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Inversion and Normalization of Time-Frequency Transform

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Abstract: This study defines a time-frequency transform (TFT) covering Gabor transform (GT) and wavelet transform (WT) and presents two inversion formulas for the TFT. By one of the inversion formulas, we find a new inversion formula for the WT, which suggests a new more general wavelet definition free from the conventional admissibility (e.g. zero integration). This study also develops a concept of normal TFT for precise time-frequency analysis, as we show that the normal TFT is unique in unbiased explication of the immediate frequency, amplitude and phase of a time harmonic. The GT should be updated (i.e. normalized) in phase to become a normal TFT, because it does not explicate the immediate but initial phase of a time harmonic. The WT should be written in L1-norm to become a normal TFT, as the non-L1-norm WT of a time harmonic yields a frequency-biased spectrum. Finally, we show how to measure the frequency resolution of a normal TFT.

Keywords: Time-frequency transform (TFT), Gabor transform (GT), Wavelet transform (WT), Inversion, Normalization

1 Introduction

For time function $f(t)\in C$ on R, its time-frequency transform (TFT) can be expressed as

$$\Psi f(\tau,\varpi)=\int_R f(t)\bar{\Psi}(t-\tau,\varpi)d\,t,\ \tau,\varpi\in R \qquad (1.1)$$

where τ is the time index and ϖ the frequency index, $\Psi(t,\varpi)$ is the transform kernel, line "–" denotes the conjugate operator, C is the complex field and R the real filed. By one to one relation $\varpi=1/a$ where a denotes the scale index, the TFT (1.1) turns into the time-scale transform. Thus the TFT and the time-scale transform are equivalent to each other. However, the TFT differs from the time-scale-frequency transform appearing in the atomic decomposition theory [1][9], because the latter employs simultaneously both scale and frequency indexes independent of each other.

Gabor transform (GT) [3][11] and Wavelet transform (WT)[4][5][10] are two typical TFTs. Letting $\Psi(t,\varpi)=w(t)\exp(i\varpi t)$, where $w(t)\in L^1(R)$ is a window, the TFT (1.1) becomes a phase-updated GT:

$$\begin{aligned}\mathrm{G}f(\tau,\varpi)&=\int_R f(t)\overline{w(t-\tau)\exp(i\varpi(t-\tau))}\,\mathrm{d}\,t\\&=\exp(i\varpi\tau)\int_R f(t)\overline{w(t-\tau)\exp(i\varpi t)}\,\mathrm{d}\,t \qquad (1.2)\\&=\exp(i\varpi\tau)\mathrm{G}^*f(\tau,\varpi)\end{aligned}$$

where G* denotes the conventional GT. Transforms G and G* differ in phase because the former translates the harmonic kernel $\exp(i\varpi t)$ and updates the harmonic phase while the latter does not. Letting $\Psi(t,\varpi)=|\varpi|\psi(\varpi t)$ where $\psi(t)$ is the fundamental wavelet and $|\cdot|$ denotes the modulus operator, the TFT (1.1) becomes a L^1-norm WT:

$$\mathrm{W}_\psi f(\tau,\varpi)=|\varpi|\int_R f(t)\bar{\psi}(\varpi(t-\tau))\mathrm{d}\,t \qquad (1.3)$$

The natural difference between the GT and the WT is that the time resolution of the former is fixed while the time resolution of the later varies with the frequency index.

There are two open questions regarding to the TFT: i) how to invert the TFT ? ii) what kind of TFT is precise for time-frequency analysis? The first question is significant to harmonic analysis theory. Although the GT and the WT, two special

TFTs, hold their respective inversion formulas [11][3], there has been no inversion formulas for the general TFT (1.1). The second question is significant to harmonic analysis application. Consider the fact that there can be numerous TFTs as the transform kernel $\Psi(t,\varpi)$ can be of numerous forms. There has been no theoretical criterion or normality for time-frequency analyzers to construct a TFT precise for time-frequency analysis.

This study (section 2) tries to answer the first question by presenting the inversion formulas for the general TFT (1.1). We give two inversion formulas, the first being general and the second being specific. By the specific inversion formula, we find a new inversion formula for the WT. A surprising fact is that the new WT inversion formula is free from the wavelet admissibility constraining the well-known WT inversion formula. Note the fact that the admissibility defines the conventional wavelet concept. For example, a conventional wavelet requires to be evenly undulant (i.e. be of zero integration) to meet the admissibility. Our finding suggests a new more general wavelet definition free from such admissibility: any local time signal undulant with a significant frequency 1 can be regarded as a wavelet, no matter its undulation is even or not.

This study (section 3) tries to answer the second question by developing the concept of normal TFT. For a time-frequency analyzer, his task is to construct a proper TFT and apply it to a time signal to identify the component harmonics and quasi-harmonics of the signal. A harmonic evolves only in phase but not in frequency or amplitude while a quasi-harmonic evolves not only in phase but also in frequency or amplitude. To identify a component harmonic or quasi-harmonic is to identify the immediate frequency, amplitude and phase of the harmonic or quasi-harmonic. So, a TFT proper for precise time-frequency analysis should be unbiased in identifying i.e. explicating the immediate frequency, amplitude and phase of a harmonic. We will show that the normal TFT is unique in unbiased explication of the immediate frequency, amplitude and phase of a harmonic, and should be precise in explicating the immediate frequency, amplitude and phase of a quasi-harmonic. The normal TFT concept suggests that the GT and the WT should be normalized when applied to time-frequency analysis. At first, the conventional GT needs to update in phase to become a normal TFT, as we show that the conventional GT of a harmonic does not explicate the immediate but initial phase of the harmonic. Consider the fact that, for a quasi-harmonic evolving in frequency, its immediate phase is much more significant than initial phase. Secondly, the WT should be written in L^1-norm to become a normal TFT, as we prove that the non-L^1-norm WT of a time harmonic yields a frequency-biased spectrum. This means that, among all the WTs, only the L^1-norm WT is precise for time-frequency analysis. However, many authors pay little attention to the WT normalization because the normalization factor enters in a simple multiplicative way in the WT. This study shows that the WT normalization is significant and necessary to precise time-frequency analysis.

Although the two open questions seem independent of each other, this study shows their inner relation. By the specific TFT inversion formula, the normal TFT finds a simple inversion formula unrelated to the TFT kernel. This means that, given the normal TFT of a time signal, one can reconstruct the time signal without knowing the TFT kernel. To say straightly, the specific TFT inversion formula introduces and justifies in theory the concept of normal TFT.

2 Inversion of TFT

Inversion Theorem 1 For time function $f(t)\in C$ on R, its TFT $\Psi f(\tau,\varpi)$ is invertible by

$$f(t)=c_\Psi^{-1}\int_R\int_R \Psi f(\tau,\varpi)I(t-\tau,\varpi)\mathrm{d}\tau\mathrm{d}\varpi,\quad t\in R \tag{2.1}$$

if the inversion kernel $I(t,\varpi)$ satisfies

$$c_\Psi=\int_R \overline{\hat{\Psi}}(\omega,\varpi)\hat{I}(\omega,\varpi)\,\mathrm{d}\varpi=\text{ constant},\ \forall\omega\in R \tag{2.2}$$

where "^" denotes the Fourier transform and "∀" means "for any".

Proof Denote $x=t'-\tau$, $y=t'-t$ and

$$\rho(y)=\int_R\int_R \overline{\Psi}(x-y,\varpi)I(x,\varpi)\mathrm{d}x\mathrm{d}\varpi \tag{2.3}$$

Note that

$$\hat{\rho}(\omega)=\int_R \overline{\hat{\Psi}}(\omega,\varpi)\hat{I}(\omega,\varpi)\mathrm{d}\varpi \tag{2.4}$$

If relation (2.2) holds, one can have that

$$\rho(y)=c_\Psi\delta(y) \tag{2.5}$$

Thus,

$$\begin{aligned}
&\int_R\int_R \Psi f(\tau,\varpi)I(t'-\tau,\varpi)\mathrm{d}\tau\mathrm{d}\varpi\\
&=\int_R\int_R\int_R f(t)\overline{\Psi}(t-\tau,\varpi)\mathrm{d}t I(t'-\tau,\varpi)\mathrm{d}\tau\mathrm{d}\varpi\\
&=\int_R\int_R f(t)\int_R \overline{\Psi}(t-\tau,\varpi)I(t'-\tau,\varpi)\mathrm{d}\tau\mathrm{d}t\mathrm{d}\varpi\\
&=\int_R\int_R f(t'-y)\int_R \overline{\Psi}(x-y,\varpi)I(x,\varpi)\mathrm{d}x\mathrm{d}y\mathrm{d}\varpi
\end{aligned}$$

$$
\begin{aligned}
&= \int_R f(t'-y) \int_R \int_R \overline{\Psi}(x-y,\varpi) I(x,\varpi) \mathrm{d}x \mathrm{d}\varpi \mathrm{d}y \\
&= \int_R f(t'-y) \rho(y) \mathrm{d}y \\
&= c_\Psi \int_R f(t'-y) \delta(y) \mathrm{d}y \\
&= c_\Psi f(t').
\end{aligned}
\tag{2.6}
$$

Here the proof ends.

The TFT inversion formula (2.1) is very general, because it does not provide a specific inversion kernel $I(t,\varpi)$ but a sufficient condition (2.2) of the inversion. Here, the inversion kernel $I(t,\varpi)$ satisfying (2.2) are regarded as admissible. Given a TFT, there may be many admissible inversion kernels $I(t,\varpi)$.

It is well known that, for $f(t) \in C$ on R, its phase-updated GT (1.2) can be inverted by

$$f(t) = c_w^{-1} \int_R \int_R \mathrm{G}f(\tau,\varpi) w(t-\tau) \exp(i\varpi(t-\tau)) \mathrm{d}\tau \mathrm{d}\varpi \tag{2.7}$$

if

$$c_w = \int_R |\hat{w}(\omega)|^2 \mathrm{d}\omega < \infty \tag{2.8}$$

In fact, the GT inversion formula (2.7) is just a simple application of the Inversion Theorem 1 in the case that $\Psi(t,\varpi)=I(t,\varpi)=w(t)\exp(i\varpi t)$.

It is also well known that, for $f(t) \in C$ on R, its WT (1.3) can be inverted by

$$f(t) = c_\psi^{-1} \int_R \int_R \mathrm{W}_\psi f(\tau,\varpi) \psi(\varpi(t-\tau)) \, \mathrm{d}\tau \mathrm{d}\varpi, \quad t \in R \tag{2.9}$$

if there is a wavelet admissibility constant,

$$c_\psi = \int_R |\hat{\psi}(\omega)|^2 |\omega|^{-1} \mathrm{d}\omega < +\infty \tag{2.10}$$

In fact, the WT inversion formula (2.9) is just a simple application of the Inversion Theorem 1 in the case that $\Psi(t,\varpi)=|\varpi|\psi(\varpi t)$ and $I(t,\varpi)=\psi(\varpi t)$. A fundamental wavelet $\psi(t)$ satisfying (2.10) is conventionally called an admissible wavelet. A necessary condition of the admissible wavelet $\psi(t)$ is that $\hat{\psi}(0)=0$, i.e. ψ(t) should be evenly undulant.

Inversion Theorem 2 For $f(t) \in L^1(R)$ satisfying $\hat{f}(\omega) \in L^1(R)$, its TFT $\Psi f(\tau,\varpi)$ is invertible by

$$f(t) = \frac{1}{2\pi} \int_R \int_R \Psi f(\tau,\varpi) [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \exp(i\varpi(t-\tau)) \mathrm{d}\tau \mathrm{d}\varpi, \quad t \in R \tag{2.11}$$

if

$$0 < |\hat{\Psi}(\varpi,\varpi)| < \infty \tag{2.12}$$

Proof

$$
\begin{aligned}
&\int_R \int_R \Psi f(\tau,\varpi) [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \exp(i\varpi(t'-\tau)) \mathrm{d}\tau \mathrm{d}\varpi \\
&= \int_R \int_R \int_R f(t) \overline{\Psi}(t-\tau,\varpi) \mathrm{d}t \exp(i\varpi(t'-\tau)) \mathrm{d}\tau [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \mathrm{d}\varpi \\
&= \int_R \int_R f(t) [\int_R \overline{\Psi}(t-\tau,\varpi) \exp(-i\varpi\tau) \mathrm{d}\tau] [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \mathrm{d}t \exp(i\varpi t') \mathrm{d}\varpi \\
&= \int_R \int_R f(t) [\overline{\hat{\Psi}}(\varpi,\varpi)] \exp(-i\varpi t) [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \mathrm{d}t \exp(i\varpi t') \mathrm{d}\varpi \\
&= \int_R \int_R f(t) \exp(-i\varpi t) \mathrm{d}t \exp(i\varpi t') \mathrm{d}\varpi \\
&= \int_R \hat{f}(\varpi) \exp(i\varpi t') \mathrm{d}\varpi \\
&= 2\pi f(t')
\end{aligned}
\tag{2.13}
$$

Here the proof ends.

In fact , Inversion Theorem 2 is a specific application of Inversion Theorem 1. The Inversion formula (2.11) has a specific inversion kernel $I(t,w) = [\overline{\hat{\Psi}}(\varpi,\varpi)]^{-1} \exp(i\varpi t)$. Such inversion kernel is global in time because it is of no time modulation. However, the Inversion formula (2.11) converges as long as $f(t) \in L^1(R)$ and $\hat{f}(\omega) \in L^1(R)$. In other words, it converges if $f(t)$'s Fourier transform exists and is invertible.

By Inversion Theorem 2, for $f(t) \in L^1(R)$ satisfying $\hat{f}(\omega) \in L^1(R)$, its phase-updated GT (1.2) can be inverted by

$$f(t) = \frac{1}{2\pi \overline{\hat{w}}(0)} \int_R \int_R \mathrm{G} f(\tau,\varpi) \exp(i\varpi(t-\tau)) \mathrm{d}\tau \mathrm{d}\varpi \tag{2.14}$$

which is an application of Inversion Theorem 2 with $\Psi(t,\varpi)=w(t)\exp(i\varpi t)$.

By Inversion Theorem 2, we find that, for $f(t)\in L^1(R)$ satisfying $\hat{f}(\omega)\in L^1(R)$, its WT (1.3) can be inverted by

$$f(t)=\frac{1}{2\pi\bar{\hat{\psi}}(1)}\int_R\int_R \mathrm{W}_\psi f(\tau,\varpi)\exp\,(i\varpi(t-\tau))\mathrm{d}\tau\mathrm{d}\varpi,$$

$$t\in R \tag{2.15}$$

To verify the formula (2.15), one just needs to notice the fact that the WT (1.3) is a special TFT with $\Psi(t,\varpi)=|\varpi|\psi(\varpi t)$.

The WT inversion formula (2.15) is new, which has not been found in the wavelet literature. It indicates two facts as follows. At first, the conventional wavelet admissibility (2.10) is not a necessary condition for WT inversion. For example, it is not necessary for a fundamental wavelet to be evenly undulant to make the WT invertible. Secondly, a WT with its fundamental wavelet having significant frequency 1 should be always invertible. This suggests a new wavelet definition free of the admissibility (2.10): **a local time function undulant with frequency 1 is called a *wavelet***. According to such definition, a wavelet $\psi(t)$ can be simply expressed as

$$\psi(t)=w(t)\exp(it) \tag{2.16}$$

where $w(t)\in L^1(R)$ is a window. Such defined wavelet is not required to be evenly undulant.

3 Normal TFT

3.1 Definition

Definition 1 A TFT Ψ is regarded as normal, if its kernel $\Psi(t,\varpi)$ satisfies

$$\bar{\hat{\Psi}}(\omega,\varpi)=\text{maximum}\{|\hat{\Psi}(\omega,\varpi)|\}=1 \;\Leftrightarrow\; \omega=\varpi \tag{3.1}$$

here "$\Leftrightarrow$" means "if and only if".

There should be two typical normal TFTs: normal GT and normal WT. The normal GT is the phase-updated GT (1.2) satisfying

$$\bar{\hat{w}}(\omega)=\text{maximum}\{|\hat{w}(\omega)|\}=1 \;\Leftrightarrow\; \omega=0 \tag{3.2}$$

Here, we called the window $w(t)$ satisfying (3.2) a normal window. In other words, the normal GT is the phase-updated GT (1.2) using a normal window. The normal WT is the L1-norm WT (1.3) satisfying

$$\bar{\hat{\psi}}(\omega)=1=\text{maixmum}\{|\hat{\psi}(\omega)|\}\Leftrightarrow\omega=1 \tag{3.3}$$

Here, we called a wavelet $\psi(t)$ satisfying (3.3) a normal wavelet. A normal wavelet $\psi(t)$ can be obtained by (2.16) with $w(t)$ being a normal window. In other words, the normal WT is the L^1-norm WT (1.3) using a normal wavelet.

In nature, the concept of normal TFT is born by the TFT inversion formula (2.11). Obviously, for $f(t)\in L^1(R)$, its normal TFT $\Psi f(\tau,\varpi)$ can be inverted simply by

$$f(t)=\frac{1}{2\pi}\int_R\int_R \Psi f(\tau,\varpi)\exp(i\varpi(t-\tau))\mathrm{d}\tau\mathrm{d}\varpi,\quad t\in R \tag{3.4}$$

In such inversion formula, the inversion kernel $I(t,\varpi)=\exp(i\varpi t)$ is independent of the TFT kernel $\Psi(t,\varpi)$. Thus, given the normal TFT of a time function, one can reconstruct the time function without knowing the TFT kernel. For example, Given the normal GT of a time function, one recover the time function without knowing the normal window $w(t)$. For another example, given the normal WT of a time function, one can recover the time function without knowing the normal wavelet $\psi(t)$. It is important to note that, the normal GT and the normal WT, though being different in time-frequency resolutions, share the same simple inversion formula (3.4).

3.2 Precision for time-frequency analysis

The normal TFT is not only simple in inversion formula but also precise for time-frequency analysis. A TFT precise for time-frequency analysis means that it can exactly identify i.e. explicate the immediate frequency, amplitude and phase of a time harmonic. A time harmonic $h(t)$ can be written as

$$h(t)=A\exp(i\beta t)=|A|\exp(i(\phi+\beta t)) \tag{3.5}$$

where A is the complex amplitude, β is the angular frequency, ϕ is the initial phase and $\phi+\beta t$ is the immediate phase. In physics, h(t) describes a circular movement of a point on a complex plane. The angular frequency β allows being negative, which refers to a retrograde (i.e. clockwise) circular movement. Applying a normal TFT Ψ to $h(t)$ yields

$$1)\;|\Psi h(\tau,\varpi)|=\text{maximum}|h(\tau)|\Leftrightarrow\varpi=\beta,\quad\forall\tau\in R \tag{3.6}$$

$$2)\;\Psi h(\tau,\beta)=h(\tau)=|A|\exp(i(\phi+\beta\tau)),\quad\forall\tau\in R \tag{3.7}$$

Relation (3.6) means that the amplitude spectrum of the normal TFT is unbiased in explicating the immediate frequency and amplitude of a harmonic. In other words, the spectral ridge of the normal TFT of a harmonic appears exactly along the harmonic frequency and the spectral ridge value is exactly the immediate amplitude of the harmonic. Relation (3.7) means that the normal TFT is unbiased in explicating the immediate (rather than initial) phase of a harmonic. Relations (3.6) and (3.7) jointly show that the normal TFT enables unbiased explication of the immediate frequency, amplitude and phase of a time harmonic, and thus should enable precise explication of the immediate frequency, amplitude and phase of a quasi-harmonic. Among all the TFTs, only the normal TFT makes relations (3.6) and (3.7) simultaneously hold. Thus, the normal TFT is unique in unbiased explication of the immediate frequency, amplitude and phase of a time harmonic, and thus is unique for precise time-frequency analysis.

However, the conventional GT G*, which is not as updated in phase as (1.2), would not make relation (3.7) hold. It is easy to prove that, if G* uses a normal window,

$$\mathrm{G}^* h(\tau,\beta) = h(0) = |A|\exp(i\phi), \quad \forall \tau \in R \quad (3.8)$$

This means that G* does not explicate the immediate but initial phase of a harmonic. In other words, G* does not show the phase evolution of a harmonic or quasi-harmonic. However, in the sense of time-frequency analysis, the immediate phase of a harmonic or quasi-harmonic is more significant than its initial phase. That is why we directly introduce in section 1 the phase-updated GT (1.2) rather than the conventional GT.

However furthermore, the WT, if not as written in L^1-norm as (1.3), would be difficult to make relation (3.6) hold. For a time function $f(t)$, its $L^{1/\gamma}$-norm WT can be expressed by

$$\mathrm{W}^{\gamma}_{\psi} f(\tau,\varpi) = |\varpi|^{\gamma} \int_R f(t)\overline{\psi}(\varpi(t-\tau))\mathrm{d}t, \quad \tau,\varpi \in R \quad (3.9)$$

Now assume that $\psi(t)$ be the normal Morlet wavelet

$$\psi(t) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{t^2}{2\sigma^2} + it) \quad (3.10)$$

where σ is the Gaussian window width parameter. It can be proved that

$$|\mathrm{W}^{\gamma}_{\psi} h(\tau,\varpi)| = \text{maximum} \Leftrightarrow \varpi = \frac{1}{1/2 + \sqrt{1/4 + (1-\gamma)\sigma^{-2}}}\beta, \quad \forall \tau \in R \quad (3.11)$$

This means that the amplitude spectrum of the $L^{1/\gamma}$-norm Morlet WT of a β-frequency harmonic gets maximum at $\varpi=\beta$ if and only if $\gamma=1$. In other words, the spectral ridge of the $L^{1/\gamma}$-norm WT of a β-frequency harmonic appears exactly at $\varpi=\beta$ if and only if $\gamma=1$. Relation (3.11) proves the frequency bias phenomenon in the amplitude spectrum of non-L^1-norm Morlet WT [Fig. 1]. Such frequency bias is not ignorable in the sense of precise time-frequency analysis. For instance, the amplitude (or energy) spectrum of the L^2-norm Morlet WT with $\sigma=2\pi$ overstates an annual signal by about 4.6 days in period. Also for instance, the amplitude spectrum of the L^∞-norm Morlet WT with $\sigma=2\pi$ overstates an annual signal in period by as large as 9.0 days. In practice, Shyu and Sun [12] experimentally found the frequency bias phenomenon in the L^2-norm Morlet WT spectrum. Liu et al [8] also experimentally found this phenomenon in the L^∞-norm Morlet WT spectrum.

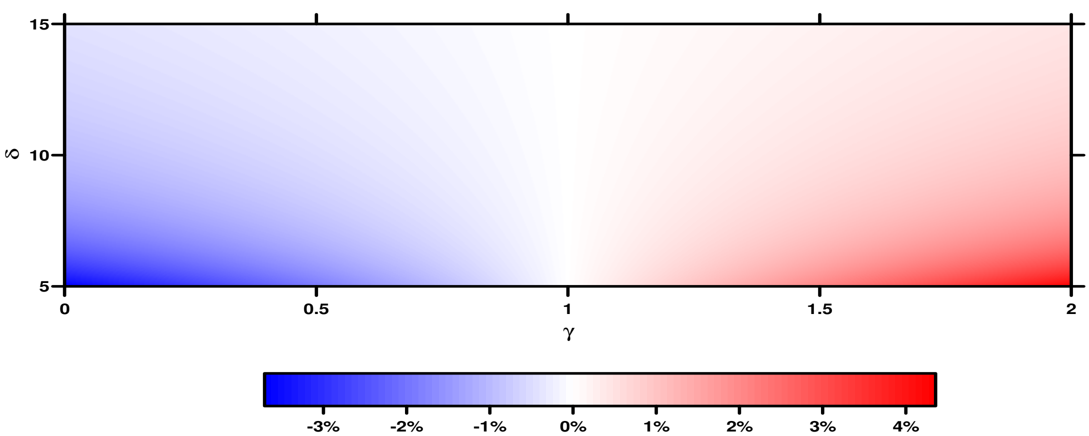

Fig. 1 Relative frequency bias in the amplitude spectrum of the $L^{1/\gamma}$-norm WT (3.9) using a normal Morlet wavelet (3.10) where σ is normal Gaussian window's width parameter.

Unfortunately, many authors pay little attention to the WT normalization because the normalization factor γ enters in a simple multiplicative way in the WT (3.9). Thus, they often applied either L2-norm

WT (e.g. [2]) or L^∞-norm WT (e.g. [8]) to do time-frequency analysis. Few of these applications have claimed or corrected the spectral frequency biases. The unfortunate is still shown throughout the internet and publications everyday! So, we strongly recommend that the L^1-norm WT should be regarded as normal in the sense of precise time-frequency analysis. Particularly, we have developed the concept of normal Morlet WT [7] to overcome the frequency bias phenomenon.

3.3 Frequency resolution

A normal TFT, when applied to analyze a time signal, requires enough fine frequency resolution to separate the component harmonics or quasi-harmonics of the signal. Then, there is a question: how to measure the frequency resolution of a normal TFT? The answer to this question will help a signal analyzer to construct a normal TFT with desired frequency resolution.

For a normal TFT Ψ, we call

$$E_\Psi(\omega,\varpi) = \overline{\hat{\Psi}}(\omega,\varpi) \tag{3.12}$$

the harmonic amplitude weight (HAW) function of Ψ. It is easy to know that

$$\begin{cases} |E_\Psi(\omega,\varpi)| < 1 & \omega \neq \varpi \\ E_\Psi(\omega,\varpi) = 1 & \omega = \varpi \end{cases} \tag{3.13}$$

The HAW function provides a measure for the frequency resolution of a normal TFT. In theory, a normal TFT Ψ with fine frequency resolution means that

$$|E_\Psi(\omega,\varpi)| << 1, \quad \text{if } \omega \neq \varpi \tag{3.14}$$

This meaning can be well understood by observing the following two applications.

At first, applying a normal TFT to a single harmonic $h(t)$ (3.5) yields

$$\Psi h(\tau,\varpi) = E_\Psi(\beta,\varpi) h(\tau) \tag{3.15}$$

Thus, $E_\Psi(\omega,\varpi)$ normally weighs the amplitude of the normal TFT of a ω-frequency harmonic at frequency index ϖ. Here, the normality means the weight maximum is exactly 1 if and only if $\omega=\varpi$, as shown by (3.13).

Secondly, applying a normal TFT to the sum of two harmonics

$$H(t) = h_1(t) + h_2(t) = A_1 \exp(i\beta_1 t) + A_2 \exp(i\beta_2 t), \ \beta_1 \neq \beta_2 \tag{3.16}$$

yields

$$\Psi H(\tau,\varpi) = E_\Psi(\beta_1,\varpi) h_1(\tau) + E_\Psi(\beta_2,\varpi) h_2(\tau) \tag{3.17}$$

and particularly along the two harmonic frequencies

$$\mathbf{\Psi} H(\tau,\beta_1) = h_1(\tau) + E_\Psi(\beta_2,\beta_1) h_2(\tau) \tag{3.18}$$

$$\mathbf{\Psi} H(\tau,\beta_2) = h_2(\tau) + E_\Psi(\beta_1,\beta_2) h_1(\tau) \tag{3.19}$$

Then, for a normal TFT Ψ to sufficiently separate the two component harmonics of $H(t)$ from each other, its HAW values at the two harmonic frequencies require to be small enough, e.g.

$$|E_\Psi(\beta_1,\beta_2)| < 1\% \quad \text{and} \quad |E_\Psi(\beta_2,\beta_1)| < 1\% \tag{3.20}$$

Upon a time signal containing two harmonics or quasi-harmonics with frequencies around about β_1 and β_2, one can use the requirements (3.20) to construct a normal TFT Ψ to do good time-frequency analysis with desired frequency resolution.

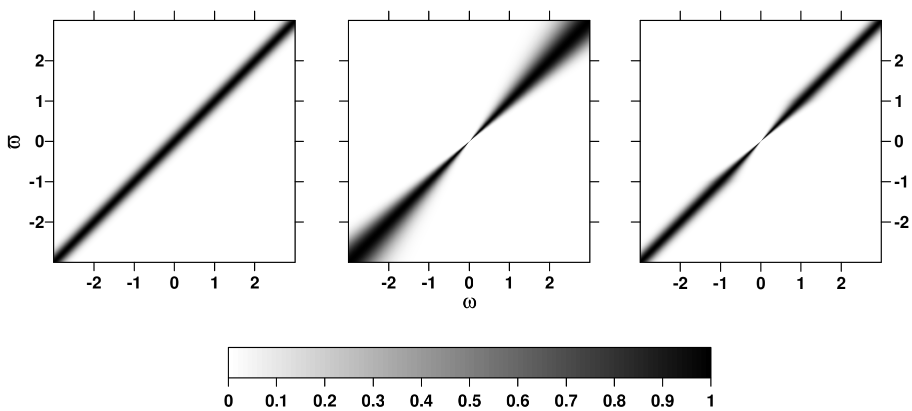

Fig. 2 HAW function of normal GT (left); HAW function of normal WT (middle) and HAW function of normal GMT (right). The three normal TFTs share a normal Gaussian window $w(t)=\exp(-t^2/(2\sigma^2))/((2\pi)^{1/2}\sigma)$ with $\sigma=2\pi$.

There are two typical HAW functions. For a normal GT G, its HAW function [Fig. 2 (left)] is

$$E_{\mathrm{G}}(\omega,\varpi) = \hat{w}(\omega - \varpi) \tag{3.21}$$

To construct a normal GT with enough good frequency resolution to separate the two component harmonics of $H(t)$ from each other means to evaluate a normal window $w(t)$ such that $E_{\mathrm{G}}(\beta_1,\beta_2)$ and $E_{\mathrm{G}}(\beta_1,\beta_2)$ are enough small. For a normal WT, its HAW function [Fig. 2 (middle)] is

$$E_{\mathrm{W}}(\omega,\varpi) = \hat{\psi}(\omega/\varpi) = \hat{w}(\omega/\varpi - 1) \tag{3.22}$$

To construct a normal WT with enough good frequency resolution to separate the two component harmonics of $H(t)$ from each other means to evaluate a normal wavelet $\psi(t)$ (i.e. a normal window $w(t)$) such that $E_W(\beta_1,\beta_2)$ and $E_W(\beta_1,\beta_2)$ are enough small.

Fig. 2 (left and middle) shows that, for the normal GT and the normal WT sharing one normal window, the former is better in frequency resolution within high frequency band (i.e. ϖ >1) than the later, and is worse in frequency resolution within low frequency band (i.e. ϖ <1). Here we defines a normal TFT Ψ with favorable frequency resolution by letting

$$\Psi(t,\varpi) = \begin{cases} w(t)\exp(i\varpi t), & |\varpi| > 1 \\ |\varpi|\, w(\varpi t)\exp(i\varpi t), & |\varpi| \le 1 \end{cases}$$

(3.23)

where $w(t)$ is a normal window. We call such normal TFT as normal Gabor-Morlet transform (GMT), because it is a combination of the normal GT (within high frequency band) and the normal WT (within low frequency band). By observing the HAW [Fig. 2 (right)] of the normal GMT, one can find that the normal GMT has the same good frequency resolution within high frequency band as the normal GT and has the same good frequency resolution within low frequency band as the normal WT. Thus, in the full-band sense, the normal GMT works better for the time-frequency analysis than an individual normal GT or normal WT. The normal GMT is invertible by (3.4).

In fact, a time-frequency analyzer can construct a normal TFT by designing a HAW function in advance. The HAW can be designed according to analyzer's particular frequency resolution requirements. Such constructed normal TFT may not be of an analytical kernel but of a numeric kernel. How to construct a normal TFT by designing a HAW with required frequency resolution is beyond the range of this study.

4 Conclusions

Two inversion formulas for the TFT are proved, one being general and the other being specific. A new inversion formula for the WT is found, and a new wavelet definition free of the conventional admissibility is made. The concept of normal TFT is developed for precise time-frequency analysis, as the normal TFT is unique in unbiased explication of the immediate frequency, amplitude and phase of a harmonic. The conventional GT should be updated in phase to become a normal TFT. The WT should be written strictly in L^1-norm to become a normal TFT. Finally, we provide a HAW function to measure the frequency resolution of the normal TFT.

Appendix: We here present a conjecture: **Except for GT (1.2) and WT (1.3), TFT (1.1) can not find the inversion way other than (2.11)**. It comes from the following considerations: GT (1.2) can be inverted by (2.7) beside by (2.14), WT (1.3) can be inverted by (2.9) besides by (2.15), but normal GMT (3.13) seems difficult to find the inversion formula other than (3.4). This conjecture concerns to the roles of the GT and the WT in the TFTs. By now, however, we cannot prove whether this conjecture is right.

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