Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110109

Stability and Uniqueness of Ulam–Hyers for Nonlinear Sequential Fractional Differential Equations with nonseparated Boundary Conditions

Hamzeh Zureigat, Areen Al-Khateeb, Shawkat Alkhazaleh and Belal Batiha*

Department of Mathematics, Faculty of Science and Technology, Jadara University, Irbid 21110, Jordan

Received: 3 Mar. 2024, Revised: 11 Jul. 2024, Accepted: 12 Aug. 2024
Published online: 1 Jan. 2025

Abstract: In various areas such as chemistry, physics, engineering, and biology, fractional-order boundary value problems are valuable tools for modelling specific phenomena. This paper addressed the uniqueness, stability, and existence of solutions for a coupled system of sequential fractional differential equations of the Caputo type, which involves non-separated boundary conditions. To establish the existence of solutions, we employ the Leray-Schauder' alternative, and to prove uniqueness, we rely on Banach's contraction principle. Furthermore, we explore the Hyers-Ulam stability of the presented system. Finally, an illustrative example is provided to demonstrate and assess relevant aspects of the problem.

Keywords: Fixed point theorem, non-separated boundary conditions, Leray-Schauder's alternative, sequential fractional differential

1 Introduction

Recently, the area of boundary value problems (BVPs) involving fractional order has been investigated by several mathematicians. This is clear via the considerable number of important studies of BVPs involving fractional-order that fundamentally focused on transforming and extending such mathematical subjects from the theoretical part to the implementation part so as to make them suitable and applicable for some of specific real-world problems. The Fractional calculus basically is integration and differentiation to an arbitrary order between $[0,1]$ where his tools have simplified the description of numerous real-world phenomena's in many fields such as chemistry, physics, and biology [1–4]. Moreover, Fractional calculus is recognized as a crucial tool in engineering, enabling the comprehensive and detailed characterization of complex models in this field. The importance of fractional calculus extends beyond scientific domains to various aspects of human civilization. As a result of ongoing research, numerous practical mathematical models have emerged, formulated using fractional differential equations, to offer comprehensive descriptions and innovative approaches applicable in diverse practical fields. This has paved the way for a fresh avenue of research, promoting increased collaboration among mathematicians and researchers from other fields. Moreover, the tangible utility of fractional-order models, as highlighted in references [5–18], stands out as a key advantage in this context.newpage Fractional-order derivatives and integrals have a wide range of applications across various fields, including applied mathematics, control theory, physics, thermodynamics, and mechanical structures [19 –21]. Unlike the classical Laplacian, which is a local operator, the fractional Laplacian represents a notable example within the broader category of nonlocal linear operators. Its implications are immediately evident in the formulation of fundamental equations such as the diffusion equation. Significant progress has been made in addressing fractional-order initial/boundary value problems, leading to a substantial body of literature on this vital subject. This literature encompasses both theoretical advancements and practical applications. The exploration of fractional-order BVP has extended to encompass various types of boundary conditions, including two-point, multi-point, nonlocal, periodic/anti-periodic, and integral conditions. For in-depth information and illustrative examples, readers are directed to a series of papers [22-26]. For additional research on sequential fractional differential equations, readers can refer to works such as [27, 28] and the associated

* Corresponding author e-mail: hamzeh.zu@jadara.edu.jo

references. In a recent study [29-32], the authors achieved significant results in establishing the existence of solutions for sequential fractional differential equations characterized by anti-periodic boundary conditions.

Motivated by the information presented above and our comprehensive literature review, this paper endeavors to examine and assess a coupled system of sequential fractional differential equations under fractional Caputo-type. Specifically, we delve into exploring the uniqueness, existence and stability of solutions for this system, which includes separated boundary conditions.

$$\begin{cases} {}^cD^{p-1}(D+\psi)u(h) = f(t,u(h),u(h)), & t\in[0,H], \quad 1<p\leq 2, \quad \psi>0, \\ {}^cD^{q-1}(D+\psi)v(h) = g(t,v(h),v(h)), & t\in[0,H], \quad 1<q\leq 2, \quad \psi>0, \end{cases} \tag{1}$$

Enhanced by incorporating boundary conditions in the following manner:

$$\begin{cases} u(0)=\lambda v'(H) \ , & v(0)=\mu u'(H) \\ u'(0)=0, & v'(0)=0 \end{cases} \tag{2}$$

Where ${}^cD^k$ Indicate the Caputo fractional derivatives with a degree of k, *where* $k=p,q$, and $f,g:[0,H]\times\mathbb{R}^2\to\mathbb{R}$, are assumed continuous functions, where the $\lambda,\ \mu$ are considered as real constants.

2 Preliminaries

In this section, the main preliminaries and theorems for this study are elaborated upon extensively. Let's revisited the definitions of fractional derivatives and integrals as mentioned in references [1, 2].

Definition 2.1. The Caputo derivative of arbitrary fractional order β , times absolutely continuous function $h:[0,\ \infty)\to\mathbb{R}$ is presented as follows:

$$ {}^cD^{\beta}h(s) = \frac{1}{\Gamma(i-\beta)}\int_0^s (s-t)^{i-\beta-1}h^{(m)}(t)dt, \quad -1<\beta<i, \quad i=[\beta]+1,$$

Definition 2.2. The expression for the fractional integral of a continuous function h with respect to the Riemann-Liouville operator of order ω is as follows:

$$I^{\omega}h(t) = \frac{1}{\Gamma(\omega)}\int_0^t \frac{h(s)}{(t-s)^{1-\omega}}ds, \qquad \omega>0,$$

Assuming that the right-hand side of Riemann-Liouville definition is defined point-wise within the interval $[0,\ \infty)$.

Definition 2.3. The definition for sequential of fractional derivative applies to adequately smooth function $h(t)$ based on Miller-Ross definition [3]

$$D^t h(s) = D^{t_1}D^{t_2}\dots D^{t_n}h(s),$$

where $t=(t_1,t_2,\dots,t_n)$ is a multi-index.

the following supplementary lemma is established in order to determine a solution for both problem (1) and problem (2)

Lemma 2.1. Assume $\Psi,\Phi\in C([0,S],\mathbb{R})$ then for the the following problem:

$$\begin{cases} {}^cD^{p-1}(D+\psi)u(h)=\Psi(h), & 1<p\leq 2, \\ {}^cD^{q-1}(D+\psi)v(h)=\Phi(h), & 1<q\leq 2, \\ u(0)=\lambda v'(H) \ , & v(0)=\mu u'(H) \\ \frac{d^2u}{dx^2}(0)=0, & \frac{d^2v}{dy^2}(0)=0 \end{cases} \quad , \quad \psi>0 \quad ,h\in[0,H] \tag{3}$$

the unique solution is:

$$u(h)=\lambda\left(\left(I^{q-1}\Phi\right)(H)-\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}\Phi\right)(t)dt\right)+\int_0^h e^{-\psi(h-t)}\left(I^{p-1}\Psi\right)(t)dt, \tag{4}$$

and

$$v(h)=\mu\left(\left(I^{p-1}\Psi\right)(H)-\psi\int_0^H e^{-\psi(H-t)}\left(I^{p-1}\Psi\right)(t)dt, \right)+\int_0^h e^{-\psi(h-t)}\left(I^{q-1}\Phi\right)(t)dt, \tag{5}$$

Proof. The solutions that encompass all possible outcomes for the sequential fractional differential equations [28] in (3) are commonly referred to as:

$$u(h) = m_0 e^{-\psi h} + m_1 + \int_0^h e^{-\psi(h-t)} \left(I^{p-1}\Psi\right)(t)dt, \tag{6}$$

$$v(h) = n_0 e^{-\psi h} + n_1 + \int_0^h e^{-\psi(h-t)} \left(I^{q-1}\Phi\right)(t)dt, \tag{7}$$

observe

$$u'(h) = -\psi m_0 e^{-\psi h} + \left(I^{p-1}\Psi\right)(h) - \psi \int_0^h e^{-\psi(h-t)} \left(I^{p-1}\Psi\right)(t)dt,$$

$$v'(h) = -\psi n_0 e^{-\psi h} + \left(I^{q-1}\Phi\right)(h) - \psi \int_0^h e^{-\psi(h-t)} \left(I^{q-1}\Phi\right)(t)dt,$$

$$u''(h) = \psi^2 m_0 e^{-\psi h} + \left(I^{p-2}\Psi\right)(h) - \psi \left(I^{p-1}\Psi\right)(h) + \psi^2 \int_0^h e^{-\psi(h-t)} \left(I^{p-1}\Psi\right)(t)dt,$$

$$v''(h) = \psi^2 n_0 e^{-\psi h} + \left(I^{q-2}\Phi\right)(h) - \psi \left(I^{q-1}\Phi\right)(h) + \psi^2 \int_0^h e^{-\psi(h-t)} \left(I^{q-1}\Phi\right)(t)dt,$$

where $m_i, n_i \in \mathbb{R}, i = 0, 1$ are arbitrary constants.
Implement the conditions

$$u''(0) = 0, \qquad v''(0) = 0,$$

Then we get:

$$m_0 = 0\ , n_0 = 0$$

In view of the conditions $u(0) = \lambda v'(H)$, $v(0) = \mu u' H)$
we get

$$u(0) = m_0 + m_1 = \lambda \left(\left(I^{q-1}\Phi\right)(H) - \psi \int_0^H e^{-\psi(H-t)} \left(I^{q-1}\Phi\right)(t)dt \right) + \int_0^j e^{-\psi(h-t)} \left(I^{p-1}\Psi\right)(t)dt,$$

so,

$$m_1 = \lambda \left(\left(I^{q-1}\Phi\right)(H) - \psi \int_0^H e^{-\psi(H-t)} \left(I^{q-1}\Phi\right)(t)dt \right) + \int_0^h e^{-\psi(h-t)} \left(I^{p-1}\Psi\right)(t)dt,$$

and

$$v(0) = n_0 + n_1 = \mu \left(\left(I^{p-1}\Psi\right)(H) - \psi \int_0^H e^{-\psi(H-t)} \left(I^{p-1}\Psi\right)(t)dt, \right) + \int_0^h e^{-\psi(h-t)} \left(I^{q-1}\Phi\right)(t)dt,,$$

So,

$$n_1 = \mu \left(\left(I^{p-1}\Psi\right)(H) - \psi \int_0^H e^{-\psi(H-t)} \left(I^{p-1}\Psi\right)(t)dt, \right) + \int_0^h e^{-\psi(h-t)} \left(I^{q-1}\Phi\right)(t)dt.$$

Substituting the values of m_0, m_1, n_0, n_1in $(6), (7)$,we get (4) and (5), which the proof is done ■

3 Existence results

Assume the space $S = \{u(s)\,|u(s) \in C[0,S]\ \}$ endowed with the norm $\|u\| = max\{|u(h)|\,,h \in [0,H]\}$. *It* is obvious that $(S,\|\ .\ \|)$ is a Banach space. Also let $T = \{v(h)\,|v(h) \in C[0,H]\ \}$ endowed with the norm $\|v\| = max\{|v(h)|\,,t \in [0,H]\}$.The product space $(S\times T,\|(u,v)\ \|)$ is also Banach space with norm $\|(u,v)\ \| = \|u\| + \|v\|$.

In view of lemma (2.1) we define the operator $\Omega : S\times T \to S\times T$ by

$$\Omega(u,v)(h) = (\Omega_1(u,v)(h), \Omega_2(u,v)(h)),$$

where

$$\Omega_1(u,v)(h) = \lambda\left(\left(I^{q-1}g\right)(H) - \psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}g\right)(t)dt\right) + \int_0^h e^{-\psi(h-t)}\left(I^{p-1}f\right)(t)dt, \tag{8}$$

and

$$\Omega_2(u,v)(h) = \mu\left(\left(I^{p-1}f\right)(H) - \psi\int_0^H e^{-\psi(H-t)}\left(I^{p-1}f\right)(t)dt,\quad\right) + \int_0^h e^{-\psi(h-t)}\left(I^{q-1}g\right)(t)dt. \tag{9}$$

Theorem 3.1. Assume $f,g : C([0,H]\times\mathbb{R}^2 \to \mathbb{R}$ They are functions that are continuous together, and there are constants such that $a_1,a_2 \in \mathbb{R}$, such that $\forall\ u_1,u_2,v_1,v_2 \in \mathbb{R}, \forall h \in [0,H]$ we have

$$|f(h,u_1,u_2) - f(h,v_1,v_2)| \le a_1(|u_2-u_1| + |v_2-v_1|),$$

$$|g(h,u_1,u_2) - g(h,v_1,v_2)| \le a_2(|u_2-u_1| + |v_2-v_1|).$$

If
$a_1(K_1+K_3) + a_2(K_2+K_4) < 1$,
Then the boundary value problem (1),(2) has a unique solution on $[0,H]$, where

$$K_1 = \frac{H^p}{\Gamma(p+1)}$$

$$K_2 = \frac{|\lambda|\left[qH^{q-1} + \psi H^q\right]}{\Gamma(q+1)}$$

$$K_3 = \frac{|\mu|\left[pH^{p-1} + \psi H^p\right]}{\Gamma(p+1)}$$

$$K_4 = \frac{H^q}{\Gamma(q+1)}$$

Proof. Define $\left(\sup_{0\le h\le H}\right) f(h,0,0) = f_0 < \infty$, $\left(\sup_{0\le h\le H}\right) g(h,0,0) = g_0 < \infty$ and $\Theta_r = \{(u,v) \in S\times T : \|(u,v)\| \le r\}$, and $r > 0$, such that

$$r \ge \frac{(K_1+K_3)f_0 + (K_2+K_4)g_0}{1 - [a_1(K_1+K_3) + a_2(K_2+K_4)]}.$$

Firstly, show that $\Omega\Theta_r \subseteq \Theta_r$.
By our assumption, for $(u,v) \in \Theta_r, h \in [0,H]$, we have

$$\begin{aligned}|f(h,u(h),v(h))| &\le |f(h,u(h),v(h)) - f(h,0,0)| + |f(h,0,0)|,\\ &\le a_1(|u(h)| + |v(h)|) + f_0 \le a_1(\|u\| + \|v\|) + f_0,\\ &\le a_1 r + f_0,\end{aligned} \tag{10}$$

and

$$|g(h,u(h),v(h))| \leq a_2(|u(h)|+|v(h)|)+g_0 \leq a_2(\|u\|+\|v\|)+g_0,$$

$$\leq a_2 r + g_0, \tag{11}$$

which lead to

$$|\Omega_1(u,v)(h)| \leq |\lambda| \left[\left(I^{q-1}|g|\right)(H) + \psi \int_0^H e^{-\psi(H-t)} \left(I^{q-1}|g|\right)(t)dt \right] + \sup_{0\leq h\leq H} \int_0^h e^{-\psi(h-t)} \left(I^{p-1}|f|\right)(t)dt \tag{12}$$

Using (10) and (11) to get

$$|\Omega_1(u,v)(h)| \leq |\lambda| \left[\left(I^{q-1}|1|\right)(H) + \psi \int_0^H e^{-\psi(H-t)} \left(I^{q-1}|1|\right)(t)dt \right] \|g\| + \left[\int_0^h e^{-\psi(h-t)} \left(I^{p-1}|1|\right)(t)dt \right] \|f\|$$

$$\leq \frac{H^p}{\Gamma(p+1)} \|f\| + \left[\frac{|\lambda| \left[qH^{q-1}+\psi H^q\right]}{\Gamma(q+1)} \right] \|g\|,$$

Hence, by (8) we have

$$\|\Omega_1(u,v)\| \leq (a_1K_1+a_2K_2)r + (K_1f_0+K_2g_0) \leq \frac{r}{2}, \tag{13}$$

In similar way we get

$$\|\Omega_1(u,v)\| \leq (a_1K_3+a_2K_4)r + (K_3f_0+K_4g_0) \leq \frac{r}{2} \tag{14}$$

From (12) and (13) we obtain $\|\Omega(u,v)\| \leq r$,

Now show that Ω is a contraction. Let$(u_1,v_1),(u_2,v_2) \in S\times T, \forall h \in [0,H]$,then we get:

$$\|\Omega_1(u_1,v_1) - \Omega_1(u_2,v_2)\| \leq a_1K_1(\|u_1-u_2\|+\|v_1-v_2\|) + a_2K_2(\|u_1-u_2\|+\|v_1-v_2\|), \tag{15}$$

$$\|\Omega_2(u_1,v_1) - Z_2(x_2,y_2)\| \leq a_1K_3(\|u_1-u_2\|+\|v_1-v_2\|) + a_2K_4(\|u_1-u_2\|+\|v_1-v_2\|). \tag{16}$$

From (14) and (15) we deduced that

$$\|\Omega(u_1,v_1)\| - \|\Omega(u_2,v_2)\| \leq (a_1(K_1+K_3)+a_2(K_2+K_4))(\|u_1-u_2\|+\|v_1-v_2\|)$$

where Ω is a contraction operator and Since $a_1(K_1+K_3)+a_2(K_2+K_4) < 1$, therefore, according to the fixed point theorem of Banach's, the operator Ω possesses a sole fixed point within the interval [0, H], serving as the exclusive solution to both problems (1) and (2), thereby concluding the proof. ■

The second outcome relies on the Leray-Schauder alternative

Lemma 3.1. "(Leray-Schauder alternative *[18]*) Let$\Delta : Z \to Z$ be a completely continuous operator (i.e., a map restricted to any bounded set in Z is compact). Let $Z(\Delta) = \{x \in Z : x = \lambda\Delta(x)\ for\ some\ 0 < \lambda < 1\}$. Then either the set $Z(\Delta)$ is unbounded or Δ has at least one fixed point."

Theorem 3.2. Assume $f,g : C([0,H] \times \mathbb{R}^2 \to \mathbb{R}$ are considered continuous functions and There is at least one positive real constant. $\eta_i, \beta_i (i=0,1,2)$ such that $\forall u_i \in \mathbb{R}, (i=1,2)$ we have

$$|f(h,u_1,u_2)| \leq \eta_0 + \eta_1|u_1| + \eta_2|u_2|,$$

$$|g(h,u_1,u_2)| \leq \beta_0 + \beta_1|u_1| + \beta_2|u_2|.$$

If

$$(K_1+K_3)\,\eta_1+(K_2+K_4)\,\beta_1<1,$$

and

$$(K_1+K_3)\,\eta_2+(K_2+K_4)\,\beta_2<1,$$

Then, there exists at least one solution to both problems (1) and (2).

Proof. The proof will be split into two stages

Step1. Demonstrate that Θ exhibits complete continuity. The operator's continuity can be established based on the continuous nature of the functions f *and* g.

Let Υ be a bounded set in $\Theta_r=\{(u,v)\in S\times T:\|(u,v)\|\le r\}$, then there exists positive constants e_1,e_2 such that

$$|f\,(h,u(h),v(h))|\le e_1, \qquad |g\,(h,u(h),v(h))|\le e_2, \quad \forall h\in[0,H],$$

then for any$(u,v)\in\Upsilon$ we have$|\Omega_1(u,v)(h)|\le K_1e_1+K_2e_2$, which implies that $\|\Omega_1(u,v)\|\le K_1e_1+K_2e_2$.

Similarly, we get $\|\Omega_2(u,v)\|\le K_3e_1+K_4e_2$.. Hence, it can be deduced from the preceding inequalities that the operator Ω is uniformly bounded, as a result

$$\|\Omega(u,v)\|\le(K_1+K_3)\,e_1+(K_2+K_4)\,e_2.$$

Next, we show that the operator is equicontinuous.

Let $h_1,h_2\in[0,H]$ with $h_1<h_2$., then we have

$$|\Omega_1(u,v)\,(h_2)-\Omega_1(u,v)\,(h_1)|\le|\lambda|\left[\left(I^{q-1}|g|\right)(H)+\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}|g|\right)(t)dt\right]+$$

$$\int_0^{h_2} e^{-\psi(h_2-s)}\left(I^{\alpha-1}|f|\right)(t)dt+\int_0^{h_1} e^{-\psi(h_1-t)}\left(I^{p-1}|f|\right)(t)dt.$$

$$\le|\lambda e_2|\left[\left(I^{q-1}|1|\right)(H)+\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}|1|\right)(t)dt\right]+e_1\,[\left(\int_0^{h_1} e^{-\mu(h_2-t)}-e^{-\mu(h_1-t)}\right)\left(I^{p-1}1\right)(t)dt+$$

$$\int_{h_1}^{h_2} e^{-\mu(h_2-t)}\left(I^{p-1}1\right)(t)dt]$$

Hence we have $\|\Omega_1(u,v)\,(h_2)-\Omega_1(u,v)\,(h_1)\|\to 0$ independent of u and v as $h_2\to h_1$. Similarly, $\|\Omega_2(u,v)\,(h_2)-\Omega_2(u,v)\,(h_1)\|\to 0$ independent of u and v as $h_2\to h_1$.

Hence, the operator $\Omega(u,v)$ exhibits equicontinuity, thereby establishing its complete continuity.

Step 2. (Boundedness of operator)

At the final, show that $\delta=\{(u,v)\in S\times T:(u,v)=K\Omega(u,v),M\in[0,1]\}$ is bounded.

$$u(h)=K\Omega_1(u,v)(h),\;\; v(h)=K\Omega_2(u,v)(h),$$

then

$$|u(h)|\le K_1\,(\eta_0+\eta_1|u|+\eta_2|v|)+K_2\,(\beta_0+\beta_1|u|+\beta_2|v|)\,,$$

and

$$|v(h)|\le K_3\,(\eta_0+\eta_1|u|+\eta_2|v|)+K_4\,(\beta_0+\beta_1|u|+\beta_2|v|)\,.$$

So we get

$$\|u\|\le K_1\,(\eta_0+\eta_1|u|+\eta_2|v|)+K_2\,(\beta_0+\beta_1|u|+\beta_2|v|)\,, \tag{17}$$

and

$$\|v\|\le K_3\,(\eta_0+\eta_1|u|+\eta_2|v|)+K_4\,(\beta_0+\beta_1|u|+\beta_2|v|)\,. \tag{18}$$

From (16), and (17) we obtain

$$\|u\|+\|v\| \leq (K_1+K_3)\,\eta_0+(K_2+K_4)\,\beta_0+((K_1+K_3)\,\eta_1+(K_2+K_4)\,\beta_1)\,\|u\|+((K_1+K_3)\,\eta_2+(K_2+K_4)\,\beta_2)\,\|v\|.$$

Therefore;

$$\|(u,v)\| \leq \frac{(K_1+K_3)\,\eta_0+(K_2+K_4)\,\beta_0}{K_0},$$

where $K_0 = min\{1-(K_1+K_3)\,\eta_1-(K_2+K_4)\,\beta_1, 1-(K_1+K_3)\,\eta_2-(K_2+K_4)\,\beta_2\}$ that is ε is the Leray-Schauder theorem establishes that the existence of solutions to boundary value problems is guaranteed within the interval $[0,\ H]$.

4 Hyers-Ulam stability

During this section, the Hyers-Ulam stability of the proposed system (1) is examined by analyzing the following inequality.

$$\begin{cases} {}^cD^{p-1}(D+\mu)u(h)-f(h,u(h),v(h)) \leq r_1, & h\in[0,H], \\ {}^cD^{q-1}(D+\mu)v(h)-g(h,u(h),v(h)) \leq r_2, & h\in[0,H]. \end{cases} \tag{19}$$

where $\mathbf{r_1},\mathbf{r_2}$ are two positive real numbers.
Provide explanations for the following non-linear operators $M_1,\ M_2\ \in C([0,H],\mathbb{R})\times C([0,H],\mathbb{R}) \to C([0,H],\mathbb{R})$;

$$\begin{aligned} {}^cD^{p-1}(D+\mu)u(h)-f(h,u(h),v(h)) = M_1(t), \quad h\in[0,H], \\ {}^cD^{q-1}(D+\mu)v(h)-g(h,u(h),v(h)) = M_2(t), \quad h\in[0,H], \end{aligned}$$

For some $r_1,r_2>0$, we consider the following inequality:

$$|M_1(h)| \leq r, \qquad |M_2(h)| \leq r_2, \qquad h\in[0,H]. \tag{20}$$

Definition 4.1. The problem in equation (1) is Hyers-Ulam stable if there exist $K_i, i=1,2,3,4$ such that for given $r_1,r_2>0$ and for each solution $(u,v)\in C([0,H]\times\mathbb{R}^2,\mathbb{R})$ of inequality(18), there exists a solution $(u^*,v^*)\in C([0,H]\times\mathbb{R}^2,\mathbb{R})$ *of* problem (1) with

$$\begin{cases} |u(h)-u^*(h)| \leq K_1\ r_1+K_2\ r_2, & h\in[0,H], \\ |v(h)-v^*(h)| \leq K_3\ r_1+K_4\ r_2, & h\in[0,H]. \end{cases}$$

Theorem 3. Provided that the conditions outlined in Theorem 3.1 are met, then the BVB (1), (2) is Hyers-Ulam stable.
Proof. Let $(u,v)\in C([0,H],\mathbb{R})\times C([0,H],\mathbb{R})$ be the solution of the problem (1), (2) satisfying (8) and (9) and let (u^*,v^*) be any solution satisfying

$$\begin{cases} {}^cD^{p-1}(D+\mu)u^*(h) = f(h,u^*(h),v^*(h))+M_1(h), & h\in[0,H], \\ {}^cD^{q-1}(D+\mu)v^*(h) = g(h,u^*(h),v^*(h))+M_2(h), & h\in[0,H]. \end{cases}$$

It follows that

$$\left| \ u(h)-|\lambda|\left[\left(I^{q-1}|g|\right)(H)+\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}|g|\right)(t)dt\right]+\int_0^h e^{-\psi(h-t)}\left(I^{p-1}|f|\right)(t)dt\right|,$$

$$\begin{aligned} &\leq |\lambda|\left[\left(I^{q-1}|M_2(h)|\right)(H)+\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}|M_2(h)|\right)(t)dt\right]+\int_0^H e^{-\psi(H-t)}\left(I^{p-1}|M_1(h)|\right)(t)dt \\ &\leq r_1\left[\int_0^H e^{-\psi(H-t)}\left(I^{p-1}|1|\right)(t)dt\right]+r_2\left[|\lambda|\left[\left(I^{q-1}|1|\right)(H)+\psi\int_0^H e^{-\psi(H-t)}\left(I^{q-1}|1|\right)(t)dt\right]\right] \\ &= r_1K_1+r_2K_2 \end{aligned} \tag{21}$$

By the same method, we can obtain that

$$|v(h)-v^*(h)| \leq K_3\ r_1+K_4\ r_2, \tag{22}$$

The prior mention of $K_i, i=1,2,3,4$ establishes that, as per equations (20) and (21), the Caputo fractional differential equations forming a nonlinear sequential coupled system exhibits Hyers-Ulam stability. Consequently, the system described by equations (1) also demonstrates Hyers-Ulam stability.

Example. Consider the following system of fractional differential equation

$$\begin{cases} {}^cD^{1/4}(D+2)u(h) = \frac{1}{4\pi\sqrt{25+t^2}}\left(\frac{|u(h)|}{|u(h)|+2}+\cos v(h)\right)+\frac{1}{5}, \quad h\in[0,1] \\ {}^cD^{1/5}(D+2)v(h) = \frac{1}{3\pi(4+t)^2}\left(\cos(u(h))+\frac{v(h)}{|u(h)|}+1\right)+2, \quad h\in[0,1] \\ u(0)=\lambda v'(1), \quad v(0)=-u'(H) \\ u''(0)=0, \quad v''(0)=0 \end{cases} \tag{23}$$

Here :$\psi=2, p=\frac{5}{4}, q=\frac{6}{5}, H=1, \lambda=2,\ \mu=-1$. We found

$$K_1=0.88261019, K_2=5.80866358, K_3=2.86848289, K_4=0.90760368, a_1=\frac{1}{20\pi}, a_2=\frac{1}{48\pi}.$$

It's evident that $f\ and\ g$ are functions that exhibit joint continuity where

$$f(h,u,v)=\frac{1}{4\pi\sqrt{25+t^2}}\left(\frac{|u(h)|}{|u(h)|+2}+\cos v(h)\right)+\frac{1}{5},$$

$$g(h,u,v)=\frac{1}{3\pi(4+t)^2}\left(\cos(u(h))+\frac{v(h)}{|u(h)|}+1\right)+2,$$

Now, check that $a_1(K_1+K_3)+a_2(K_2+K_4)<1$,
Hence

$$\frac{1}{20\pi}(3.75109308)+\frac{1}{48\pi}(6.71626726)=0.1042391312<1,$$

If all the requirements outlined in Theorem 3.1 are met, then there exists a sole solution to the problem (22) within the interval [0, 1].

5 Conclusions

This paper explores the significance of fractional-order BVPs across multiple scientific disciplines and investigates a coupled system of sequential fractional differential equations with non-separated boundary conditions. The study establishes the existence and uniqueness of solutions using Leray-Schauder's alternative and Banach's contraction principle, respectively, while also examining the Hyers-Ulam stability of the system. Through a concrete example, the research illustrates the practical relevance of these findings, offering valuable insights for sequential fractional differential equations with non-separated boundary conditions.

References

[1] I. Podlubny, *Fractional differential equations*, Academic Press 1999, San Diego.

[2] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 2006, 204, Elsevier Science B.V., Amsterdam.

[3] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons 1993, New York.

[4] H. Zureigat, M. A. Tashtoush, A. F. A. Jassar, E. A. Az-Zo'bi and M. W. Alomari, A solution of the complex fuzzy heat equation in terms of complex Dirichlet conditions using a modified Crank–Nicolson method, Adv. Math. Phys.textbf2023, (2023)

[5] B. Batiha, New solution of the Sine-Gordon equation by the Daftardar-Gejji and Jafari method, *Symmetry* **14**(1), 57 (2022).

[6] H. Zureigat, M. Al-Smadi, A. Al-Khateeb, S. Al-Omari and S. E. Alhazmi, Fourth-order numerical solutions for a fuzzy time-fractional convection–diffusion equation under Caputo generalized Hukuhara derivative, *Frac. Fract.* **7**(1), 47 (2022).

[7] H. H. Zureigat and A. I. M. Ismail, Numerical solution of fuzzy heat equation with two different fuzzifications. In 2016 SAI Computing Conference (SAI) (pp. 85-90), IEEE.

[8] W. Al-Omeri and T. Noiri, On almost e-$\mathscr{I}$-continuous functions. *Demonstr. Math.* **54**(1), 168-177 (2021).
[9] E. Malkawi and D. Baleanu, Fractional killing-yano tensors and killing vectors using the caputo derivative in some one-and two-dimensional curved space, *Abstr. Appl. Anal* **2014**, Article ID 290694, pp. 4.
[10] H. Zureigat, A. I. Ismail and S. Sathasivam, A compact Crank–Nicholson scheme for the numerical solution of fuzzy time fractional diffusion equations. *Neural Comput. Appl.*, **32**(10), 6405-6412 (2020).
[11] B. Ahmad and S. K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ* **2011**, (2011b) Article ID 107384, pp. 11 (2011).
[12] B. Ahmad and S. K. Ntouyas, A four-point nonlocal integral boundary value problem for fractional differential equations of arbitrary order, *Electron. J. Qual. Theor. Differ. Equ.* **22**, 15 (2011).
[13] H. Zureigat, M. Al-Smadi, A. Al-Khateeb, S. Al-Omari and S. Alhazmi, Numerical solution for fuzzy time-fractional cancer tumor model with a time-dependent net Killing rate of cancer cells, *Int. J. Environ. Res. Public Health*, **20**(4), 3766 (2023).
[14] A. Hazaymeh, Time effective fuzzy soft set and its some applications with and without a neutrosophic, *Int. J. Neutr. Sci.* **23**(2), 129-29 (2023).
[15] H. Zureigat, M. A. Tashtoush, A. F. A. Jassar, E. A. Az-Zo'bi and M. W. Alomari, A solution of the complex fuzzy geat equation in terms of complex Dirichlet conditions using a modified Crank–Nicolson method, *Adv. Math. Phys.* **2023**(1), (2023).
[16] A. Al-Quran and S. Alkhazaleh, Relations between the complex neutrosophic sets with their applications in decision-making. Axioms, **7**(3), 64 (2018).
[17] C. Zhai and L. Xu, Properties of positive solutions to a class of four-point boundary value problem of caputo fractional differential equations with a parameter, *Commun. Nonlinear Sci. Numer. Simul.* **19**, 2820–2827 (2014).
[18] A. Granas and J. Dugundji, Fixed point theory, Springer-Verlag 2005, New York.
[19] S. Alkhazaleh and E. Beshtawi, Effective fuzzy soft expert set theory and its applications, International Journal of Fuzzy Logic and Intelligent Systems, 23(2), 192-204 ()2023, Klafter, J, Lim, SC, Metzler, R (eds.): Fractional Dynamics in Physics. World Scientific, Singapore (2011).
[20] X. J. Yang, Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat-transfer problems, *Therm. Sci.* **1**, (2016)
[21] G. Wang, R. P. Agarwal and A. Cabada, Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations, *Appl. Math. Lett.* **25**, 1019-1024 (2012).
[22] L. Zhang, B. Ahmad, G. Wang and R. P. Agarwal, Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, J. Comput. Appl. Math., **249**, 51-56 (2013).
[23] J. Henderson and N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.* **17**, 872-880 (2014).
[24] W. Zhai and L. Xu, Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter, *Commun. Nonlinear Sci. Numer. Simul.* **19**, 2820-2827 (2014).
[25] F. Punzo and G. Terrone, On the Cauchy problem for a general fractional porous medium equation with variable density, *Nonlin. Anal.* **98**, 27-47 (2014).
[26] A. Bashir and S. K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, *Appl. Math. Comput.* **266**, 615-622 (2015).
[27] A. Bashir and J. J. Nieto, Boundary value problems for a class of sequential integrodifferential equations of fractional order, *J. Funct. Spac. Appl.* **2013**, (2013).
[28] M. I. Nazim, S. Bawaneh and A. Al-Khateeb, On a coupled system of fractional differential equations with four point integral boundary conditions, *Mathematics* **7**(3), 270 (2019).
[29] M. I. Nazim and A. Al-Khateeb, Existence and stability results on Hadamard type fractional time-delay semilinear differential equations, *Mathematics* **8**(8), 1242 (2020).
[30] M. I. Nazim and A. Al-Khateeb, Existence and Ulam–Hyers stability of coupled sequential fractional differential equations with integral boundary conditions, *J. Ineq. Appl.* **2019**(1), 1-15 (2019).
[31] M. I. Nazim and A. Al-Khateeb, Stability, existence and uniqueness of boundary value problems for a coupled system of fractional differential equations, *Mathematics* **7**(4), 354 (2019).