}(-1)^m\binom{\theta+1}{m}\theta\lambda g(x;\xi)\,(2-G(x;\xi))\,[G(x;\xi)]^{2\lambda m+2\lambda-1}[1-G(x;\xi)]^{-(\lambda m+\lambda+1)} \\
&= \sum_{m=v=0}^{\infty}\frac{(-1)^m\Gamma(\lambda m+\lambda+1+v)\theta\lambda}{v!\Gamma(\lambda m+\lambda+1)}\binom{\theta+1}{m} g(x;\xi)\,(2-G(x;\xi))\,[G(x;\xi)]^{v+2\lambda m+2\lambda-1} \\
&= \sum_{m=v=r=0}^{\infty}\frac{(-1)^{m+r}\Gamma(\lambda m+\lambda+1+v)\theta\lambda}{v!\Gamma(\lambda m+\lambda+1)}\binom{\theta+1}{m}\binom{v+2\lambda m+2\lambda-1}{r}g(x;\xi) \times (2-G(x;\xi))\,[1-G(x;\xi)]^r \\
&= \sum_{m=v=r=j=0}^{\infty}\frac{(-1)^{m+r+j}\Gamma(\lambda m+\lambda+1+v)\theta\lambda}{v!\Gamma(\lambda m+\lambda+1)}\binom{\theta+1}{m}\binom{r}{j}\binom{v+2\lambda m+2\lambda-1}{r}\times g(x;\xi)\,(2-G(x;\xi))\,[G(x;\xi)]^j
\end{aligned}$$

$$= \sum_{r=j=0}^{\infty}\psi_{m,v,r,j}2g(x;\xi)[G(x;\xi)]^j - \sum_{r=j=0}^{\infty}\psi_{m,v,r,j}g(x;\xi)[G(x;\xi)]^{j+1}. \quad (12)$$

Now, substituting Equations (1) and (2) into equation (12)

$$\begin{aligned}
k(x)_{OBXII-IR} &= \sum_{j=0}^{\infty}\psi_{m,v,r,j}\frac{4\delta}{x^3}e^{-\frac{\delta}{x^2}}\left[e^{-\frac{\delta}{x^2}}\right]^j - \sum_{j=0}^{\infty}\psi_{m,v,r,j}\frac{2\delta}{x^3}e^{-\frac{\delta}{x^2}}\left[e^{-\frac{\delta}{x^2}}\right]^{j+1} \\
&= \sum_{j=0}^{\infty}\psi_{m,v,r,j}\frac{4\delta}{x^3}e^{-\frac{\delta(j+1)}{x^2}} - \sum_{j=0}^{\infty}\psi_{m,v,r,j}\frac{2\delta}{x^3}e^{-\frac{\delta(j+2)}{x^2}}
\end{aligned}$$

where

$$\psi_{m,v,r,j} = \sum_{m=v=r=0}^{\infty}\frac{(-1)^{m+r+j}\Gamma(\lambda m+\lambda+1+v)\theta\lambda}{v!\Gamma(\lambda m+\lambda+1)}\binom{\theta+1}{m}\binom{v+2\lambda m+2\lambda-1}{r}\binom{r}{j}.$$

4 Moment

Theorem 4.1. Let X be a random variable that follows an OBXII-IR distribution with parameters θ, λ, and δ. The $r \sim th$ moment of X about the origin is denoted by $\dot{\mu}_r$:

$$\dot{\mu}_r = \sum_{j=0}^{\infty}\psi_{m,v,r,j}\delta^{\frac{r}{2}}\,\Gamma\left(1-\frac{r}{2}\right)\left[2(j+1)^{\frac{r}{2}-1}-(j+2)^{\frac{r}{2}-1}\right]\ r<2. \quad (13)$$

Proof.

$$\acute{\mu}_r = E\left(X^r\right)_{OBXII-IR} = \int_0^\infty x^r k\left(x\right)_{OBXII-IR} dx, \quad (14)$$

substituting Equation (11) into Equation (14)

$$\acute{\mu}_r = \sum_{j=0}^\infty \psi_{m,v,r,j} 4\delta \int_0^\infty x^{r-3} e^{-\frac{\delta(j+1)}{x^2}} dx - \sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta \int_0^\infty x^{r-3} e^{-\frac{\delta(j+2)}{x^2}} dx$$

$$= 2\sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta\, T_1 - \sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta T_2$$

Then

$$T_1 = \int_0^\infty x^{r-3} e^{-\frac{\delta(j+1)}{x^2}} dx$$

$$= \left(y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}}, dx = -\frac{1}{2} y^{-\frac{3}{2}} dy\right)$$

$$= \frac{1}{2}\int_0^\infty y^{-\frac{r}{2}} e^{-\delta(j+1)y} dy$$

$$= \frac{1}{2}\Gamma\left(1-\frac{r}{2}\right)\left(\delta\left(j+1\right)\right)^{\frac{r}{2}-1}$$

$$T_2 = \int_0^\infty x^{r-3} e^{-\frac{\delta(j+2)}{x^2}} dx = \frac{1}{2}\Gamma\left(1-\frac{r}{2}\right)\left(\delta\left(j+2\right)\right)^{\frac{r}{2}-1}.$$

Now,

$$\acute{\mu}_r = \sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta^{\frac{r}{2}}\, \Gamma\left(1-\frac{r}{2}\right)\left(j+1\right)^{\frac{r}{2}-1} - \sum_{j=0}^\infty \psi_{m,v,r,j} \delta^{\frac{r}{2}}\, \Gamma\left(1-\frac{r}{2}\right)\left(j+2\right)^{\frac{r}{2}-1}$$

$$= \sum_{j=0}^\infty \psi_{m,v,r,j} \delta^{\frac{r}{2}}\, \Gamma\left(1-\frac{r}{2}\right)\left[2\left(j+1\right)^{\frac{r}{2}-1} - \left(j+2\right)^{\frac{r}{2}-1}\right].$$

From equation (8), we can compute the $\acute{\mu}_1, \acute{\mu}_2, \acute{\mu}_3, \acute{\mu}_4$, variance $V = E(X^2) - (E\left(X\right))^2$, skewness $SK = \frac{\mu_3}{\mu_2^{\left(\frac{3}{2}\right)}}$, and kurtosis $Ku = \frac{\mu_4}{\mu_2^2}$ of the OBXII-IR distribution as follows:

5 Incomplete moments

Theorem 5.1. Let X be a random variable that follows the OBXII-IR distribution, characterized by the parameters θ, λ, and δ. The incomplete moments $\acute{\mu}_\rho\left(t\right)$ of X can be computed using the following formula:

$$\acute{\mu}_\rho\left(t\right) = 2\delta\left(\delta\left(j+1\right)\right)^{\frac{\rho}{2}-1} \sum_{j=0}^\infty \psi_{m,v,r,j} \gamma\left(\Gamma\left(1-\frac{\rho}{2}\right), \frac{\delta(j+1)}{t^2}\right) - \delta\left(\delta\left(j+2\right)\right)^{\frac{\rho}{2}-1} \quad (15)$$

$$\sum_{j=0}^\infty \psi_{m,v,r,j} \gamma\left(\Gamma\left(1-\frac{\rho}{2}\right), \frac{\delta(j+2)}{t^2}\right). \quad (16)$$

Proof.

$$\acute{\mu}_\rho\left(t\right) = \int_{-\infty}^t x^\rho g\left(x\right)_{OBXIIEW} dx$$

$$= \int_0^t \sum_{j=0}^\infty \psi_{m,v,r,j} 4\delta x^{\rho-3} e^{-\frac{\delta(j+1)}{x^2}} - \int_0^t \sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta x^{\rho-3} e^{-\frac{\delta(j+2)}{x^2}} dx$$

$$= \left(y = \frac{\delta(j+1)}{x^2} \Rightarrow x = \left(\frac{\delta(j+1)}{y}\right)^{\frac{1}{2}}\right)$$

$$= \left(when\ x = 0 \Rightarrow y = 0\ , and\ if\ \ x = t\ \Rightarrow y = \frac{\delta(j+1)}{t^2}\ \Rightarrow then\ dt = -\frac{1}{2} y^{-\frac{3}{2}}\left(\delta\left(j+1\right)\right)^{\frac{1}{2}}\right)$$

$$= \left(\delta\left(j+1\right)\right)^{\frac{\rho}{2}-1} \int_0^{\frac{\delta(j+1)}{t^2}} \sum_{j=0}^\infty \psi_{m,v,r,j} 2\delta y^{-\frac{\rho}{2}} e^{-y} dy - \left(\delta\left(j+2\right)\right)^{\frac{\rho}{2}-1} \int_0^{\frac{\delta(j+2)}{t^2}} \sum_{j=0}^\infty \psi_{m,v,r,j} \delta y^{-\frac{\rho}{2}} e^{-y} dy$$

$$= 2\delta\left(\delta\left(j+1\right)\right)^{\frac{\rho}{2}-1} \sum_{j=0}^\infty \psi_{m,v,r,j} \gamma\left(\Gamma\left(1-\frac{\rho}{2}\right), \frac{\delta(j+1)}{t^2}\right) - \delta\left(\delta\left(j+2\right)\right)^{\frac{\rho}{2}-1} \sum_{j=0}^\infty \psi_{m,v,r,j} \gamma\left(\Gamma\left(1-\frac{\rho}{2}\right), \frac{\delta(j+2)}{t^2}\right),$$

where $\Gamma\left(\rho,t\right) = \int_0^t x^{\rho-1} e^{-x} dx.$

Table 2: Show the OBXII-IR distribution $\acute{\mu}_1$, $\acute{\mu}_2$, $\acute{\mu}_3$, $\acute{\mu}_4$, V, SK, and Ku

θ	λ	δ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	V	SK	Ku
2	1.5	1.6	1.623	2.815	5.289	11.10	0.1818	1.1198	1.4007
2	1.5	1.8	1.722	3.167	6.311	14.05	0.2017	1.1197	1.4008
2	1.7	1.6	1.635	2.812	5.128	10.04	0.1387	1.0874	1.2697
2	1.7	1.8	1.734	3.164	6.119	12.71	0.1572	1.0872	1.2686
2.5	1.5	1.6	1.539	2.494	4.275	7.826	0.1254	1.0854	1.2581
2.5	1.5	1.8	1.632	2.805	5.101	9.905	0.1415	1.0858	1.2588
2.5	1.7	1.6	1.560	2.536	4.303	7.656	0.1024	1.0654	1.1904
2.5	1.7	1.8	1.655	2.853	5.134	9.689	0.1139	1.0653	1.1903
3	1.5	1.6	1.480	2.289	3.704	6.296	0.0098	1.0695	1.2016
3	1.5	1.8	1.570	2.576	4.420	7.969	0.1111	1.0690	1.2009
3	1.7	1.6	1.507	2.354	3.810	6.400	0.0829	1.0549	1.1549
3	1.7	1.8	1.599	2.648	4.546	8.100	0.0911	1.0550	1.1551

6 Quantile function

Theorem 6.1. The quantile function of the OBXII-IR distribution, characterized by the parameters θ, λ, and δ, may be expressed as follows:

$$Q(u) = \left(\frac{1}{\delta}\left(-ln\frac{1}{2}\left(-\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}} + \left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{2}{\lambda}} -4\left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{2}}\right)\right)\right)^{-\frac{1}{2}}. \quad (17)$$

Proof. We proceed with the equation $K(x)_{OBXII-IR} = u$, where K is the cumulative distribution function

$$1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta} = u$$

$$\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda} = (1-u)^{-\frac{1}{\theta}}-1$$

$$\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}} = \left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}}$$

Let $v = e^{-\frac{\delta}{x^2}}$, we get a quadratic equation using v as the variable:

$$v^2 + sv - s = 0$$

Solving for the given

$$v = \frac{-\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}} + \left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{2}{\lambda}}\right.}{2} \frac{\left.-4\left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{2}}}{2}$$

we get the quantile function by substituting v back into x_u

$$x_u = Q(u) = \left(\frac{1}{\delta}\left(-ln\frac{1}{2}\left(-\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}} + \left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{2}{\lambda}} -4\left(\left((1-u)^{-\frac{1}{\theta}}-1\right)^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{2}}\right)\right)\right)^{-\frac{1}{2}}.$$

7 Order statistics

In this part, we'll use the OBXII-IR distribution to determine how the order statistic falls. Regarding applicability, reliability, and survival analysis, order statistics is a natural advantage.

When $X_{i:n}$ is the ith order statistic and $X_1, X_2, \ldots, X_n$ is a random sample from the OBXII-IR distribution, the pdf $k_{i:n}(x)$ of $X_{i:n}$ from the same distribution is given by [25]:

$$k_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}k(x)_{OBXII-IR} \left[K(x)_{OBXII-IR}\right]^{i-1}\left[1-K(x)_{OBXII-IR}\right]^{n-i}. \quad (18)$$

Inserting the Equations (8) and (9) into (18)

$$k_{i:n}(x) = \frac{2n!\delta\theta\lambda e^{-\frac{2\delta}{x^2}}\left(2-e^{-\frac{\delta}{x^2}}\right)\left(1-e^{-\frac{\delta}{x^2}}\right)^{-2}}{(i-1)!\,(n-i)!x^3} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda-1}\left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right]^{i-1} \times\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta(n+1-i)-1}.$$

The pdf of the smallest order statistic for OBXII-IR distribution:

$$k_{1:n}(x) = nk(x)_{OBXII-IR}\left[1-K(x)_{OBXII-IR}\right]^{n-1}$$

$$= \frac{2n\delta\theta\lambda}{x^3}e^{-\frac{2\delta}{x^2}}\left(2-e^{-\frac{\delta}{x^2}}\right)\left(1-e^{-\frac{\delta}{x^2}}\right)^{-2} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda-1}\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta n-1}. \quad (19)$$

The pdf of the largest order statistic for OBXII-IR distribution:

$$k_{n:n}(x) = nk(x)_{OBXII-IR}\left[K(x)_{OBXII-IR}\right]^{n-1}$$

$$= \frac{2n\delta\theta\lambda}{x^3}e^{-\frac{2\delta}{x^2}}\left(2-e^{-\frac{\delta}{x^2}}\right)\left(1-e^{-\frac{\delta}{x^2}}\right)^{-2} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda-1}\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-(\theta+1)} \left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right]^{n-1} \quad (20)$$

8 Harmonic mean

Theorem 8.1. suppose X be a random variable with OBXII-IR distribution parameters θ, λ, and δ. The

Table 3: Displays the quantiles for the OBXII-IR distribution's specified parameter value

u	$(\theta, \lambda, \delta)$				
	(0.5, 1.7, 1.1)	(0.6, 1.3, 0.9)	(1.2, 1.5,1.6)	(1.4, 2, 1.8)	(2, 3, 1.9)
0.1	0.6288	0.6510	1.1274	1.2160	1.3461
0.2	0.7353	0.7696	1.2689	1.3456	1.4522
0.3	0.8367	0.8843	1.3858	1.4476	1.5305
0.4	0.9485	1.0125	1.4982	1.5417	1.5985
0.5	1.0844	1.1702	1.6166	1.6365	1.6634
0.6	1.2659	1.3830	1.7512	1.7395	1.7300
0.7	1.5383	1.7044	1.9187	1.8608	1.8035
0.8	2.0257	2.2799	2.1561	2.0217	1.8937
0.9	3.2592	3.7304	2.5937	2.2910	2.0288

harmonic mean (HM) of X obtained as follows:

$$HM = \sum_{j=0}^{\infty} \psi_{m,v,r,j} 2\delta^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right)(j+1)^{\frac{3}{2}} - \sum_{j=0}^{\infty} \psi_{m,v,r,j} \delta^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right)(j+2)^{\frac{3}{2}}. \quad (21)$$

Proof.

$$HM = E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} k(x)_{OBXII-IR}\, dx$$
$$= \sum_{j=0}^{\infty} \psi_{m,v,r,j} 4\delta \int_0^{\infty} x^{-4} e^{-\frac{\delta(j+1)}{x^2}} dx - \sum_{j=0}^{\infty} \psi_{m,v,r,j} 2\delta \int_0^{\infty} x^{-4} e^{-\frac{\delta(j+2)}{x^2}} dx$$
$$= \left(y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}},\ dx = -\frac{1}{2} y^{-\frac{3}{2}} dy \right)$$
$$= \sum_{j=0}^{\infty} \psi_{m,v,r,j} 2\delta \int_0^{\infty} y^{\frac{3}{2}-1}\, e^{-\delta(j+1)y} dy - \sum_{j=0}^{\infty} \psi_{m,v,r,j} \delta \int_0^{\infty} y^{\frac{3}{2}-1}\, e^{-\delta(j+2)y} dy$$

$$= \sum_{j=0}^{\infty} \psi_{m,v,r,j} 2\delta^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right)(j+1)^{\frac{3}{2}} - \sum_{j=0}^{\infty} \psi_{m,v,r,j} \delta^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right)(j+2)^{\frac{3}{2}}.$$

9 Rényi entropy

Theorem 9.1. suppose X be a random variable with OBXII-IR distribution parameters θ, λ, and δ. The rényi entropy $T_R(\varphi)$ of X obtained as follows:

$$T_R(s) = \frac{1}{1-s} \log\left(\frac{1}{2} \sum_{m=0}^{\infty} \omega_{i,p,m} \Gamma\left(\frac{3s-5}{2}\right) (\delta(2\lambda(s+i)+m))^{-\frac{(3s+2)}{2}} \right), \quad (22)$$

where

$$\omega_{i,p,m} = \sum_{i=p=0}^{\infty} \frac{\Gamma(s(1+\lambda)-\lambda i+p+m)(-1)^i}{m!\Gamma(s(1+\lambda)-\lambda i+p)} \binom{s(\theta+1)}{i} \binom{s}{p} (2\delta\theta\lambda)^s.$$

Proof.

$$T_R(s) = \frac{1}{1-s} \log \int_0^{\infty} \left(k(x)_{OBXII-IR}\right)^s dx,\ s>0,\ s \neq 1$$
$$= \frac{1}{1-s} \log \int_0^{\infty} (2\delta\theta\lambda)^s x^{-3s} e^{-\frac{2s\delta}{x^2}} \left(2-e^{-\frac{\delta}{x^2}}\right)^s \left(1-e^{-\frac{\delta}{x^2}}\right)^{-2s} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{s(\lambda-1)}$$

$$\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-s(\theta+1)} dx$$

$$= \left(\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-s(\theta+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{s(\theta+1)}{i} \left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda i} \right)$$

$$= \left(\left(2-e^{-\frac{\delta}{x^2}}\right)^s = \left(1+\left(1-e^{-\frac{\delta}{x^2}}\right)\right)^s \right.$$

$$\left. = \sum_{p=0}^{\infty} \binom{s}{p} \left(1-e^{-\frac{\delta}{x^2}}\right)^p \right)$$

$$= \frac{1}{1-s} \log \int_0^\infty \sum_{i=p=0}^{\infty} (-1)^i \binom{s(\theta+1)}{i} \binom{s}{p} (2\delta\theta\lambda)^s x^{-3s}$$
$$e^{-\frac{2s\delta\lambda(s+i)}{x^2}} \left(1-e^{-\frac{\delta}{x^2}}\right)^{-s(1+\lambda)-\lambda i+p} dx$$

$$= \frac{1}{1-s} \log \int_0^\infty \sum_{m=0}^{\infty} \omega_{i,p,m} x^{-3s}\, e^{-\frac{\delta(2\lambda(s+i)+m)}{x^2}} dx$$
$$= \left(y = \frac{1}{x^2} \;\Rightarrow x = \frac{1}{\sqrt{y}},\; dx = -\frac{1}{2} y^{-\frac{3}{2}} dy\right)$$
$$= \frac{1}{1-s} \log \int_0^\infty \frac{1}{2} \sum_{m=0}^{\infty} \omega_{i,p,m} y^{\frac{3}{2}(s-1)}\, e^{-\delta(2\lambda(s+i)+m)y} dy$$
$$= \frac{1}{1-s} \log \left(\frac{1}{2} \sum_{m=0}^{\infty} \omega_{i,p,m} \Gamma\left(\frac{3}{2}(s-1)-1\right.\right.$$
$$\left.\left. (\delta(2\lambda(s+i)+m))^{1-\frac{3}{2}(s-1)} \right)\right.$$
$$= \frac{1}{1-s} \log \left(\frac{1}{2} \sum_{m=0}^{\infty} \omega_{i,p,m} \Gamma\left(\frac{3s-5}{2}\right) \right.$$
$$\left. (\delta(2\lambda(s+i)+m))^{-\frac{(3s+2)}{2}} \right),$$

where

$$\omega_{i,p,m} = \sum_{i=p=0}^{\infty} \frac{\Gamma(s(1+\lambda)-\lambda i+p+m)(-1)^i}{m!\Gamma(s(1+\lambda)-\lambda i+p)} \binom{s(\theta+1)}{i} \binom{s}{p} (2\delta\theta\lambda)^s.$$

10 Different methods of estimation

In this section, we will discuss the maximum likelihood (MLE) estimation procedure, Least Squares Estimation (LSE), Weighted Least Squares Estimation (WLSE), Anderson-Darling Estimation (ADE) and Right-Tailed Anderson-Darling Estimation (RTADE) to estimate the unknown parameter.

10.1 Maximum likelihood estimation

The MLE approach is generally acknowledged for its effectiveness in estimating parameters inside distributions. Let $\Theta = (\theta,\lambda,\delta)^T$, obtaining the log-likelihood function for Θ is as follows [26,27]:

$$\ell(\Theta) = n\log 2 + n\log\theta + n\log\lambda + n\log\delta$$
$$-3\sum_{i=0}^{n}\log x_i - 2\delta\sum_{i=0}^{n}\frac{1}{x_i^2}$$
$$+\sum_{i=0}^{n}\log\left(2-e^{-\frac{\delta}{x_i^2}}\right) - 2\sum_{i=0}^{n}\log\left(1-e^{-\frac{\delta}{x_i^2}}\right)$$
$$+(\lambda-1)\sum_{i=0}^{n}\log\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)$$
$$-(\theta+1)\sum_{i=0}^{n}\log\left(1+\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{--\frac{\delta}{x_i^2}}}\right)^\lambda\right). \quad (23)$$

They can estimate the probability of parameters θ, λ, and δ by using the first partial derivative of the log-likelihood function.

$$\frac{\partial\ell(\Theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=0}^{n}\log\left(1+\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^\lambda\right), \quad (24)$$

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=0}^{n}\log\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)$$
$$-(\theta+1)\sum_{i=0}^{n}\frac{\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^\lambda \log\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)}{1+\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^\lambda}, \quad (25)$$

$$\frac{\partial\ell(\Theta)}{\partial\delta} = \frac{n}{\delta} - 2\sum_{i=0}^{n}\frac{1}{x_i^2}$$
$$+\sum_{i=0}^{n}\frac{\frac{1}{x_i^2}e^{-\frac{\delta}{x_i^2}}}{\left(2-e^{-\frac{\delta}{x_i^2}}\right)} - 2\sum_{i=0}^{n}\frac{\frac{1}{x_i^2}e^{-\frac{\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)}$$
$$+(\lambda-1)\sum_{i=0}^{n} e^{\frac{2\delta}{x_i^2}}\left(1-e^{-\frac{\delta}{x_i^2}}\right)$$
$$\left(-\frac{\frac{2}{x_i^2}e^{-\frac{2\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)} - \frac{\frac{1}{x_i^2}e^{-\frac{3\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)^2}\right)$$

$$-(\theta+1)\sum_{i=0}^{n}\frac{\lambda e^{\frac{2\delta}{x_i^2}}\left(1-e^{-\frac{\delta}{x_i^2}}\right)\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^{\lambda}}{\left(1+\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^{\lambda}\right)}.\frac{\left(-\frac{\frac{2}{x_i^2}e^{-\frac{2\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)}-\frac{\frac{1}{x_i^2}e^{-\frac{3\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)^2}\right)}{\left(1+\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right)^{\lambda}\right)} \tag{26}$$

While the Newton-Raphson method may be used for numerical calculations, it is not feasible to compute the MLE analytically for the parameters θ, λ, and δ. The procedure entails numerically solving Equations (24)-(26) by making them equal to zero.

10.2 Least squares estimation (LSE)

By [28], the LSE method had first appeared. These are the parameter estimates that were derived from the objective function. The LSE function is equivalent to this:

$$L(\theta,\lambda,\delta)=\sum_{i=1}^{n}\left(K(x_i,\theta,\ \lambda,\ \delta)-\frac{i}{n+1}\right)^2$$
$$=\sum_{i=1}^{n}\left(\left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right]-\frac{i}{n+1}\right)^2 \tag{27}$$

We use the following

$$K_i=K(x_i,\theta,\ \lambda,\ \delta),$$

$$K_{\theta_i}=\left[1+\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\right]^{-\theta}\ln\left(1+\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\right), \tag{28}$$

$$K_{\lambda_i}=\theta\left[1+\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\right]^{-\theta-1}\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\ln\left(\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right), \tag{29}$$

$$K_{\delta_i}=\lambda\theta\left(1-e^{-\frac{\delta}{x_i^2}}\right)\left[1+\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\right]^{-\theta-1}\left[\frac{e^{-\frac{2\delta}{x_i^2}}}{1-e^{-\frac{\delta}{x_i^2}}}\right]^{\lambda}\left(-\frac{\frac{2}{x_i^2}e^{-\frac{2\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)}-\frac{\frac{1}{x_i^2}e^{-\frac{3\delta}{x_i^2}}}{\left(1-e^{-\frac{\delta}{x_i^2}}\right)^2}\right) \tag{30}$$

Where $K_{\theta_i}=\frac{\partial K(x_i,\theta,\ \lambda,\ \delta)}{\partial\theta}$, $K_{\lambda_i}=\frac{\partial K(x_i,\theta,\ \lambda,\ \delta)}{\partial\lambda}$, and $K_{\delta_i}=\frac{\partial K(x_i,\theta,\ \lambda,\ \delta)}{\partial\delta}$, in the first step, we calculate the partial derivatives and initialize them to zero. Use numerical techniques to resolve the Equations (28), (29), and (30) since it cannot be defined in closed form. Partially, we get the following by using (28) to (30).

$$\frac{\partial L(\theta,\lambda,\delta)}{\partial\theta}=2\sum_{i=1}^{n}K_{\theta_i}\left(K_i-\frac{i}{n+1}\right)^2,$$
$$\frac{\partial L(\theta,\lambda,\ \delta)}{\partial\lambda}=2\sum_{i=1}^{n}K_{\lambda_i}\left(K_i-\frac{i}{n+1}\right)^2,$$
$$\frac{\partial L(\theta,\lambda,\delta)}{\partial\delta}=2\sum_{i=1}^{n}K_{\delta_i}\left(K_i-\frac{i}{n+1}\right)^2.$$

10.3 Weighted least squares estimation (WLSE)

The WLSE method estimates θ, λ, and δ. To estimate parameters, minimize the WLSE function. Target function definition [29]:

$$W(\theta,\lambda,\delta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(K(x_i,\theta,\ \lambda,\ \delta)-\frac{i}{n+1}\right)^2$$
$$=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right]-\frac{i}{n+1}\right)^2. \tag{31}$$

By using the same methodology as in the LSE, we get the partial derivatives.

$$\frac{\partial W(\theta,\lambda,\delta)}{\partial\theta}=2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}K_{\theta_i}\left(K_i-\frac{i}{n+1}\right)^2,$$
$$\frac{\partial W(\theta,\lambda,\delta)}{\partial\lambda}=2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}K_{\lambda_i}\left(K_i-\frac{i}{n+1}\right)^2,$$
$$\frac{\partial W(\theta,\lambda,\delta)}{\partial\delta}=2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}K_{\delta_i}\left(K_i-\frac{i}{n+1}\right)^2.$$

10.4 Anderson-Darling estimation (ADE)

The ADE method minimizes the AD function to estimate θ, λ, and δ. The formal function is [30]:

$$A(\theta,\lambda,\delta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2!-1)\left(\ln\left(K\left(x_i,\theta,\lambda,\delta\right)\right)+\ln\left(S\left(x_i,\theta,\ \lambda,\ \delta\right)\right)\right)$$
$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2!-1)\left(\ln\left(\left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right]\right)+\ln\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right). \quad (32)$$

To estimate θ, λ, and δ, we use Equations (28) to (30) to determine the partial derivatives of $A(\theta,\lambda,\delta)$.

$$\frac{\partial A(\theta,\ \lambda,\ \delta)}{\partial\theta} = -\frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\theta_i}}{K(x_i,\theta,\ \lambda,\ \delta)} - \frac{K_{\theta_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right],$$

$$\frac{\partial A(\theta,\ \lambda,\ \delta)}{\partial\lambda} = -\frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\ \lambda_i}}{K(x_i,\theta,\ \lambda,\ \delta)} - \frac{K_{\ \lambda_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right],$$

$$\frac{\partial A(\theta,\ \lambda,\ \delta)}{\partial\delta} = -\frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\delta_i}}{K(x_i,\theta,\ \lambda,\ \delta)} - \frac{K_{\delta_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right].$$

10.5 Right-Tailed Anderson-Darling estimation (RTADE)

The RTADE of the OBXII-IR distribution parameters θ, λ, and δ may be obtained by minimizing the function R(θ, λ, δ) in relation to these parameters [31].

$$R(\theta,\ \lambda,\ \delta) = \frac{n}{2} - 2\sum_{i=1}^{n}K\left(x_i,\theta,\ \lambda,\ \delta\right) - \frac{1}{n}\sum_{i=1}^{n}(2!-1)\ln\left(S\left(x_i,\theta,\ \lambda,\ \delta\right)\right)$$
$$= \frac{n}{2} - 2\sum_{i=1}^{n}\left[1-\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}\right] - \frac{1}{n}\sum_{i=1}^{n}(2!-1)\ln\left[1+\left[\frac{e^{-\frac{2\delta}{x^2}}}{1-e^{-\frac{\delta}{x^2}}}\right]^{\lambda}\right]^{-\theta}. \quad (33)$$

Equations (28) to (30) and the previously discussed method for determining the partial derivatives of RTADE enable us to obtain

$$\frac{\partial R(\theta,\ \lambda,\ \delta)}{\partial\theta} = -2\sum_{i=1}^{n}K_{\theta_i} + \frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\theta_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right],$$

$$\frac{\partial R(\theta,\ \lambda,\ \delta)}{\partial\lambda} = -2\sum_{i=1}^{n}K_{\lambda_i} + \frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\ \lambda_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right],$$

$$\frac{\partial R(\theta,\lambda,\delta)}{\partial\delta} = -2\sum_{i=1}^{n}K_{\delta_i} + \frac{1}{n}\sum_{i=1}^{n}(2!-1)\left[\frac{K_{\delta_{n+1-i}}}{1-K(x_{n+1-i},\theta,\ \lambda,\ \delta)}\right].$$

Table 4: Mean, RMSE, and Bias with the MLE, LSE, WLSE, ADE, and RTADE for the OBXII-IR model with θ= 0.7, λ =0.6, δ=0.05

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
30	Mean	$\hat{\theta}$	0.93538	0.83328	0.83868	0.82641	0.82533
		$\hat{\lambda}$	0.80641	0.66804	0.65691	0.66883	0.79625
		$\hat{\delta}$	0.08398	0.08215	0.07932	0.07607	0.08163
	RMSE	$\hat{\theta}$	0.50786	0.47630	0.48313	0.46209	0.49216
		$\hat{\lambda}$	0.87575	0.56998	0.58631	0.56758	0.81292
		$\hat{\delta}$	0.07102	0.08483	0.07690	0.06845	0.07714
	Bias	$\hat{\theta}$	0.23582	0.13328	0.13868	0.12641	0.12533
		$\hat{\lambda}$	0.20641	0.06804	0.05691	0.06883	0.19625
		$\hat{\delta}$	0.03398	0.03215	0.02932	0.02607	0.03163
60	Mean	$\hat{\theta}$	0.87633	0.82065	0.83430	0.82111	0.80763
		$\hat{\lambda}$	0.60864	0.61166	0.59441	061378	0.69644
		$\hat{\delta}$	0.07374	0.07234	0.07226	0.07046	0.07284
	RMSE	$\hat{\theta}$	0.41634	0.40386	0.41143	0.39987	0.41881

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
		$\hat{\lambda}$	0.26349	0.34036	0.26441	0.30815	0.50407
		$\hat{\delta}$	0.05134	0.05567	0.05454	0.05146	0.05631
	Bias	$\hat{\theta}$	0.17633	0.12065	0.13430	0.12111	0.10763
		$\hat{\lambda}$	0.01913	0.01166	0.00558	0.01978	0.09644
		$\hat{\delta}$	0.02374	0.02234	0.02226	0.02046	0.02284
120	Mean	$\hat{\theta}$	0.78339	0.78308	0.77801	0.76571	0.76436
		$\hat{\lambda}$	0.61775	0.59994	0.60441	0.61787	0.66422
		$\hat{\delta}$	0.06210	0.06413	0.06265	0.06141	0.06440
	RMSE	$\hat{\theta}$	0.31335	0.31931	0.30880	0.30351	0.33528
		$\hat{\lambda}$	0.24048	0.21728	0.21157	0.22714	0.33424
		$\hat{\delta}$	0.03357	0.03778	0.03500	0.03363	0.03947
	Bias	$\hat{\theta}$	0.08339	0.08308	0.07801	0.06571	0.06436
		$\hat{\lambda}$	0.01775	0.00517	0.00441	0.01787	0.06422
		$\hat{\delta}$	0.01210	0.01413	0.01265	0.01141	0.01408
200	Mean	$\hat{\theta}$	0.74956	0.76043	0.75682	0.74769	0.76314
		$\hat{\lambda}$	0.62006	0.59747	0.60009	0.60855	0.62391
		$\hat{\delta}$	0.05740	0.05994	0.05872	0.05782	0.06149
	RMSE	$\hat{\theta}$	0.26015	0.27044	0.25595	0.25534	0.28592
		$\hat{\lambda}$	0.20619	0.17704	0.17536	0.17797	0.23758
		$\hat{\delta}$	0.02622	0.03003	0.02714	0.02649	0.03223
	Bias	$\hat{\theta}$	0.04926	0.06043	0.05682	0.04769	0.06314
		$\hat{\lambda}$	0.01006	0.00452	0.00094	0.01455	0.02391
		$\hat{\delta}$	0.00740	0.00994	0.00872	0.00782	0.01149
300	Mean	$\hat{\theta}$	0.73566	0.74545	0.73990	0.73622	0.74170
		$\hat{\lambda}$	0.62239	0.60360	0.60782	0.61121	0.63036
		$\hat{\delta}$	0.05534	0.05734	0.05620	0.05585	0.05820
	RMSE	$\hat{\theta}$	0.2348	0.24366	0.23080	0.23099	0.26156
		$\hat{\lambda}$	0.19319	0.16646	0.16623	0.16677	0.21856
		$\hat{\delta}$	0.02226	0.02519	0.02294	0.02270	0.02770
	Bias	$\hat{\theta}$	0.03566	0.04545	0.03990	0.03622	0.04170
		$\hat{\lambda}$	0.00239	0.00360	0.00078	0.01121	0.02036
		$\hat{\delta}$	0.00534	0.00734	0.00620	0.00585	0.00820

11 Simulation

In this section, we perform a simulation to assess and compare the effectiveness of the aforementioned estimations in terms of their mean, root mean squared errors (RMSE), and average bias (Bias). For each value of n=30, 60, 120, 200, and 300, we produce 1000 random samples of size n from the OBXII-IR distribution. Four distinct sets of parameters have been considered: (θ= 0.7, λ =0.6, δ=0.05), (θ= 0.8, λ =0.5, δ=0.05), (θ= 0.8, λ =0.6, δ=0.04), and (θ= 0.9, λ =0.6, δ=0.03).

i. $RMSE\left(\hat{\sigma}\right)=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\sigma}_i-\sigma)^2}{N}}$.

ii. $bias\left(\hat{\sigma}\right)=\frac{\sum_{i=1}^{N}\hat{\sigma}_i}{N}-\sigma$.

Tables 4-7 include the mean, RMSE, and Bias estimates. To compare the various techniques of estimation. These tables show:

1. The estimated parameters θ, λ, and δ approach their initial values as n increases.
2. The values of RMSE and Bias decrease as n increases.
3. Out of all the approaches, MLE has the lowest result. Thus, when evaluating θ, λ, and δ, MLE outperforms all other approaches that were taken into consideration.

12 Application

In this part, we demonstrate the adaptability of the OBXII-IR model by examining two real-world datasets, referred to as D1 and D2, which are presented as follows:

For the first data (D1), we use the lifespan datasets [32,33] provided. Their datasets comprise the recorded durations of pain alleviation experienced by 20 individuals who were administered an analgesic. The data are as follows: 1.1,1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. The second data (D2), according to [34,35], the D2 examines strength measurements taken in GPA for single-carbon fibers with 20 mm gauge lengths were evaluated. 1.312,1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055,2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684,2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096,3.128, 3.233, 3.433, 3.585, 3.585.

The fitting behavior of the OBXII-IR model is compared to that of (BeIR) Beta Inverse Rayleigh distribution [1], (GoIR) Gompertz Inverse Rayleigh distribution [36], (TEEIR) Truncated Exponentiated Exponential Inverse Rayleigh distribution, (TEMOIR) Truncated Exponential Marshall-Olkin Inverse Rayleigh distribution, (KuIR) Kumaraswamy Inverse Rayleigh distribution [37], (EGIR) Exponentiated Generalized Inverse Rayleigh distribution [38], (OLxIR) Odd Lomax Inverse Rayleigh distribution.

1.BeIR

$$K(x) = pbeta\left(e^{-\frac{\delta}{x^2}}, \lambda, \theta\right).$$

2.GoIR

$$K(x) = 1 - e^{\left(\left(\frac{\lambda}{\theta}\right)\left(1-\left(1-e^{-\frac{\delta}{x^2}}\right)^{-\theta}\right)\right)}.$$

3.TEEIR

$$K(x) = \frac{\left(\left(1-e^{\left(-\lambda e^{-\frac{\delta}{x^2}}\right)}\right)\right)^{\theta}}{\left(1-e^{(-\lambda)}\right)^{\theta}}.$$

4.TEMOIR

$$K(x) = \frac{1-e^{\left(\frac{-\lambda e^{-\frac{\delta}{x^2}}}{\theta+\left((1-\theta)e^{-\frac{\delta}{x^2}}\right)}\right)}}{1-e^{(-\lambda)}}.$$

5.KuIR

$$K(x) = 1 - \left(1 - \left(e^{-\frac{\delta}{x^2}}\right)^{\lambda}\right)^{\theta}.$$

6.EGIR

$$K(x) = \left(1 - \left(1 - e^{-\frac{\delta}{x^2}}\right)^{\lambda}\right)^{\theta}.$$

7.OLxIR

$$K(x) = 1 - \left(1 - e^{-\frac{\delta}{x^2}} \frac{log\left(1-e^{-\frac{\delta}{x^2}}\right)}{\theta}\right)^{-\lambda}.$$

Table 5: Mean, RMSE, and Bias with the MLE, LSE, WLSE, ADE, and RTADE for the OBXII-IR model with θ= 0.8, λ =0.5, δ=0.05

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
30	Mean	$\hat{\theta}$	0.97034	0.90974	0.88218	0.89252	0.96178
		$\hat{\lambda}$	0.61898	0.52036	0.54847	0.56231	0.54379
		$\hat{\delta}$	0.07833	0.08182	0.07530	0.07426	0.08935
	RMSE	$\hat{\theta}$	0.48007	0.48473	0.47180	0.47071	0.52382
		$\hat{\lambda}$	1.84908	0.37594	0.34865	0.67142	0.40263
		$\hat{\delta}$	0.06664	0.08791	0.07506	0.06927	0.08837
	Bias	$\hat{\theta}$	0.17034	0.10974	0.08218	0.04529	0.16178
		$\hat{\lambda}$	0.11898	0.04036	0.04847	0.06235	0.04379
		$\hat{\delta}$	0.02833	0.03182	0.02530	0.02526	0.03935
60	Mean	$\hat{\theta}$	0.87973	0.85162	0.83644	0.84200	0.89154
		$\hat{\lambda}$	0.55851	0.53140	0.54648	0.54188	0.54028
		$\hat{\delta}$	0.06325	0.06440	0.06137	0.06102	0.07129
	RMSE	$\hat{\theta}$	0.36427	0.38706	0.37364	0.36698	0.41747

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
		$\hat{\lambda}$	0.94349	0.22738	0.23274	0.21586	0.26957
		$\hat{\delta}$	0.04010	0.04883	0.04363	0.04047	0.05738
	Bias	$\hat{\theta}$	0.07973	0.05162	0.03644	0.04200	0.09154
		$\hat{\lambda}$	0.05851	0.03140	0.04688	0.04188	0.04028
		$\hat{\delta}$	0.01325	0.01440	0.01137	0.01102	0.02129
120	Mean	$\hat{\theta}$	0.84158	0.82844	0.82186	0.82213	0.85492
		$\hat{\lambda}$	0.53916	0.52500	0.53302	0.53465	0.52656
		$\hat{\delta}$	0.05693	0.05685	0.05551	0.05555	0.06083
	RMSE	$\hat{\theta}$	0.29864	0.30077	0.29214	0.29325	0.32279
		$\hat{\lambda}$	0.24711	0.18303	0.18841	0.18765	0.20008
		$\hat{\delta}$	0.02788	0.02944	0.02712	0.02715	0.03529
	Bias	$\hat{\theta}$	0.04158	0.02844	0.02186	0.02213	0.05492
		$\hat{\lambda}$	0.03916	0.02500	0.03302	0.03465	0.02656
		$\hat{\delta}$	0.00693	0.00685	0.00551	0.00555	0.01083
200	Mean	$\hat{\theta}$	0.82607	0.81945	0.81660	0.81269	0.84660
		$\hat{\lambda}$	0.53166	0.52450	0.52777	0.53191	0.51869
		$\hat{\delta}$	0.05441	0.05486	0.05397	0.05367	0.05822
	RMSE	$\hat{\theta}$	0.25139	0.25974	0.24957	0.25060	0.27688
		$\hat{\lambda}$	0.18480	0.16618	0.16703	0.17032	0.17099
		$\hat{\delta}$	0.02176	0.02397	0.02203	0.02187	0.02770
	Bias	$\hat{\theta}$	0.02607	0.01945	0.01660	0.01269	0.04660
		$\hat{\lambda}$	0.03166	0.02450	0.02777	0.03191	0.01869
		$\hat{\delta}$	0.00441	0.00486	0.00397	0.00367	0.00822
300	Mean	$\hat{\theta}$	0.80801	0.79556	0.80075	0.80594	0.82606
		$\hat{\lambda}$	0.53285	0.53433	0.53054	0.53481	0.52091
		$\hat{\delta}$	0.05264	0.05244	0.05237	0.05195	0.05571
	RMSE	$\hat{\theta}$	0.22266	0.23576	0.22290	0.22453	0.24075
		$\hat{\lambda}$	0.17043	0.16308	0.15680	0.15982	0.15654
		$\hat{\delta}$	0.01871	0.02089	0.01915	0.01903	0.02304
	Bias	$\hat{\theta}$	0.00801	0.00443	0.00075	0.00405	0.02606
		$\hat{\lambda}$	0.03085	0.01433	0.02054	0.03011	0.01691
		$\hat{\delta}$	0.00264	0.00244	0.00237	0.00195	0.00571

Table 6: Mean, RMSE, and Bias with the MLE, LSE, WLSE, ADE, and RTADE for the OBXII-IR model with θ= 0.8, λ =0.6, δ=0.04

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
30	Mean	$\hat{\theta}$	1.05826	0.94273	0.94718	0.94525	0.96598
		$\hat{\lambda}$	0.60297	0.65263	0.63025	0.64880	0.70797
		$\hat{\delta}$	0.06612	0.06473	0.06366	0.06172	0.06835
	RMSE	$\hat{\theta}$	0.56235	0.51247	0.52328	0.52156	0.54361
		$\hat{\lambda}$	0.93043	0.74755	0.41473	0.48980	0.67585
		$\hat{\delta}$	0.05622	0.06375	0.06317	0.05847	0.07097
	Bias	$\hat{\theta}$	0.25826	0.14278	0.14718	0.14525	0.16598
		$\hat{\lambda}$	0.09978	0.05263	0.03025	0.04880	0.10797
		$\hat{\delta}$	0.02612	0.02473	0.02366	0.02172	0.02835
60	Mean	$\hat{\theta}$	0.97080	0.91596	0.92372	0.92372	0.93315
		$\hat{\lambda}$	0.59187	0.60914	0.60432	0.60432	0.64690
		$\hat{\delta}$	0.05643	0.05615	0.05513	0.05513	0.05836
	RMSE	$\hat{\theta}$	0.43968	0.44145	0.43672	0.43672	0.46115

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
		$\hat{\lambda}$	0.25992	0.28995	0.26301	0.26301	0.35883
		$\hat{\delta}$	0.36741	0.04232	0.03922	0.03922	0.04480
	Bias	$\hat{\theta}$	0.17080	0.11596	0.12372	0.12372	0.13315
		$\hat{\lambda}$	0.08125	0.00914	0.00439	0.04329	0.04690
		$\hat{\delta}$	0.01643	0.01615	0.01513	0.01513	0.01836
120	Mean	$\hat{\theta}$	0.90625	0.88048	0.86989	0.86640	0.90161
		$\hat{\lambda}$	0.59003	0.59514	0.60369	0.60784	0.61047
		$\hat{\delta}$	0.05026	0.05025	0.04844	0.04819	0.05278
	RMSE	$\hat{\theta}$	0.33504	0.34414	0.32426	0.32667	0.36360
		$\hat{\lambda}$	0.20016	0.20362	0.20554	0.20300	0.25393
		$\hat{\delta}$	0.26197	0.02852	0.02512	0.02523	0.03140
	Bias	$\hat{\theta}$	0.10625	0.08048	0.06989	0.06640	0.10161
		$\hat{\lambda}$	0.07996	0.00485	0.00369	0.02784	0.03047
		$\hat{\delta}$	0.01026	0.01025	0.00844	0.00819	0.01278
200	Mean	$\hat{\theta}$	0.84625	0.84490	0.83902	0.83601	0.86143
		$\hat{\lambda}$	0.62523	0.61122	0.61671	0.61898	0.62479
		$\hat{\delta}$	0.04515	0.04597	0.04504	0.04476	0.04797
	RMSE	$\hat{\theta}$	0.28932	0.29631	0.28159	0.27923	0.32075
		$\hat{\lambda}$	0.20655	0.17990	0.18135	0.17964	0.22410
		$\hat{\delta}$	0.01991	0.02203	0.02002	0.01960	0.02462
	Bias	$\hat{\theta}$	0.04625	0.04490	0.03902	0.03601	0.06143
		$\hat{\lambda}$	0.02523	0.00312	0.00267	0.01898	0.02479
		$\hat{\delta}$	0.00751	0.00597	0.00504	0.00476	0.00797
300	Mean	$\hat{\theta}$	0.84089	0.84160	0.84251	0.83657	0.86145
		$\hat{\lambda}$	0.61777	0.60688	0.60582	0.61105	0.61077
		$\hat{\delta}$	0.04424	0.04507	0.04454	0.04418	0.04707
	RMSE	$\hat{\theta}$	0.25700	0.26815	0.25207	0.25225	0.28736
		$\hat{\lambda}$	0.18364	0.16280	0.15814	0.16152	0.19050
		$\hat{\delta}$	0.01729	0.01905	0.01722	0.01710	0.02131
	Bias	$\hat{\theta}$	0.04089	0.04160	0.04251	0.03557	0.06135
		$\hat{\lambda}$	0.01777	0.00188	0.00182	0.01105	0.01077
		$\hat{\delta}$	0.00424	0.00507	0.00454	0.00418	0.00707

Table 7: Mean, RMSE, and Bias with the MLE, LSE, WLSE, ADE, and RTADE for the OBXII-IR model with θ= 0.9, λ =0.6, δ=0.03

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
30	Mean	$\hat{\theta}$	1.16434	1.03598	1.03778	1.04660	1.0968
		$\hat{\lambda}$	0.68422	0.64359	0.66599	0.63990	0.66493
		$\hat{\delta}$	0.04806	0.05006	0.04740	0.04587	0.05188
	RMSE	$\hat{\theta}$	0.59425	0.58895	0.58047	0.57847	0.61265
		$\hat{\lambda}$	1.05648	0.56634	0.99709	0.40608	0.59367
		$\hat{\delta}$	0.03858	0.05644	0.04813	0.04330	0.05312
	Bias	$\hat{\theta}$	0.26434	0.13598	0.13778	0.14660	0.19684
		$\hat{\lambda}$	0.08422	0.04359	0.06599	0.03990	0.06493
		$\hat{\delta}$	0.01806	0.02006	0.01740	0.01587	0.02188
60	Mean	$\hat{\theta}$	1.05505	1.01162	0.99612	1.00039	1.05244
		$\hat{\lambda}$	0.59705	0.60541	0.61690	0.61723	0.62031
		$\hat{\delta}$	0.04036	0.04178	0.03979	0.03924	0.04393
	RMSE	$\hat{\theta}$	0.46242	0.48604	0.47048	0.46655	0.50293

n		Est. Par.	MLE	LSE	WLSE	ADE	RTADE
		$\hat{\lambda}$	0.24046	0.28006	0.25183	0.24536	0.31164
		$\hat{\delta}$	0.02597	0.03314	0.02993	0.02769	0.03348
	Bias	$\hat{\theta}$	0.15505	0.11162	0.01612	0.10039	0.15242
		$\hat{\lambda}$	0.02947	0.03541	0.01690	0.01723	0.02031
		$\hat{\delta}$	0.01036	0.01178	0.01979	0.01924	0.01393
120	Mean	$\hat{\theta}$	1.00448	0.99315	0.99070	0.98012	1.00920
		$\hat{\lambda}$	0.60945	0.59874	0.60120	0.61106	0.61766
		$\hat{\delta}$	0.03666	0.03766	0.03672	0.03610	0.03875
	RMSE	$\hat{\theta}$	0.39001	0.40243	0.38490	0.38227	0.41721
		$\hat{\lambda}$	0.22144	0.20869	0.20017	0.20817	0.25671
		$\hat{\delta}$	0.01913	0.02200	0.01974	0.01905	0.02287
	Bias	$\hat{\theta}$	0.10448	0.09315	0.09070	0.08012	0.10920
		$\hat{\lambda}$	0.00944	0.00425	0.00320	0.04106	0.01766
		$\hat{\delta}$	0.00666	0.00766	0.00672	0.00610	0.00875
200	Mean	$\hat{\theta}$	0.94956	0.94881	0.93579	0.93149	0.97705
		$\hat{\lambda}$	0.62067	0.60966	0.62005	0.62330	0.61075
		$\hat{\delta}$	0.03383	0.03468	0.03368	0.03341	0.03633
	RMSE	$\hat{\theta}$	0.31878	0.33737	0.32375	0.32061	0.35392
		$\hat{\lambda}$	0.19744	0.18074	0.18532	0.18538	0.20468
		$\hat{\delta}$	0.01475	0.01687	0.01531	0.01496	0.01824
	Bias	$\hat{\theta}$	0.04956	0.04881	0.03579	0.03149	0.07705
		$\hat{\lambda}$	0.00606	0.00366	0.00205	0.00233	0.01075
		$\hat{\delta}$	0.00383	0.00468	0.00368	0.00341	0.00633
300	Mean	$\hat{\theta}$	0.90307	0.94215	0.93968	0.93023	0.97195
		$\hat{\lambda}$	0.62904	0.60699	0.61005	0.61746	0.60256
		$\hat{\delta}$	0.03253	0.03355	0.03307	0.03267	0.03526
	RMSE	$\hat{\theta}$	0.29023	0.29300	0.27810	0.28009	0.31191
		$\hat{\lambda}$	0.19258	0.16111	0.16098	0.16593	0.17319
		$\hat{\delta}$	0.01287	0.01370	0.01241	0.01234	0.01553
	Bias	$\hat{\theta}$	0.03070	0.04215	0.03968	0.03023	0.07195
		$\hat{\lambda}$	0.00290	0.00299	0.00105	0.00174	0.00256
		$\hat{\delta}$	0.00253	0.00355	0.00307	0.00267	0.00526

Table 8: Descriptive analysis of the two data

Dataset	N	Min.	Median	Mean	SD	Max.	SK	KU
D1	20	1.1	1.7	1.9	0.7	4.1	1.59	2.35
D2	69	1.31	2.48	2.45	0.5	3.58	-0.03	0.14

We use many well-known criteria to make sure that our comparisons are fair. These include the log-likelihood-based criteria minus estimated $-\ell$, Akaike information criterion (AIC), consistent AIC (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC), as well as Cramer Von-Mises (W), Anderson-Darling (A), Kolmogorov-Smirnov (KS), and its corresponding p-value. According to Tables 9 and 10, the OBXII-IR model is the most suitable for D1 due to its lowest values of AIC, CAIC, BIC, HQIC, W, A, and KS and the highest p-value. Additionally, the OBXII-IR model's p-value is close to 1, making it difficult to surpass using the D1 comparison. The OBXII-IR model shows excellent fits in Figures 4, 5, and 6. The curves closely match the corresponding empirical ones in all five plots, demonstrating the suitability of the OBXII-IR model for D1. According to Tables **??** and 12, the OBXII-IR model is the most suitable for D2 due to its lowest values of AIC, CAIC, BIC, HQIC, W, A, and KS and the highest p-value. Additionally, the OBXII-IR model's p-value is close to 1, making it difficult to surpass using the D2 comparison. The OBXII-IR model shows excellent fits in Figures 7, 8, and 9. The curves closely match the corresponding empirical ones in all five plots, demonstrating the suitability of the OBXII-IR model for D2.

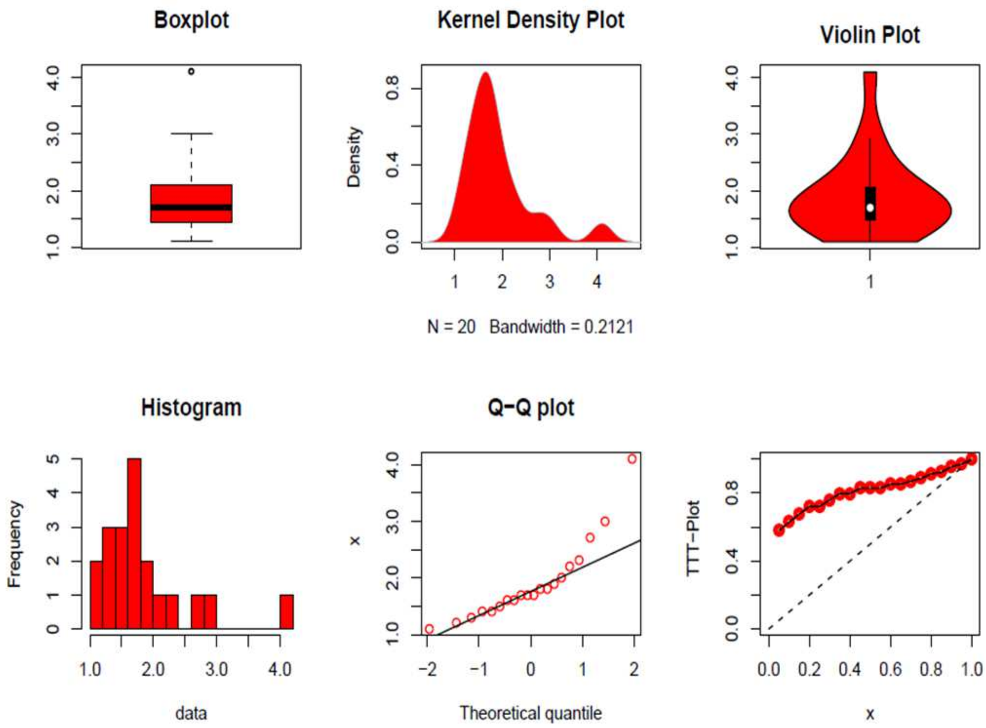

Fig. 2: Show D1 and its visualization

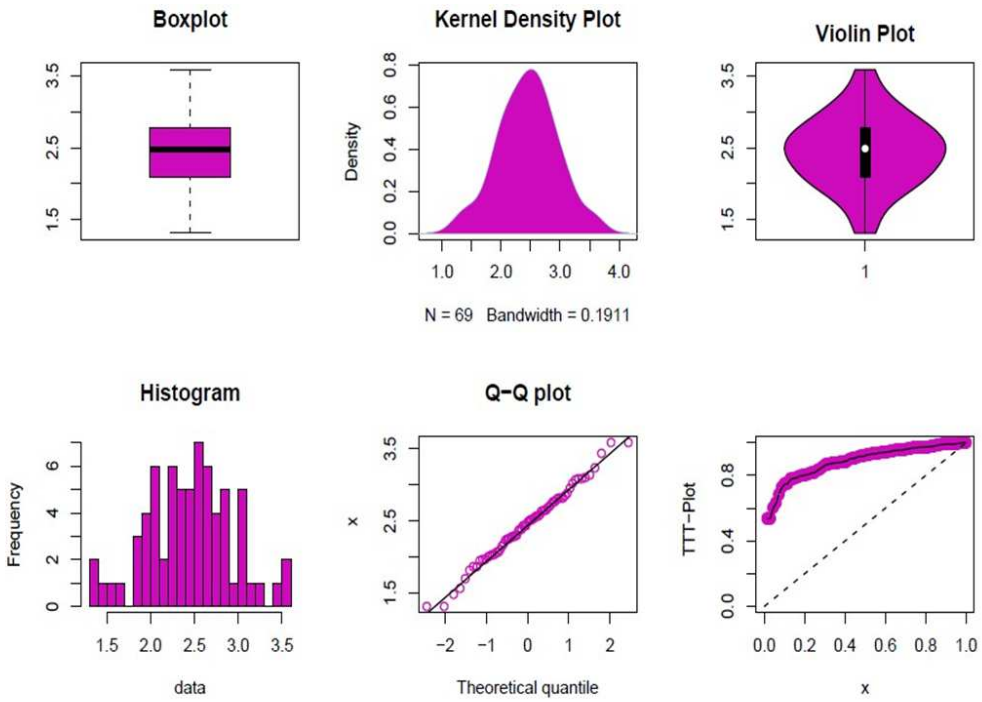

Fig. 3: Show D2 and its visualization

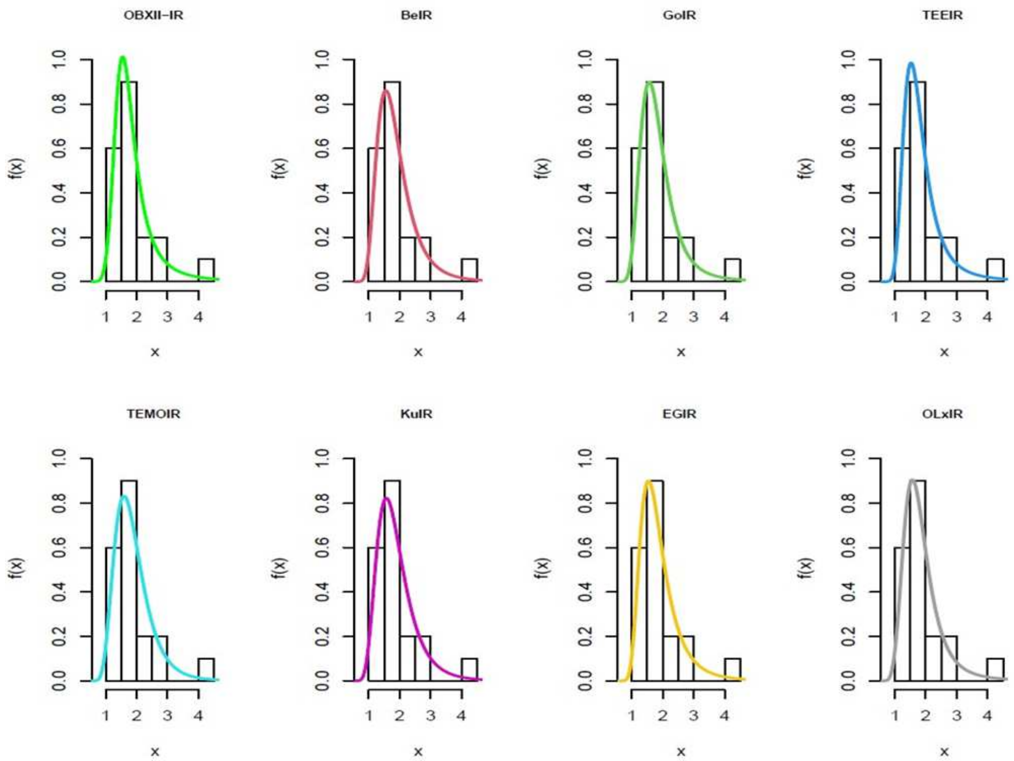

Fig. 4: The plot displays the epdf for the comparison models and the OBXII-IR model for D1

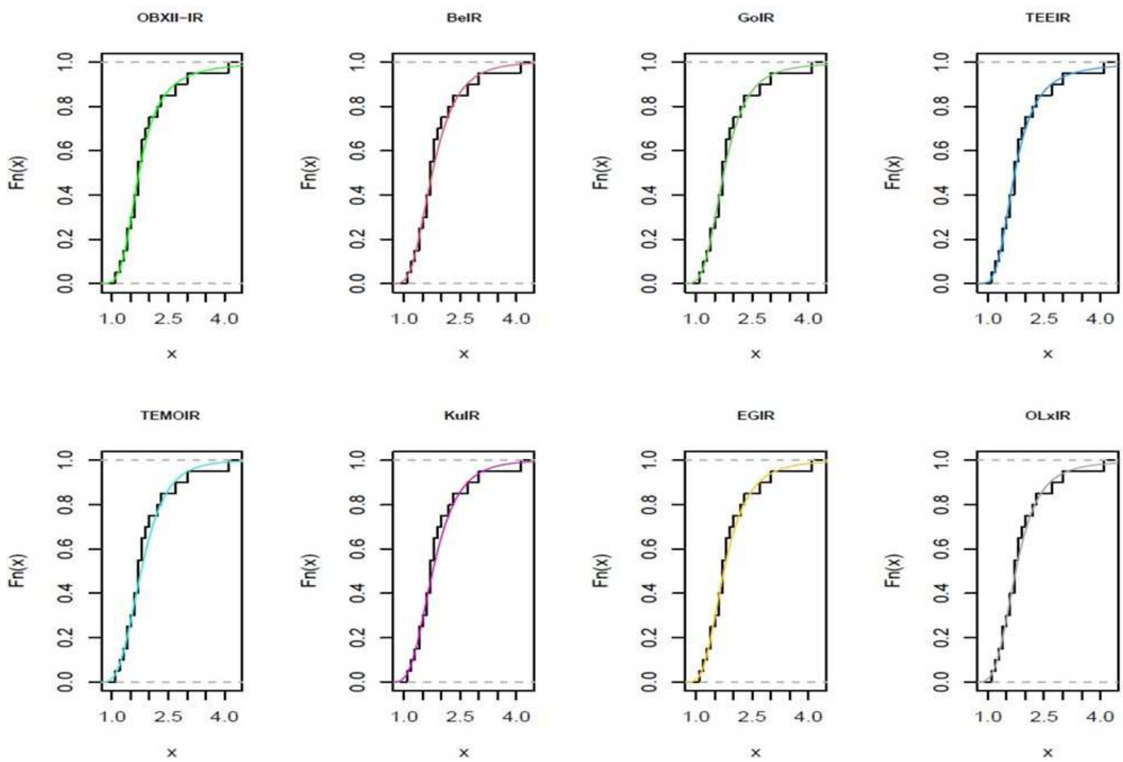

Fig. 5: The plot displays the ecdf for the comparison models and the OBXII-IR model for D1

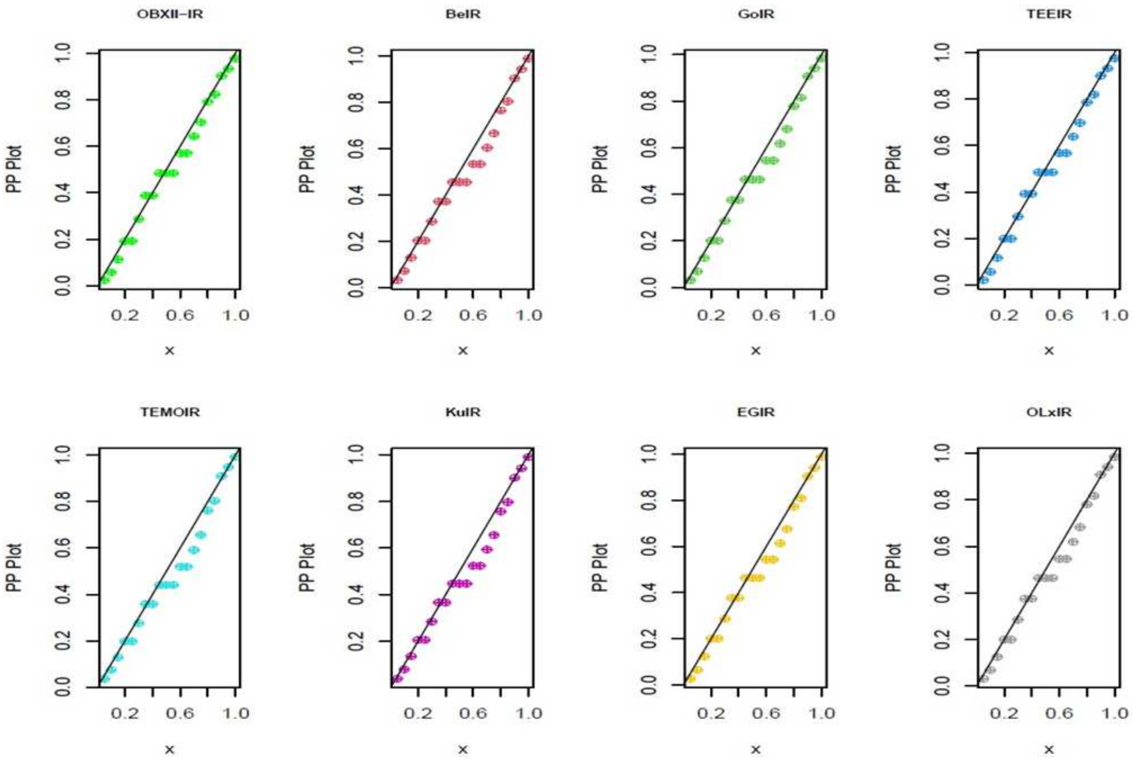

Fig. 6: Estimated PP plots for D1

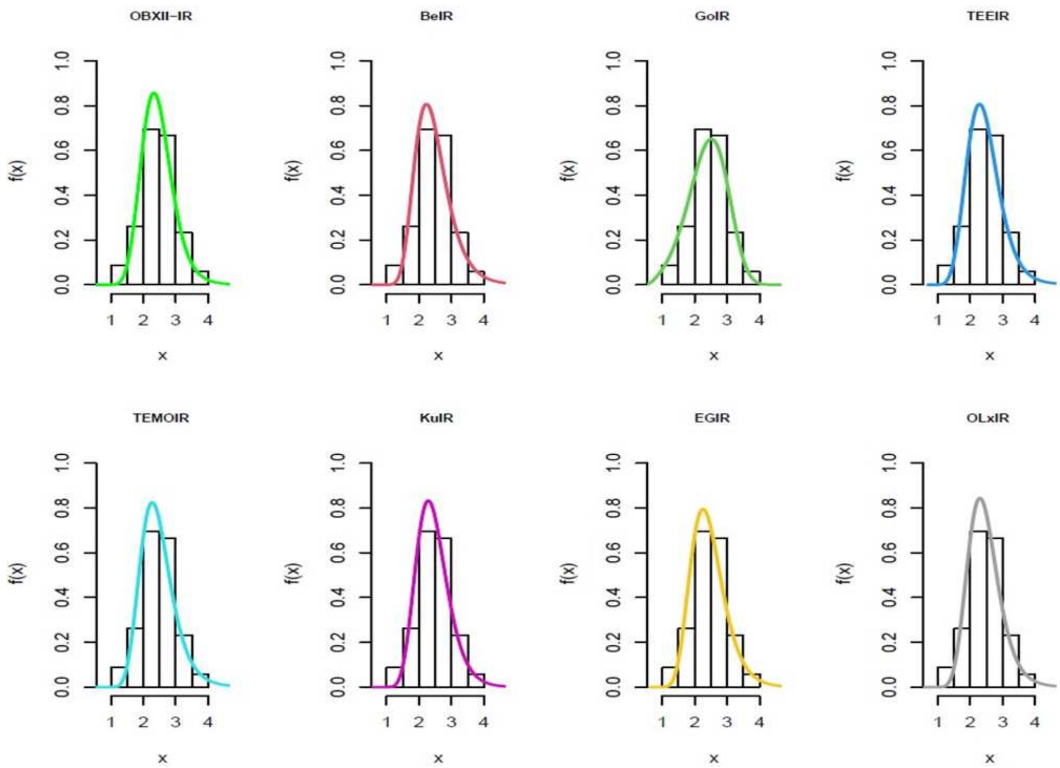

Fig. 7: The plot displays the epdf for the comparison models and the OBXII-IR model for D2

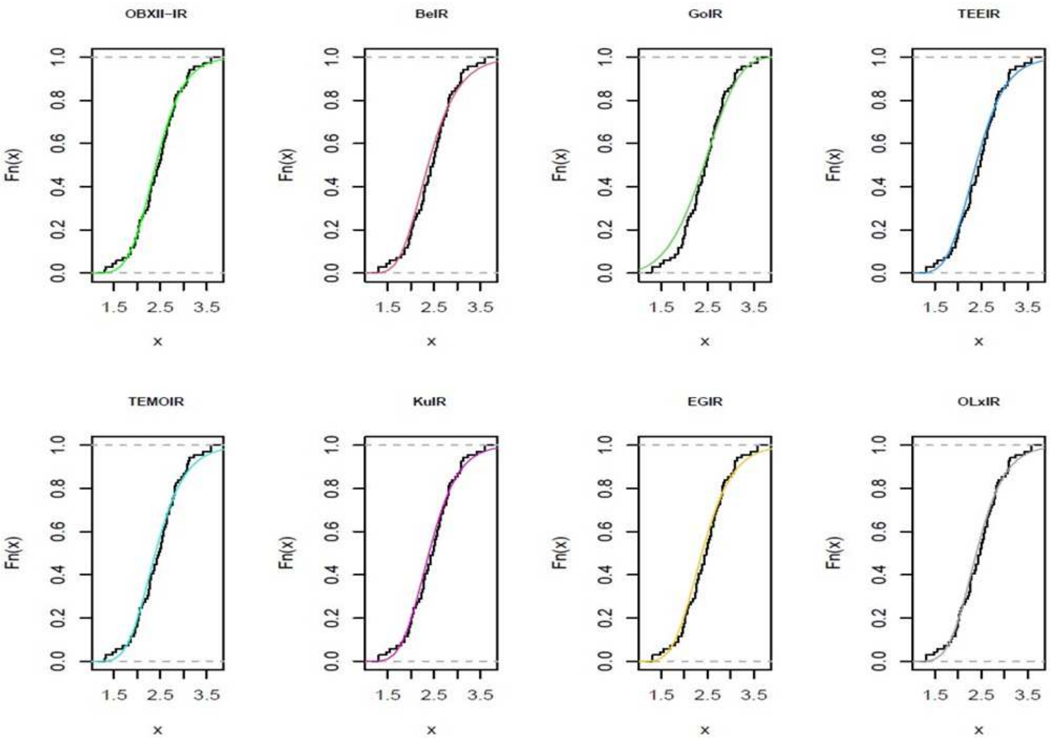

Fig. 8: The plot displays the ecdf for the comparison models and the OBXII-IR model for D2

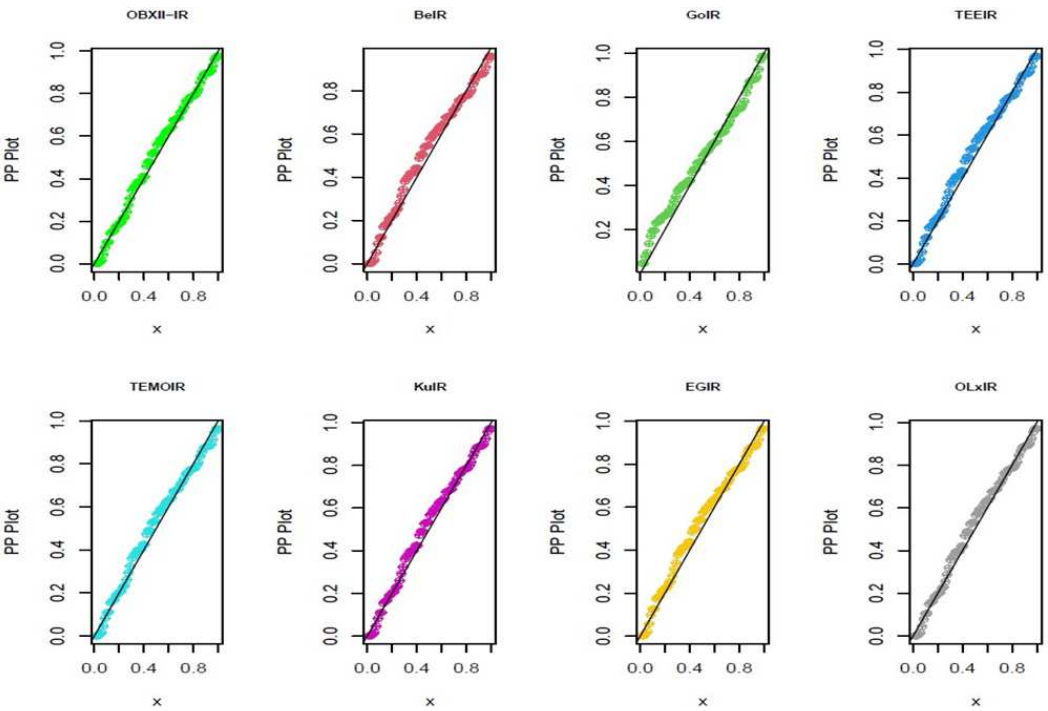

Fig. 9: Estimated PP plots for D2

Table 9: MLEs, and statistic measures for different models for D1

Model	MLEs	W	A	KS	P-value
OBXII-IR	$\widehat{\theta}$:0.72660 $\widehat{\lambda}$:2.09433 $\widehat{\delta}$:1.24409	0.02210	0.13172	0.08779	0.99786
BeIR	$\widehat{\theta}$:2.45646 $\widehat{\lambda}$:3.34255 $\widehat{\delta}$:2.71783	0.03666	0.21063	0.11661	0.94849
GoIR	$\widehat{\theta}$:5.14200 $\widehat{\lambda}$:0.72114 $\widehat{\delta}$:6.15427	0.02900	0.16818	0.10535	0.97946
TEEIR	$\widehat{\theta}$:1.94723 $\widehat{\lambda}$:5.15697 $\widehat{\delta}$:4.32062	0.02383	0.14002	0.09205	0.99579
TEMOIR	$\widehat{\theta}$:1.25616 $\widehat{\lambda}$:2.87414 $\widehat{\delta}$:5.53980	0.04478	0.25922	0.13099	0.88245
KuIR	$\widehat{\theta}$:2.44178 $\widehat{\lambda}$:3.60980 $\widehat{\delta}$:2.23415	0.04207	0.24358	0.12642	0.90654
EGIR	$\widehat{\theta}$:2.77566 $\widehat{\lambda}$:3.26838 $\widehat{\delta}$:2.43669	0.03104	0.17727	0.10699	0.97605
OLxIR	$\widehat{\theta}$:4.91534 $\widehat{\lambda}$:1.16251 $\widehat{\delta}$:2.98105	0.02860	0.16629	0.10407	0.98181

Table 10: The log-likelihood and goodness-of-fit measures for different models for D1

model	$-\ell$	AIC	CAIC	BIC	HQIC
OBXII-IR	15.4817	36.9634	38.4634	39.9506	37.5465
BeIR	15.6864	37.3731	38.8731	40.3603	37.9562
GoIR	15.6778	37.3556	38.8556	40.3428	37.9388
TEEIR	15.5232	37.0465	38.5465	40.0337	37.6296
TEMOIR	15.9887	37.9832	39.4832	40.9704	38.5663
KuIR	15.8679	37.7359	39.2359	40.7231	38.3190
EGIR	15.5301	37.0606	38.5606	40.0478	37.6437
OLxIR	15.6459	37.2920	38.7920	40.2792	37.8751

Table 11: MLEs, and statistic measures for different models for D2

Model	MLEs	W	A	KS	P-value
OBXII-IR	$\widehat{\theta}$:4.59054 $\widehat{\lambda}$:1.25114 $\widehat{\delta}$:5.58651	0.07104	0.51064	0.06730	0.91337
BeIR	$\widehat{\theta}$:1.57190 $\widehat{\lambda}$:6.96084 $\widehat{\delta}$:1.35851	0.12968	0.89399	0.09077	0.62035
GoIR	$\widehat{\theta}$:0.07430 $\widehat{\lambda}$:2.56989 $\widehat{\delta}$:1.97667	0.09729	0.72668	0.11446	0.32654
TEMOIR	$\widehat{\theta}$:3.19275 $\widehat{\lambda}$:1.26660 $\widehat{\delta}$:5.60693	0.10853	0.75643	0.07630	0.81657
KuIR	$\widehat{\theta}$:3.93545 $\widehat{\lambda}$:1.07189 $\widehat{\delta}$:3.94398	0.09195	0.64913	0.07732	0.80376

Model	MLEs	W	A	KS	P-value
EGIR	$\widehat{\theta}$:2.77767 $\widehat{\lambda}$:0.95988 $\widehat{\delta}$:1.84846	0.09770	0.68224	0.08766	0.66397
OLxIR	$\widehat{\theta}$:1.01262 $\widehat{\lambda}$:2.02790 $\widehat{\delta}$:1.68896	0.09209	0.64941	0.07549	0.82648

Table 12: The log-likelihood and goodness-of-fit measures for different models for D2

model	$-\ell$	AIC	CAIC	BIC	HQIC
OBXII-IR	51.4150	108.830	109.199	115.532	111.489
BeIR	53.4917	112.989	113.358	119.691	115.648
GoIR	51.8931	109.801	110.171	116.504	112.460
TEEIR	51.7705	109.576	109.946	116.279	112.235
TEMOIR	52.7616	111.526	111.896	118.229	114.185
KuIR	52.0717	110.143	110.512	116.846	112.802
EGIR	52.1980	110.399	110.768	117.101	113.058
OLxIR	52.1812	110.362	110.731	117.065	113.021

13 Conclusions

This article introduced a modified version of the inverse Rayleigh model by including the Odd Burr XII-G family of distributions. The study focused on analyzing statistical properties such as moments, incomplete moments, quantile function, order statistics, harmonic mean, and Rényi entropy for the proposed OBXII-IR distribution. This study included five distinct estimating methodologies to examine the behavior of the parameters in the OBXII-IR model. To ascertain the most effective estimators, researchers used Monte Carlo simulation. Based on an empirical analysis of actuarial statistics, the OBXII-IR distribution exhibits a higher degree of tail heaviness compared to the BeIR, GoIR, TEEIR, TEMOIR, KuIR, EGIR, and OLxIR distributions. The OBXII-IR distribution demonstrates superior performance compared to seven other models in real-world scenarios, namely in medical and carbon fiber data.

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References

[1] J. Leao, H. Saulo, M. Bourguignon, R.J. Cintra, L.C. Rego, and G.M. Cordeiro, On some properties of the beta inverse Rayleigh distribution, *arXiv*, **4**, 111-131 (2022).

[2] A. Ahmad, S.P. Ahmad, A. Ahmed, Transmuted inverse Rayleigh distribution: A generalization of the inverse Rayleigh distribution, *Math. Theory Model.*, **4**(7), 90-98 (2014).

[3] K. Fatima and S.P. Ahmad, Weighted inverse Rayleigh distribution, *Int. J. Stat. Syst.*, **12**(1), 119-137 (2017).

[4] M. Elgarhy, S. Alrajhi, The odd Fréchet inverse Rayleigh distribution: Statistical properties and applications, *J. Nonlinear Sci. Appl.*, **12**, 291-299 (2019).

[5] H. F. Mohammed, N. Yahia, On type II topp-Leone inverse Rayleigh distribution, *Applied Mathematical Sciences*, **13**(13), 607-615 (2019).

[6] G. S. Rao, S. Mbwambo, Exponentiated inverse Rayleigh distribution and an application to coating weights of iron sheets data, *Journal of Probability and Statistics*, **2019**(1), 7519429 (2019).

[7] A. M. Almarashi, M. M. Badr, M. Elgarhy, F. Jamal, C. Chesneau, Statistical inference of the half-logistic inverse Rayleigh distribution, *Entropy*, **22**(4), 449 (2020).

[8] M. Ali, A. Khalil, M. Ijaz, N. Saeed, Alpha-Power Exponentiated Inverse Rayleigh distribution and its applications to real and simulated data, *PloS One*, **16**(1), e0245253 (2021).

[9] E. M. Almetwally, Extended odd Weibull inverse Rayleigh distribution with application on carbon fibres, *Math. Sci. Lett.*, **10**(1), 5-14 (2021).

[10] E. S. A. El-Sherpieny, H. Z. Muhammed, E. M. Almetwally, A new inverse Rayleigh distribution with applications of COVID-19 data: Properties, estimation methods and censored sample, *Electronic Journal of Applied Statistical Analysis*, **16**(2), 449-472 (2023).

[11] A. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of continuous distributions, *Metron*, **71**(1), 63-79 (2013).

[12] E. Eugene, C. Lee, F. Famoye, Beta-normal distribution and its applications, *Commun. Stat.-Theory Meth.*, **31**(4), 497-512 (2002).

[13] G. M. Cordeiro, M. De Castro, A new family of generalized distributions, *J. Stat. Comput. Simul.*, **81**(7), 883-898 (2011).

[14] K. Jayakumar, T. Mathew, On a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution, *Stat. Pap.*, **49**(3), 421 (2008).

[15] B. Dias, C. R. Cordeiro, G. M. Cordeiro, M. Alizadeh, P. R. Diniz Marinho, H. F. Campos Coélho, Exponentiated Marshall-Olkin family of distributions, *J. Stat. Distrib. Appl.*, **3**, 1-21 (2016).
[16] R. C. Gupta, R. D. Gupta, Proportional reversed hazard rate model and its applications, *J. Stat. Plann. Inference*, **137**(11), 3525-3536 (2007).
[17] M. M. Risti?, N. Balakrishnan, The gamma-exponentiated exponential distribution, *J. Stat. Comput. Simul.*, **82**(8), 1191-1206 (2012).
[18] K. Zografos, N. Balakrishnan, On families of beta-and generalized gamma-generated distributions and associated inference, *Stat. Methodol.*, **6**(4), 344-362 (2009).
[19] H. Torabi, N. M. Hedesh, The gamma-uniform distribution and its applications, *Kybernetika*, **48**(1), 16-30 (2012).
[20] H. Torabi, N. H. Montazeri, The logistic-uniform distribution and its applications, *Commun. Stat.-Simul. Comput.*, **43**(10), 2551-2569 (2014).
[21] A. A. Khalaf, The New Strange Generalized Rayleigh Family: Characteristics and Applications to COVID-19 Data, *Iraqi J. Comput. Sci. Math.*, **5**(3), 92-107 (2024).
[22] R. S. Gomaa, A. M. Magar, N. Alsadat, E. M. Almetwally, A. H. Tolba, The Unit Alpha-Power Kum-Modified Size-Biased Lehmann Type II Distribution: Theory, Simulation, and Applications, *Symmetry*, **15**(6), 1283 (2023).
[23] D. D. Ahmed, M. A. Khaleel, The Gompertz Nadarajah-Haghighi (GoNH) Distribution Properties with Application to Real Data, *Iraqi J. Comput. Sci. Math.*, **5**(3) (2024).
[24] A. H. Tolba, A. H. Muse, A. Fayomi, H. M. Baaqeel, E. M. Almetwally, The Gull Alpha Power Lomax distributions: Properties, simulation, and applications to modeling COVID-19 mortality rates, *PloS One*, **18**(9), e0283308 (2023).
[25] S. O. Bashiru, A. A. Khalaf, A. M. Isa, A. Kaigama, ON MODELING OF BIOMEDICAL DATA WITH EXPONENTIATED GOMPERTZ INVERSE RAYLEIGH DISTRIBUTION, *Reliab. Theory Appl.*, **19**(3 (79)), 460-475 (2024).
[26] N. A. Nooria, A. A. Khalafb, M. A. Khaleelc, A New Generalized Family of Odd Lomax-G Distributions Properties and Applications, *Advances in the Theory of Nonlinear Analysis and Its Applicationm*, **7**, 1-16 (2023).
[27] G. A. Mahdi, M. A. Khaleel, A. M. Gemeay, M. Nagy, A. H. Mansi, M. M. Hossain, E. Hussam, A new hybrid odd exponential-Φ family: Properties and applications, AIP Adv. **14**(4) (2024).
[28] A. A. Khalaf, [0, 1] Truncated exponentiated exponential gompertz distribution: Properties and applications, *AIP Conf. Proc.*, **2394**(1), 070035 (2022).
[29] H. S. Jabarah, A. H. Tolba, A. T. Ramadan, A. I. El-Gohary, The Truncated Unit Chris-Jerry Distribution and Its Applications, *Appl. Math.*, **18**(6), 1317-1330 (2024).
[30] A. H. Tolba, C. K. Onyekwere, A. R. El-Saeed, N. Alsadat, H. Alohali, O. J. Obulezi, A New Distribution for Modeling Data with Increasing Hazard Rate: A Case of COVID-19 Pandemic and Vinyl Chloride Data, *Sustainability*, **15**(17), 12782 (2023).
[31] K. Aidi, N. Seddik-Ameur, A. Ahmed, M. A. Khaleel, The Topp-Leone Extended Exponential Distribution: Statistical properties, different estimation methods and applications to life time data, *Pak. J. Stat. Oper. Res.*, **18**(4), 817-836 (2022).
[32] J. J. Swain, S. Venkatraman, J. R. Wilson, Least-squares estimation of distribution functions in Johnson's translation system, *J. Stat. Comput. Simul.*, **29**(4), 271-297 (1988).
[33] M. E. Mead, On five-parameter Lomax distribution: Properties and applications, Pak. J. Stat. Oper. Res., **12**(1), 185-199 (2016).
[34] S. O. Bashiru, A. A. Khalaf, A. M. Isa, Topp-Leone Exponentiated Gompertz Inverse Rayleigh Distribution: Properties and Applications, *Reliab. Theory Appl.*, **19**(3) (79), 59-77 (2024).
[35] D. Kundu, M. Z. Raqab, Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution, *Stat. Prob. Lett.*, **79**(17), 1839-1846 (2009).
[36] H. O. Yusuf, S. O. Bashiru, A Classical and Bayesian Estimation Techniques for Gompertz Inverse Rayleigh Distribution: Properties and Application, *Pak. J. Stat.*, **38**(1), 49-76 (2022).
[37] M. Hussian, E. A. Amin, Estimation and Prediction for the Kumaraswamy-Inverse Rayleigh Distribution Based on Records, *Int. J. Adv. Stat. Prob.*, **2**(1), 21-27 (2014).
[38] K. Fatima, S. Naqash, S. P. Ahmad, Exponentiated Generalized Inverse Rayleigh Distribution with Applications in Medical Sciences, *Pak. J. Stat.*, **34**(5), 425-439 (2018).

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