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A New Bivariate Distribution with Frêchet and Burr-Type XII as Marginals

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Abstract: In this work, an explicit formula to derive bivariate distributions is discussed, and a new bivariate distribution is characterized by using the concomitant of k^{th} upper record values with Frêchet and Burr type XII distributions as marginals. Some properties of the new distribution are studied. Estimation of the parameters is discussed using maximum likelihood and pseudo-likelihood methods. In addition, a Monte Carlo simulation study is carried out to investigate the estimates of the parameters and their biases and MSE, and the applicability of the constructed distribution is tested by comparing with a competitive distribution using real data sets.

Keywords: Bivariate Distribution, Concomitant of upper Record Values, Local Measures of Dependence, Maximum likelihood estimation

1 Introduction

Developing new classes of bivariate distributions and examining their properties is essential to provide models for contemporary applications and phenomena. Although there have been many contributions in the last two decades to this matter, there are still many limitations in using these methodologies; one of these limitations is the available information about the bivariate population. For instance, if conditional probability distribution functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ of the bivariate distribution are provided, then the methodology of conditionally specified distributions [2] can be used. In which conditional probability distribution functions are used in $f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y)$ to find the marginals and the desired bivariate distribution. However, in order to use this method, the compatibility of conditional functions must be satisfied, which represents a limitation in using this method. Also, if the cumulative probability distribution functions (CDF's) $F_X(x)$ and $F_Y(y)$ of the bivariate distribution are provided, then any function $C(u,v)$ defined from I^2 to I and satisfying the copula properties [14] can be used. In which $C(F_X(x),F_Y(y))$ represents the bivariate CDF $F_{X,Y}(x,y)$. However, it is challenging to select a suitable copula among infinitely many for the bivariate population of interest. For more methodologies for developing new classes of bivariate distributions and their related limitations, see [12].

A recently used methodology to uniquely specify the bivariate distribution, in which the above mentioned restrictions are taken into consideration, if the concomitant of order statistics $f_{[r:n]}(x)$ or upper record values $f_{[U_k(n)]}(x)$ (see [5] and [15]) is provided for one variable and the marginal for the other one, in which T. G. Veena and P. Yageen Thomas [17] used the concomitant function to derive an inversion formula to construct new bivariate classes of distributions. This work has motivated us to apply this method to develop a new class of bivariate distribution using the k^{th} upper record values, in which one of the marginals and the concomitant of the k^{th} upper record values of the other distribution are provided. As shown in the following theorem:

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Theorem 1. *A new bivariate distribution can be uniquely specified by one of its marginals and the concomitant of k^{th} upper record of the other random variable, using the following inversion formula:*

$$f(u|y)u^{n-1} = \mathscr{L}^{-1}\left(\frac{f_{[U_n(k)]}(y)\Gamma(n)}{f(y)k^n}\right), \tag{1}$$

where $\mathscr{L}, f_{[U_n(k)]}$ and u are the Laplace transformation,the density function of the concomitant of k^{th} upper records and cumulative hazard rate function respectively.

Proof. Let (X_i,Y_i), where $i = 1,2,\ldots,$ be a sample from a bivariate distribution, with a absolutely continuous joint probability density $f_{X,Y}(x,y)$ and, $f_X(x)$ and $f_Y(y)$ as marginals of X and Y respectively, if $\{X_{U_k(n)}^{U_k(n)+k-1}\}$ is the sequence of the k-upper records of the random variable X, then the other component Y is denoted as the concomitant of k^{th} upper records (for details see [15]), with probability density function (PDF):

$$f_{[U_n(k)]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{x_{U(n)}}(x)dx \tag{2}$$

And, the PDF of the k^{th} upper record values is

$$f_{[U_n(k)]}(x) = \frac{k^n}{\Gamma(n)}(R(x))^{(n-1)} f(x)(1-F(x))^{(k-1)},$$

where, $R(x) = -\log[1-F(x)]$, the complete hazard function, Then, equation (2) can be rewritten as:

$$\begin{aligned} f_{[U_n(k)]}(y) &= f(y)\frac{k^n}{\Gamma(n)}\int_{-\infty}^{\infty} f(x|y)(R(x))^{(n-1)}e^{(-(k-1)R(x))}dx \\ &= f(y)\frac{k^n}{\Gamma(n)}E\left[(R(x))^{(n-1)}e^{-(k-1)R(x)}|y\right] \end{aligned}$$

For simplicity, let $R(x) = u$ and $k-1 = s$,

$$f_{[U_n(k)]}(y) = f(y)\frac{k^n}{\Gamma(n)}E\left[u^{(n-1)}e^{-su}|y\right], \tag{3}$$

where $u \sim exp(1)$, On the other hand, the analogy between equation (3) and Laplace transformation is noted (using[8]) as:

$$\begin{aligned} E\left[u^{(n-1)}e^{-su}|y\right] &= \int_0^{\infty} f(u|y)u^{(n-1)}e^{-su}dx \\ &= \mathscr{L}[f(u|y)u^{(n-1)}] \end{aligned} \tag{4}$$

From equation (3) and equation (4), the desired formula can be derived using the uniqueness property of the Laplace transformation.

2 The Bivariate Frêchet Burr-Type XII Distribution

In this section, a new class of bivariate distributions with Frêchet and Burr-type XII as marginals (FBBD) is derived.
Using the concomitant of k^{th} upper record of the constructed Weibull –Burr impounded bivariate distribution (WBBD)(see[11]), which is given by:

$$f_{[U_n(k)]}(y) = \frac{k^n}{\Gamma(n)}\frac{p}{\lambda}\left(\frac{y}{\lambda}\right)^{-(2p+1)}\frac{\Gamma(n+2)}{\left(k+\left(\frac{y}{\lambda}\right)^{-p}\right)^{n+2}}, \tag{5}$$

where, the random variable Y has a Burr type XII distribution with PDF (see [1]):

$$f_Y(y) = \frac{2p}{\lambda}\left(\frac{y}{\lambda}\right)^{p-1}\left(1+\left(\frac{y}{\lambda}\right)^{p}\right)^{-3} \qquad y \geq 0, p\, and\, \lambda > 0 \tag{6}$$

And, assume that the random variable X has a Frêchet distribution which PDF and CDF are given by Elbatal, Asha, and Raja [6]

$$f_X(x) = \frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{(\beta+1)} e^{-\left(\frac{\theta}{x}\right)^{\beta}} \qquad x \geq 0, \beta\, and\, \theta > 0 \tag{7}$$

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{\beta}} \qquad x \geq 0, \beta\, and\, \theta > 0 \tag{8}$$

respectively, which will be denoted as $X \sim F(\beta, \theta)$, and $Y \sim B - XII(P, \lambda)$, the following theorem can be derived.

Theorem 2. *The bivariate distribution with given marginal and concomitant of k^{th} upper record values in equation (6) and equation (7) is:*

$$f_{(X,Y)}(x,y) = \frac{p}{\lambda}\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{(\beta+1)} e^{-\left(\frac{\theta}{x}\right)^{\beta}}\left(\frac{y}{\lambda}\right)^{-(2p+1)}\left(-\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{2}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-p}}$$
$$x \geq 0, y \geq 0, p, \beta, \theta\, and\, \lambda > 0 \tag{9}$$

Proof. By substituting with equation (6) and equation (5) in the inversion formula (1), the following formula is obtained:

$$f(u|y)u^{n-1} = \mathcal{L}^{-1}\left(\left(\frac{y}{\lambda}\right)^{-3p}\frac{\Gamma(n+2)}{2}\frac{(1+(\frac{y}{\lambda})^{p})^{3}}{\left(s+1+(\frac{y}{\lambda})^{-p}\right)^{n+2}}\right)$$

Then, by applying the Laplace inverse for the R.H.S., to obtain:

$$\begin{aligned} f(u|y) &= \left(\frac{y}{\lambda}\right)^{-3p}\frac{1}{2}\left(1+\left(\frac{y}{\lambda}\right)^{p}\right)^{3} u^{2} e^{-u\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)} \\ &= \frac{1}{2}\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{3} u^{2} e^{-u\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)} \end{aligned}$$

now, a re-transform variable u to the original variable x is done by letting $u = -\log[1-F(x)]$.

$$\begin{aligned} f(x|y) &= 1/2\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{3}(-\log[1-F(x)])^{2} e^{\log[1-F(x)]\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)}\frac{f(x)}{1-F(x)} \\ &= 1/2 f(x)\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{3}(-\log[1-F(x)])^{2}(1-F(x))^{\left(\frac{y}{\lambda}\right)^{-p}} \end{aligned}$$

And, the new bivariate distribution $f_{(X,Y)}(x,y)$ can be easily constructed by multiplying this equation by $f_Y(y)$

$$\begin{aligned} f_{(X,Y)}(x,y) &= \frac{1}{2}f(x)\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{3}(-\log[1-F(x)])^{2}(1-F(x))^{\left(\frac{y}{\lambda}\right)^{-p}}\frac{2p}{\lambda}\left(\frac{y}{\lambda}\right)^{p-1}\left(1+\left(\frac{y}{\lambda}\right)^{p}\right)^{-3} \\ &= \frac{p}{\lambda}f(x)\left(\frac{y}{\lambda}\right)^{-(2p+1)}(-\log[1-F(x)])^{2}(1-F(x))^{\left(\frac{y}{\lambda}\right)^{-p}} \end{aligned}$$

Eventually, the given bivariate distribution with parameters $p, \beta, \theta, and\, \lambda$ can be derived by substituting with equation (7) and equation (8).

Figure 1 below shows the curve of the new distribution given by equation (9).

3 Some Properties of the Bivariate Distribution

In this section, some properties of the new bivariate distribution are exhibited.

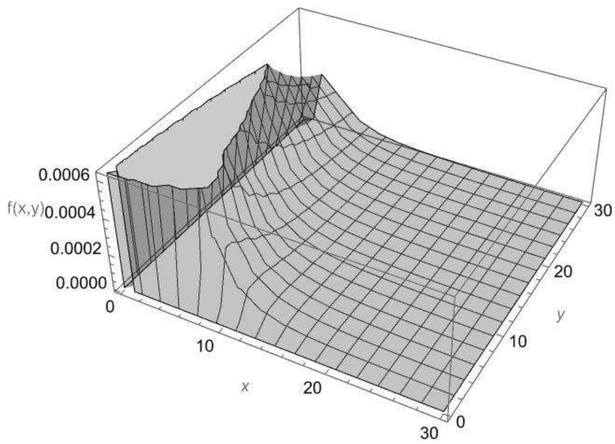

Fig. 1: The PDF given by equation (9) with parameters $p = 0.5$, $\lambda = 3$, $\beta = 2$, and $\theta = 1.5$.

3.1 Conditional Densities

The conditional PDF of X given $Y = y$ is:

$$f(x|y) = \frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{\beta+1} e^{-\left(\frac{\theta}{x}\right)^{\beta}}\left(-\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{2}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-p}}\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{3}, \quad x \geq 0, y \geq 0, p, \beta, \theta \; and \; \lambda > 0 \tag{10}$$

and, the conditional PDF of Y given $X = x$ is:

$$f(y|x) = \frac{p}{\lambda}\left(\frac{y}{\lambda}\right)^{-(2p+1)}\left(-\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{2}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-p}}, \quad x \geq 0, y \geq 0, p, \beta, \theta \; and \; \lambda > 0 \tag{11}$$

3.2 Cumulative Distribution Function

The bivariate CDF is given by

$$F(x,y) = \frac{1+2\left(\frac{y}{\lambda}\right)^{-p}}{\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)^{2}} - \frac{\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{1+\left(\frac{y}{\lambda}\right)^{-p}}}{1+\left(\frac{y}{\lambda}\right)^{-p}}\left(1-\left(\frac{y}{\lambda}\right)^{-p}\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]+\frac{\left(\frac{y}{\lambda}\right)^{-p}}{1+\left(\frac{y}{\lambda}\right)^{-p}}\right) \quad x \geq 0, y \geq 0, p, \beta, \theta \; and \; \lambda > 0$$

Figure 2 below shows the curve of the CDF.

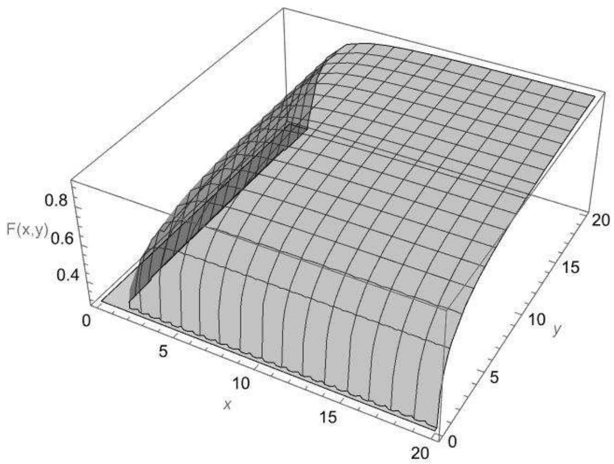

Fig. 2: the cumulative distribution function with parameters $p = 0.5, \lambda = 3, \beta = 2, and\, \theta = 1.5$.

3.3 Survival Function

Also, the formula of the survival function is computed by

$$\overline{F}(x,y) = \left(1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right) + \frac{\left(\frac{y}{\lambda}\right)^{-p}\left(1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{1+\left(\frac{y}{\lambda}\right)^{-p}}}{1 + \left(\frac{y}{\lambda}\right)^{-p}}\left(\log\left[1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right] - \left(\frac{y}{\lambda}\right)^{p} - \frac{1}{1 + \left(\frac{y}{\lambda}\right)^{-p}}\right)$$
$$x \geq 0, y \geq 0, p, \beta, \theta\, and\, \lambda > 0$$

3.4 The Moment Functions

The first moment of the bivariate distribution is given by:

$$E(XY) = \theta\lambda\left(\frac{p}{p+1}\right)\Gamma(2 - \frac{1}{p})\int_0^{\infty}\left(-\log\left[1 - e^{-u^{\frac{p}{p+1}}}\right]\right)^{\frac{-1}{\beta}} e^{-u^{\frac{p}{p+1}}}\,du$$

Also, the first moments of the conditional distributions are:

$$E(X|Y = y) = \theta\left(1 + \left(\frac{y}{\lambda}\right)^{-p}\right)^{3}\int_0^{\infty}\left(-\log\left[1 - e^{-u}\right]\right)^{\frac{-1}{\beta}} u^{2} e^{-u\left(1+\left(\frac{y}{\lambda}\right)^{-p}\right)}\,du$$

And

$$E(Y|X = x) = p\Gamma(2 - \frac{1}{p})\left(-\log\left[1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{\frac{1}{p}}$$

3.5 The Hazard Rate Function

The hazard rate function of f(x,y) is denoted by h(x,y) and given by

$$h(x,y) = \frac{f(x,y)}{\overline{F}(x,y)}$$

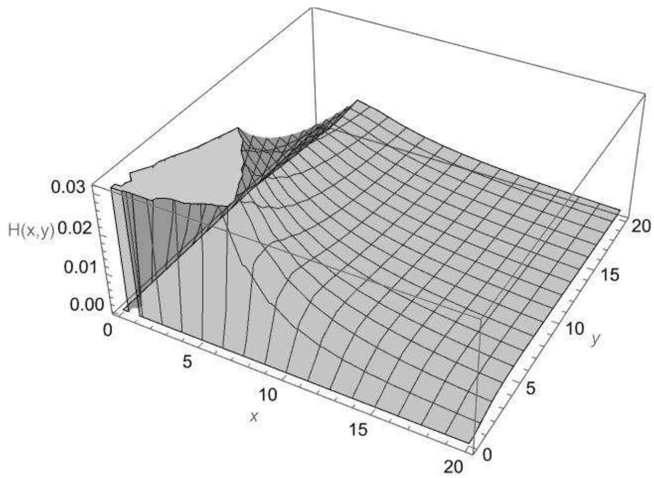

Fig. 3: the hazard rate function with parameters $p = 0.5, \lambda = 3, \beta = 2, and\, \theta = 1.5$

or,

$$h(x,y) = \frac{\frac{p}{\lambda}\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{(\beta+1)} e^{-\left(\frac{\theta}{x}\right)^{\beta}} \left(\frac{y}{\lambda}\right)^{-(2p+1)} \left(-\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{2} \left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-p}}}{\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right) + \frac{\left(\frac{y}{\lambda}\right)^{-p}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{1+\left(\frac{y}{\lambda}\right)^{-p}}}{1+\left(\frac{y}{\lambda}\right)^{-p}} \left(\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right] - \left(\frac{y}{\lambda}\right)^{p} - \frac{1}{1+\left(\frac{y}{\lambda}\right)^{-p}}\right)}$$

$$x \geq 0, y \geq 0, p, \beta, \theta\, and\, \lambda > 0$$

Figure 3 shows the curve of the hazard rate function.

4 Related Distribution

In this section, related classes of the constructed bivariate distribution are exhibited according to, selections of $p, \lambda, \beta\, and\, \theta$.

4.1 The Uniform Burr type XII bivariate distribution: $\left(u = e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)$

the bivariate distribution (9) reduces to:

$$f_{(X,Y)}(x,y) = \frac{p}{\lambda}\left(\frac{y}{\lambda}\right)^{-(2p+1)} (-\log[1-u])^2 (1-u)^{\left(\frac{y}{\lambda}\right)^{-p}}$$

$$u \in (0,1), y \geq 0, p, and\, \lambda > 0$$

the two RV's X and Y has marginal densities:

$$f_X(x) = 1 \qquad , u \in (0,1)$$

which is the uniform distribution $U(0,1)$, and

$$f_Y(y) = \frac{2p}{\lambda}\left(\frac{y}{\lambda}\right)^{(p-1)}\left(1+\left(\frac{y}{\lambda}\right)^{p}\right)^{-3}, \qquad y > 0, p\, and\, \lambda > 0.$$

which is the Burr-type XII distribution.

4.2 The Weibull Burr type XII bivariate distribution: $(u=\frac{1}{x})$

the bivariate distribution (9) reduces to:

$$f_{(X,Y)}(x,y)=\frac{p}{\lambda}(\beta\theta)(\theta u)^{(\beta-1)}e^{-(\theta u)^{\beta}}\left(\frac{y}{\lambda}\right)^{-(2p+1)}\left(-\log\left[1-e^{-(\theta u)^{\beta}}\right]\right)^{2}\left(1-e^{-(\theta u)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-p}}$$
$$u>0,y>0,p,\beta,\theta\; and\; \lambda>0$$

which is a generalization of [16],
the two RV's X and Y has marginal densities:

$$f_X(x)=\beta\theta(\theta u)^{(\beta-1)}e^{-(\theta u)^{\beta}}, \qquad u\geq 0,\beta\; and\; \theta>0,$$

which is the Weibull (β,θ) distribution, and

$$f_Y(y)=\frac{2p}{\lambda}\left(\frac{y}{\lambda}\right)^{(p-1)}\left(1+\left(\frac{y}{\lambda}\right)^{p}\right)^{-3}, \qquad y\geq 0,p\; and\; \lambda>0.$$

which is the Burr-type XII distribution.

4.3 The Frêchet Lomax bivariate distribution: $(p=1)$

the bivariate distribution (9) reduces to:

$$f_{(X,Y)}(x,y)=\frac{1}{\lambda}\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{(\beta+1)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}\left(\frac{\lambda}{y}\right)^{3}\left(-\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right)^{2}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{\lambda}{y}\right)},$$
$$x\geq 0,y\geq 0,\beta,\theta\; and\; \lambda>0$$

the two RV's X and Y has marginal densities:

$$f_X(x)=\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{(\beta+1)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}, \qquad x\geq 0,\beta\; and\; \theta>0,$$

which is the Frêchet distribution, and

$$f_Y(y)=\frac{2}{\lambda}\left(1+\frac{y}{\lambda}\right)^{-3}, \qquad y\geq 0,\lambda>0.$$

which is the Lomax $(2,\lambda)$ distribution.

4.4 The Uniform Lomax bivariate distribution: $\left(u=e^{-\left(\frac{\theta}{x}\right)^{\beta}},p=1\right)$

the bivariate distribution (9) reduces to:

$$f_{(X,Y)}(x,y)=\frac{1}{\lambda}\left(\frac{\lambda}{y}\right)^{3}(-\log[1-u])^{2}(1-u)^{\left(\frac{\lambda}{y}\right)},$$
$$u\in(0,1),y\geq 0,\lambda>0$$

the two RV's X and Y has marginal densities:

$$f_X(x)=1 \qquad ,u\in(0,1)$$

which is the uniform distribution $U(0,1)$, and

$$f_Y(y)=\frac{2}{\lambda}\left(1+\frac{y}{\lambda}\right)^{-3}, \qquad y\geq 0,\lambda>0.$$

which is the Lomax $(2,\lambda)$ distribution.

4.5 The Weibull Lomax bivariate distribution: $(u = \frac{1}{x}, p = 1)$

the bivariate distribution (9) reduces to:

$$f_{(X,Y)}(x,y) = \frac{1}{\lambda}(\beta\theta)(\theta u)^{\beta-1}e^{-(\theta u)^{\beta}}\left(\frac{\lambda}{y}\right)^{3}\left(-\log\left[1-e^{-(\theta u)^{\beta}}\right]\right)^{2}\left(1-e^{-(\theta u)^{\beta}}\right)^{\frac{\lambda}{y}},$$
$$u \geq 0, y \geq 0, \beta, \theta \, and \, \lambda > 0$$

the two RV's X and Y has marginal densities:

$$f_X(x) = \beta\theta(\theta u)^{\beta-1}e^{-(\theta u)^{\beta}}, \qquad x \geq 0, \beta \, and \, \theta > 0,$$

which is the Weibull (β, θ) distribution, and

$$f_Y(y) = \frac{2}{\lambda}\left(1+\frac{y}{\lambda}\right)^{-3}, \qquad y \geq 0, \lambda > 0.$$

which is the Lomax $(2, \lambda)$ distribution.

5 Dependence

In this section the correlation between X and Y is studied, using *Local Measures of Dependence*, by defining the local dependence function (see [10]):

$$\gamma_f(x,y) = \lim_{dx,dy\to 0}\frac{\log[\alpha(x,y;x+dx,y+dy)]}{dxdy}$$
$$= \frac{\sigma^2}{\sigma_x\sigma_y}\log[f_{(X,Y)}(x,y)],$$

where $\alpha(x,y;u,v)$ is the cross-product ratio of the bivariate distribution function.
The importance of the local dependence function is to study the dependence without using its margins, as illustrated in the following theorem.

Theorem 3. *The bivariate distribution (9) is Totally Positive of Order 2* (TP_2).

Proof. By deriving the local dependence function, to obtain:

$$\gamma_f(x,y) = \frac{p}{\lambda}\left(\frac{y}{\lambda}\right)^{-(p+1)}\frac{\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{\beta+1}e^{-\left(\frac{\theta}{x}\right)^{\beta}}}{1-x^{-\left(\frac{\theta}{x}\right)^{\beta}}}. \tag{12}$$

Hence, $\gamma_f(x,y) > 0$ for any choice of parameters $p, \lambda, \beta, and \, \theta$, the given distribution is TP_2, according to Erdélyi [7] definitions; $\gamma_f(x,y) > 0$ means more positive dependence (positively correlated) than independence.

Remark. The local dependence function (12) can be used with the marginals to uniquely specify the bivariate distribution.

6 Estimation of the Parameters

Suppose that a sample $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ of size n is observed from the bivariate distribution (9), then parameter estimation can be done using one of the following methods:

6.1 Maximum Likelihood Estimation (MLE)

First method to estimate the parameters is by using the likelihood function, which given by:

$$L(p,\beta,\theta,\lambda)=\prod_{i=1}^{n}f_{(x,y)}(x_i,y_i) \tag{13}$$

By substituting with the bivariate distribution (9):

$$L(p,\beta,\theta,\lambda)=\prod_{i=1}^{n}\frac{p}{\lambda}\frac{\beta}{\theta}\left(\frac{\theta}{x_i}\right)^{(\beta+1)}e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\left(\frac{y_i}{\lambda}\right)^{-(2p+1)}\left(-\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\right)^{2}\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\left(\frac{y_i}{\lambda}\right)^{-p}}$$

Taking Log on both sides,

$$\begin{aligned}log[L(p,\beta,\theta,\lambda)]&=\sum_{i=1}^{n}\left(\log\left[\frac{p\beta}{\lambda\theta}\right]-(2p+1)\log\left[\frac{y_i}{\lambda}\right]+(\beta+1)\log\left[\frac{\theta}{x_i}\right]-\left(\frac{\theta}{x_i}\right)^{\beta}+2\log\left[\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\right]\right.\\&\quad\left.+\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\right)\\&=n\log\left[\frac{p\beta}{\lambda\theta}\right]-(2p-1)\sum_{i=1}^{n}\log\left[\frac{y_i}{\lambda}\right]+(\beta+1)\sum_{i=1}^{n}\log\left[\frac{\theta}{x_i}\right]-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta}+2\sum_{i=1}^{n}\log\left[\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\right]\\&\quad+\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right],\end{aligned}$$

Where $log[L(p,\beta,\theta,\lambda)]$ is the log-likelihood function, differentiating it w.r.t. the parameters $p,\beta,\theta,and\,\lambda$ respectively, to obtain:

$$\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta p}=\frac{n}{p}-2\sum_{i=1}^{n}\log\left[\frac{y_i}{\lambda}\right]-\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[\frac{y_i}{\lambda}\right]\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right] \tag{14}$$

$$\begin{aligned}\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\beta}&=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[\frac{\theta}{x_i}\right]-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]+2\sum_{i=1}^{n}\frac{e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]}{\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]}\\&\quad+\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\frac{e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]}{1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\theta}&=\frac{n\beta}{\theta}-\sum_{i=1}^{n}\frac{\beta}{x_i}\left(\frac{\theta}{x_i}\right)^{\beta-1}+2\sum_{i=1}^{n}\frac{\frac{\beta}{x_i}\left(\frac{\theta}{x_i}\right)^{\beta-1}e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]}\\&\quad+\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\frac{\frac{\beta}{x_i}\left(\frac{\theta}{x_i}\right)^{\beta-1}e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}\end{aligned} \tag{16}$$

$$\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\lambda}=\frac{2pn}{\lambda}+\sum_{i=1}^{n}\frac{p}{y_i}\left(\frac{y_i}{\lambda}\right)^{-p+1}\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right] \tag{17}$$

Then after setting the partial derivatives equal to zero, a numerical method such as Newton-Raphson Method is applied to solve equations (14), (15), (17), and (16), due to their implicit nature of them.

6.2 Pseudo Likelihood Estimation (MPLE)

Another method to estimate the parameters is by using the Pseudo Likelihood function (see[4] and [3]):

$$PL(p,\beta,\theta,\lambda)=\prod_{i=1}^{n} f_{(x|y)}(x_i|y_i)f_{(y|x)}(y_i|x_i) \tag{18}$$

By substituting with conditional densities (10) and (11), then taking Log on both sides:

$$\begin{aligned}\log[PL(p,\beta,\theta,\lambda)] &= n\log\left[\frac{p\beta}{\theta\lambda}\right]-(2p+1)\sum_{i=1}^{n}\log\left[\frac{y_i}{\lambda}\right]+(\beta+1)\sum_{i=1}^{n}\log\left[\frac{\theta}{x_i}\right]-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta}\\ &+4\sum_{i=1}^{n}\log\left[-\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\right]+2\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\\ &+3\sum_{i=1}^{n}\log\left[1+\left(\frac{y_i}{\lambda}\right)^{-p}\right]\end{aligned}$$

By differentiating w.r.t. the parameters $p,\beta,\theta, and\, \lambda$ respectively, to obtain the Pseudo Likelihood equations as:

$$\begin{aligned}\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta p} &= \frac{n}{p}-2\sum_{i=1}^{n}\log\left[\frac{y_i}{\lambda}\right]-2\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[\frac{y_i}{\lambda}\right]\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\\ &-3\sum_{i=1}^{n}\frac{\left(\frac{y_i}{\lambda}\right)^{-p}\log\left[\frac{y_i}{\lambda}\right]}{1+\left(\frac{y_i}{\lambda}\right)^{-p}}\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\beta} &= \frac{n}{\beta}+\sum_{i=1}^{n}\log\left[\frac{\theta}{x_i}\right]-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]-4\sum_{i=1}^{n}\frac{\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)}\\ &+2\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\frac{\left(\frac{\theta}{x_i}\right)^{\beta}\log\left[\frac{\theta}{x_i}\right]e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)}\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\theta} &= (\beta+2)\frac{n}{\theta}-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta-1}\left(\frac{\beta}{x_i}\right)-4\sum_{i=1}^{n}\frac{\left(\frac{\theta}{x_i}\right)^{\beta-1}\left(\frac{\beta}{x_i}\right)e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)}\\ &+2\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p}\frac{\left(\frac{\theta}{x_i}\right)^{\beta-1}\left(\frac{\beta}{x_i}\right)e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}}{\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right)}\end{aligned} \tag{21}$$

$$\frac{\delta\log[PL(p,\beta,\theta,\lambda)]}{\delta\lambda}=(2p)\frac{n}{\lambda}+2\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{-p+1}\left(\frac{p}{y_i}\right)\log\left[1-e^{-\left(\frac{\theta}{x_i}\right)^{\beta}}\right]+3\sum_{i=1}^{n}\frac{\left(\frac{y_i}{\lambda}\right)^{-p+1}\left(\frac{p}{y_i}\right)}{1+\left(\frac{y_i}{\lambda}\right)^{-p}} \tag{22}$$

Then setting the partial derivatives equal to zero and a numerical method such as Newton-Raphson Method is applied to solve equations (19), (22), (20), and (21), due to their implicit nature of them.

7 Simulation

The Monte Carlo simulation study is done to compare the estimation methods of the parameters of the probability distribution function given in (9). The study relies on samples generated using a conditional distribution, with sizes 30, 50, and 100. The methodology for generating bivariate random variables, using Wolfram Mathematica software, is as follows:

1. Generate two sets of uniform random numbers $(0,1)$, say, u and v.
2. Let X follow Frêchet (β,θ). Using inverse CDF transformation, to generate X using the expression $X=\theta\left(-\log[u]\right)^{\frac{-1}{\beta}}$.
3. Then, by using the CDF of the conditional distribution $Y|X$, the generated X value, and the generated random number v, to generate Y using the expression

$$v=\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\left(\frac{y}{\lambda}\right)^{-P}}\left(1-\left(\frac{y}{\lambda}\right)^{-P}\log\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]\right).$$

4. After generating the bivariate data of X and Y. It was used to compute maximum likelihood and pseudo-likelihood estimates for $p,\lambda,\beta, and\, \theta$, then the Mean Square Error (MSE) and average bias are computed, which occurred in the estimates when compared to the true values of $p,\lambda,\beta, and\, \theta$ in 1000 Monte Carlo iterations.

Eventually table 1 is obtained.

Table 1: Estimation of parameters

Sample Size	Parameters	Given value	MLE	MSE	BIAS	MPLE	MSE	BIAS
50	p	0.5	0.5214	0.0018	0.0214	0.5204	0.0031	0.0204
	λ	3	3.0383	0.0093	0.0383	3.0033	0.0095	0.0033
	β	2	1.9768	0.0084	-0.0231	2.0197	0.0147	0.0197
	θ	1.5	1.5435	0.0049	0.0435	1.5213	0.0091	0.0213
100	p	0.5	0.5223	0.0011	0.0223	0.4981	0.0011	-0.0018
	λ	3	3.0838	0.0096	0.0838	2.9766	0.0096	-0.0233
	β	2	1.9308	0.0104	-0.0691	2.0314	0.0076	0.0314
	θ	1.5	1.5709	0.0068	0.0709	1.4784	0.0042	-0.0215
200	p	0.5	0.4970	0.0002	-0.0029	0.5048	0.0006	0.0048
	λ	3	3.0533	0.0089	0.0533	2.9959	0.0093	-0.0040
	β	2	1.9472	0.0050	-0.0527	2.0028	0.0048	0.0028
	θ	1.5	1.5102	0.0007	0.0102	1.5060	0.0025	0.0060

From Table 1, we observe that MSE and BIAS tend to zero in both methods as the sample size increases. Also, we can verify that MPLE is better than MLE due to better accuracy and lower MSE and BIAS values. Consequently, we conclude that MLE and MPLE can be used to estimate the parameters.

8 Application

In this section, two real data sets are used to compare the constructed distribution (FBBD) and one of its related distributions (WBBD) [16] The first data set as reported in [13] which consists of 77 observations on (X,Y), where X represents the aquifer resistivity (Ωm) and Y represents the coefficient of anisotrophy (λ) as:
(105, 0.25), (130, 0.1), (115, 0.11), (115, 0.01), (145, 0.08), (120, 0.01), (128, 0.12), (140, 0.06), (125, 0.04), (105, 0.13), (85, 0.37), (140, 0.22), (115, 0.1), (100, 0.14), (110, 0.08), (75, 0.51), (125, 0.06), (128, 0.2), (122, 0.07), (135, 0.13), (136, 0.11), (225, 0.78), (135, 0.44), (65, 0.84), (125, 0.31), (65, 0.49), (95, 0.38), (170, 0.59), (105, 0.22), (168, 0.6), (125, 0.42), (156, 0.39), (132, 0.11), (75, 0.42), (148, 0.42), (122, 0.06), (106, 0.32), (380, 1.43), (118, 0.3), (148, 0.45), (240, 0.65), (320, 1.23), (152, 0.49), (131, 0.13), (115, 0.31), (120, 0.28), (112, 0.25), (121, 0.28), (110, 0.27), (109, 0.1), (116, 0.18), (105, 0.18), (135, 0.46), (100, 0.08), (127, 0.37), (342, 1.2), (230, 0.11), (165, 0.18), (185, 0.24), (259, 0.14), (127, 0.14), (247, 0.12), (340, 0.84), (380, 1.18), (161, 0.51), (143, 0.01), (456, 1.7), (390, 1.9), (198, 0.24), (225, 0.5), (106, 0.16), (207, 0.43), (320, 1.22), (134, 0.26), (245, 0.4), (270, 0.52), (125, 0.04).

The second data set reported in [9] which consists of 41 observations in (X,Y), where X represents the SO_2 content of air in micrograms per cubic meter and Y represents the average annual temperature in F as:
(10, 70.3), (13, 61.0), (12, 56.7), (17, 51.9), (56, 49.1), (36, 54.0),(29, 57.3), (14, 68.4), (10, 75.5), (24, 61.5), (110, 50.6), (28, 52.3), (17, 49.0), (8, 56.6), (30, 55.6), (9, 68.3), (47, 55.0), (35, 49.9), (29, 43.5), (14, 54.5), (56, 55.9), (14, 51.5), (11, 56.8), (46, 47.6), (11, 47.1), (23, 54.0), (65, 49.7), (26, 51.5), (69, 54.6), (61, 50.4), (94, 50.0), (10, 61.6), (18, 59.4), (9, 66.2), (10, 68.9), (28, 51.0), (31, 59.3), (26, 57.8), (29, 51.1), (31, 55.2), (16, 45.7).
The methodology of comparison, using Mathematica, as:

1. Given data is used to compute maximum likelihood estimates for the parameters of each distribution.
2. Calculate the log-likelihood function for each distribution.
3. Compare between distributions, using AIC (Akaike information criterion) and BIC (Bayesian information criterion), which defined as:

$$AIC = 2K - 2l \text{ and } ,BIC = k\log[n] - 2l,$$

where, l denotes the log–likelihood function, k is the number of parameters and n is the sample size.

by comparing AIC and BIC in Table 2, $FBBD$ is observed to provide a better fit for given data sets.

Table 2: Log-likelihood, AIC, and BIC of FBBD, and WBBD distributions.

	Distribution	Parameters	Estimation	$-l$	AIC	BIC
First Data Set	FBBD	p λ β θ	0.6623 0.6237 2.0765 109.2724	452.8859	913.7718	913.3177
	WBBD	p λ β σ	0.6623 0.6237 2.0765 109.2724	452.8859	913.7718	913.3177
Second Data Set	FBBD	p λ β θ	6.0509 58.4142 0.9642 10.9661	342.0984	692.1968	690.6479
	WBBD	p λ β σ	6.0849 58.4786 0.7332 28.0988	344.3735	696.747	695.1981

9 Conclusion

The concomitant of the k^{th} upper record values is used to derive an explicit formula to generate bivariate distributions; the Frêchet and Burr-type XII bivariate distribution is introduced in the present paper. Conditional densities, survival function, moments, are among the presented properties. Two methods are used, maximum likelihood and pseudo-likelihood, to estimate the parameters of the introduced distribution. Also, a Mont Carlo simulation study was done to observe that MPLE is better than MLE at estimating parameters. Finally, two data sets are used to compare between Frêchet and Burr-type XII and Weibull and Burr-type XII distributions, and observe that Frêchet and Burr-type XII provide a better fit for both. Wishing that the present study will serve as a reference and help in future work in this field.

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