Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/140605

Estimation of Financial Risk using the Archimedean Gumbel Copula with Log-Normal Distributed Marginals

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Received: 7 Jun. 2025, Revised: 21 Sep. 2025, Accepted: 2 Oct. 2025
Published online: 1 Nov. 2025

Abstract: This study uses the Archimedean Gumbel copula to model the dependence between the South African Industrial Index (J520) and South African Financial Index (J580) growth rates. This study aims to estimate extreme portfolio risk and account for diversification benefits of investing in a portfolio consisting of the growth rates from the two assets. This study uses growth rates, which contain only positive values, in contrast to financial returns used in other studies. Growth rates align with the requirements of copula models, which accept input values only within the (0, 1) interval, whereas financial returns can be either negative or positive. The Log-Normal distribution is used as the marginal distribution of each of the two growth rates, instead of the Normal distribution, to characterise better the heavy tails, a typical stylised fact prevalent in financial returns and growth rates. The scatterplots and AIC/BIC indicate that the dependence for the financial growth rates is best captured using the Archimedean Gumbel copula. The Archimedean Gumbel copula is used to capture extreme risk in the data sets. Monte Carlo simulation is used to quantify the risk of the portfolio. The results show minimal diversification benefits when investing in the two assets. The observation of minimal diversification benefits can suggest that the South African economy may be more stable, with limited contagion between the Industrial and Financial sectors. This information has important implications for both local and international investors regarding their strategies for investment, diversification, contagion management, and hedging decisions.

Keywords: Diversification benefits, Expected shortfall, Growth rates, Log-Normal distribution, Value-at-Risk

1 Introduction

Stock market participants often favour portfolios containing multiple assets to leverage the benefits of diversification, which then helps lower risk. This underscores the importance of accurately estimating portfolio risk, as univariate models tend not to effectively capture risk when the risk factors are dependent. Therefore, in situations where assets exhibit dependence, copula models can be used to assess the dependence structure, portfolio risk and evaluate the benefits and advantages of diversification with greater precision. In light of this, understanding the nature and extent of dependence between assets becomes particularly crucial, especially in increasingly interconnected markets such as those influenced by globalisation, financial integration and contagion.

According to [1], financial integration and contagion have been of growing interest to investors and practitioners since the Global Financial Crisis (GFC) of 2008-2009 as it reflects the increased inter-connectedness of Global stock markets arising from rapid globalisation. Global stock markets have become highly dependent on each other. However, developing countries' stock markets are less dependent on developed stock markets. Furthermore, the high dependence between the assets reduces the benefit of diversification to a minimum [2]. This study estimates the portfolio risk and diversification benefits of the growth rates of the South Africa Industrial Index (J520) and Financial Index (J580) growth rates.

One of the main challenges in risk management is the aggregation of individual risk factors. The problem becomes much more challenging when modelling dependent random variables. It is especially difficult when one does not know the joint distribution needed to determine the dependence structure (co-movement) between the given individual financial risk factors. A copula model is generally used to model the dependence structure between individual risk factors and

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estimate portfolio risk and diversification effects. A diversified portfolio will have less risk than the univariate sum of its constituent stocks [3].
According to [4], there is higher dependence between developed stock markets (which mainly excludes developing countries' stock markets). Stock market participants can mitigate the impact of rising dependence and inflation between developed stock markets by investing in portfolios, including developing countries' stock markets, such as the South African stock market financial assets in the portfolio. International diversification can benefit investors in potentially improving investment diversification, as developing countries' markets are less dependent on developed stock markets. Efforts have been made to investigate the inter-dependence of financial assets at industrial sector level [5]. In this study, two industrial sectors, viz: the South African Industrial and Financial sectors, are investigated. This study is confined to a bivariate case, although in principle the ideas discussed can be applied to higher-dimension models. The portfolio risk measures adopted in this study are the Value at Risk (VaR) and Expected Shortfall (ES). According to [6], VaR is the primary risk metric used by investors and practitioners to estimate the market risk, due to its simplicity, ease of computation and applicability. According to [7], VaR is not sub-additive and therefore is not a coherent risk measure. However, VaR is sub-additive in most practical situations and is therefore considered coherent in those situations. This property of sub-additivity is mathematical description of diversification and represents the benefits of diversification for a portfolio [2]. In this study, the Log-Normal distribution marginals are used to estimate individual risks of the financial growth rates of the two assets. This distribution is sufficient to describe different degrees of skewness, leptokurtosis and heavy-tailedness of the given growth rates. [8] proposed the use of the Archimedean Gumbel copula to overcome the shortcomings of the Gaussian copula. The Gaussian copula underestimates extreme risk. [9] used Archimedean Gumbel copula which offers investors and practitioners, a powerful tool to model the portfolio's extreme risk. The Archimedean Gumbel copula with Log-Normal distributed marginals model is adopted in this paper and used to estimate portfolio risk and account for diversification benefits thereof.

1.1 The aim of the study

The main aim of the study is to estimate portfolio risk and evaluating diversification benefits using data from two major South African equity indices viz: the South African Industrial Index (J520) and the South African Financial Index (J580). Diversification, a strategy aimed at reducing exposure to individual assets, industries, or countries, plays a key role in managing portfolio volatility. A portfolio's diversification level depends on both the number of assets and their dependence structure. However, research specifically addressing investment risk estimation for these South African indices using growth rates is limited. This study aims to fill that gap by estimating the portfolio risk of an equally weighted portfolio comprising the growth rates of these two indices, thereby assessing the potential diversification benefits. The specific objectives are: (i) to identify the best-fitting marginal distribution for the growth rate data among the Log-Normal, Exponential, Weibull, and Pareto-type tail distributions; and (ii) to estimate univariate risk using Value at Risk (VaR) and Expected Shortfall (ES). The study determines that the Archimedean Gumbel copula best models the dependence between the indices. A joint distribution is then fitted to the best-fitting marginal distribution. Portfolio risk is subsequently estimated using the copula-based approach, and the resulting VaR and ES measures are interpreted. Finally, the study quantifies the diversification benefits. The findings contribute to the empirical literature by offering a robust statistical framework for modelling dependence structures and calculating portfolio risk, particularly useful for market participants dealing with extreme risk scenarios in the South African equity market.

2 Review of Literature

[10] investigated the estimation of portfolio VaR using an approach combining Copula functions, EVT and GARCH-GJR models. The study investigated the dependence structure between Tehran Stock Exchange Price Index (TEPIX) (Iran) and Composite NASDAQ Index (USA). The study used an asymmetric GARCH model and an EVT method to model the marginal distributions of each log returns series and then using Copula functions (Gaussian, Student's t, Clayton, Gumbel and Frank) to link the marginal distributions together into a bivariate joint distribution. The portfolio VaR was then estimated. To check the goodness of fit of the approach, backtesting methods were used. The empirical results show that the copula model captured the dependence more accurately when compared with traditional methods.
[11] estimated equity portfolio risk using the Generalised Pareto Distribution (GPD)-Archimedean copula approach in South Africa. The study used eight sets of data consisting of the following stock market index returns, viz: the JSE/FTSE All Share Index (South Africa), Bovespa Index (Brazil), Indice de Precios Cotizaciones Index (Mexico), Shanghai Composite Index (China), SP500 (United States), FTSE100 (United Kingdom), DAX (Germany) and France's CAC40 (France). The Gumbel copula was shown to be good in modelling the upper tail of the return distribution in bullish

markets, whereas the Clayton copula was useful in modelling the lower tail dependency structure in bearish markets.
[6] proposed a method for estimating portfolio VaR using the GPD-Gumbel copula model in Brazil. The study used a portfolio of two stock market Indices, viz: IBOVESPA and MERVAL from Brazil. The losses/left tail were modelled using the GPD marginal distributions. The Gumbel copula was used to estimate the dependency structure/joint probability distribution of the two Indices. The model was used to estimate portfolio VaR of the two Indices and compared to other traditional methods. The results revealed that the GPD-Gumbel copula model provided more accurate portfolio risk estimates than other traditional methods.
[12] used the Gaussian copula and the Student-t-copula with Normal distribution and GPD marginals returns to estimate portfolio VaR in Italy. The researchers used the Monte Carlo method in estimating and optimising the portfolio risk of 20 Italian equities. The aim was to obtain an accurate estimate of the portfolio VaR. Their findings were that the GPD-copula model provided better VaR estimates than other traditional VaR models.
[13] applied the GPD-copula model to model a non-linear dependence structure (co-movement) of financial returns and estimated diversification effects in the USA. The GPD was used as the marginal distribution for the lower and upper tails separately and quantify the VaR and ES. The simulated sample consisted of financial returns data from United States of America financial institutions. The results revealed that the GPD-copula model provides more accurate portfolio VaR and ES estimates than historical simulation traditional methods and that the diversification benefits can be harvested. The diversification benefits ranged from 0.5 % to 10.7 %. [14] in Brazil, used copulas to model the dependence structure of a portfolio consisting of two financial Indices returns, viz: IBOVESPA (Brazil) and SP500 (USA). When compared to the Student t-copula under heavier-tails, the study found little difference in the benefits of diversification with the Gaussian copula.
There are many other studies in literature that have applied the copula models with the different statistical distributions as marginals to estimate portfolio risk, but this study extends these studies by not only estimating portfolio VaR but also estimating diversification benefits on a South Africa dataset. There are limited studies that used the Archimedean Gumbel copula with the Log-Normal distributed marginals to estimate portfolio risk and even diversification benefits in South Africa. The study uses the full dataset in modelling the Log-Normal distribution as marginal distributions for the South African Industrial and Financial Indices growth rates.

3 Methodology

This study uses the Archimedean Gumbel copula with a statistical parent distribution as marginals to estimate portfolio risk and calculate diversification benefits. These two components are discussed in the sections below.

3.1 Statistical Parent Distributions

There are many different statistical parent distributions, listed in order of increasing tail heaviness (including the Weibull with $\tau > 1$, Exponential, Weibull with $\tau < 1$, Log-Normal, Burr, and Pareto distributions) that can be used to describe financial growth rate distributions.

The statistical parent distributions model the main body of the data, and the selected distributions are light to moderately heavy-tailed. The Log-Normal distribution is shown to provide the best fit for the two indices' growth rates. The Maximum Likelihood Estimate (MLE) method is used to estimate the parameters of the fitted financial growth rate distributions.

3.2 The Log-Normal Distribution

The Log-Normal distribution is a probability distribution commonly used to model variables whose logarithm is Normally distributed. According to [15], the probability density function (PDF) and cumulative distribution function (CDF) of the Log-Normal distribution are given by:

$$f(x) = \frac{1}{\sigma x}\phi\left(\frac{\log x - \mu}{\sigma}\right), \tag{1}$$

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right), \tag{2}$$

where Φ is the standard Normal cumulative distribution function, and μ and σ represent the shape and scale parameters, respectively. The Log-Normal distribution is heavier-tailed than the Weibull distribution [16].

Theorem 1. *The log-likelihood function for a sample $\{x_1,\dots,x_m\}$ from a Log-Normal distribution with parameters μ and σ is:*

$$L=-\frac{m}{2}\ln(2\pi)-\frac{m}{2}\ln(\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(\ln x_i-\mu)^2-\sum_{i=1}^{m}\ln x_i. \tag{3}$$

The MLE estimates for the parameters are:

$$\hat{\mu}=\frac{1}{m}\sum_{i=1}^{m}\ln x_i, \tag{4}$$

$$\hat{\sigma}^2=\frac{1}{m}\sum_{i=1}^{m}(\ln x_i-\hat{\mu})^2. \tag{5}$$

3.3 Log-Normal Distribution Risk Measures

In this section, the formulas used to calculate and quantify risk in the South African Industrial and Financial Indices growth rate data, namely Value-at-Risk (VaR) and Expected Shortfall (ES), are discussed.

According to [15], the VaR and ES for the Log-Normal distribution are calculated using the following formulas:

$$\mathrm{VaR}_p(X)=\exp[\mu+\sigma\phi^{-1}(p)], \tag{6}$$

$$\mathrm{ES}_p(X)=\frac{\exp(\mu)}{p}\int_0^p\exp[\sigma\phi^{-1}(v)]\,dv, \tag{7}$$

where $x>0$, $0<p<1$, $-\infty<\mu<\infty$ is the location parameter, and $\sigma>0$ is the scale parameter.

3.4 Modelling Tail Dependence

The concept of tail dependence deals with the joint probability of extreme events occurring in the upper or lower tails of a bivariate distribution. In this study, the Archimedean Gumbel copula with selected marginal distributions is used to analyze the bivariate portfolio composed of the South African Industrial and Financial Indices growth rates.

The Archimedean Gumbel copula is relatively simple as it has a closed-form expression and can be applied to marginal distributions of several variables [17]. As noted in [18], it is an extreme value copula that is asymmetric and exhibits greater upper tail dependence. It is found appropriate for modeling the data used in this study. The bivariate Archimedean Gumbel copula is also referred to as the Gumbel-Hougaard copula.

According to [19], the Archimedean Gumbel copula is defined as:

$$C(u_1,u_2;\theta)=\exp\left(-\left[(-\log u_1)^{\theta}+(-\log u_2)^{\theta}\right]^{1/\theta}\right), \tag{8}$$

where $\theta\in[1,\infty)$. The parameter θ estimates the degree of dependency: when $\theta=1$, independence is obtained; when $\theta>1$, upper tail dependence is present; and as $\theta\to\infty$, the Gumbel copula converges to perfect positive dependence.

According to [20], the Archimedean Gumbel copula is attractive because its parameter θ is related to both the tail dependence coefficient and Kendall's tau. The relationship between θ and Kendall's tau τ is summarized in Table **??**.

3.5 Parameter Estimation

The Archimedean Gumbel copula is characterised by a single dependence parameter θ that must be estimated. The inference for margins (IFM) approach is employed for this estimation. Table 1 shows the relationship of the Archimedean Gumbel copula parameter θ with Kendall's tau (τ) and tail dependence coefficients.

Table 1: The Archimedean copula parameter θ and its relationship with tail dependence coefficients: Kendall's tau (τ), upper tail coefficient (λ_U), and lower tail coefficient (λ_L).

Copula	Range of θ	Kendall's tau τ	Upper Tail λ_U	Lower Tail λ_L
Gumbel	$[1,\infty]$	$\tau = \frac{\theta-1}{\theta}$	$\lambda_U = 2-2^{-\theta}$	$\lambda_L = 0$

In applied statistics, engineering, and mathematical finance, the Archimedean Gumbel copula is widely used as an extreme value copula [18], [21], and [22].

3.6 Estimation Procedure

According to [6], a five-step estimation procedure is followed when fitting the Archimedean Gumbel copula with Log-Normal distributed marginals:

1. **Fit Marginals:** Calculate financial growth rates and fit a Log-Normal distribution to the growth rates of the two indices.
2. **Transform to Uniform:** Transform the series into standard uniform $(0,1)$ variates u_1 and u_2, assuming i.i.d. observations. Plotting these marginals helps determine a suitable copula for the data.
3. **Copula Selection and Parameter Estimation:** Use the scatterplots of the uniform marginals along with AIC/BIC to select the most appropriate bivariate copula model. Estimate Kendall's tau τ, and use the IFM method to estimate θ.
4. **Simulation:** Simulate $N = 10{,}000$ values from the estimated copula model (Gumbel) to obtain joint uniform marginals u_1 and u_2. Transform these to the original scale using the inverse CDF of the Log-Normal distribution. Use the average of the input parameters as the new parameters.
5. **Portfolio Risk Estimation:** Estimate Value-at-Risk (VaR) and Expected Shortfall (ES) for the equally weighted portfolio. Portfolio VaR and ES are calculated via Monte Carlo simulation and then used to estimate diversification benefits.

3.7 Diversification Benefits

To capture the benefits of diversification, the Archimedean Gumbel copula with Log-Normal marginals is applied, and diversification effects are estimated. According to [23], diversification benefits are calculated using the following formulas:

$$\text{Diversification Benefit}_{\text{VaR}} = \frac{\text{Simple Sum of VaR} - \text{Aggregate VaR}}{\text{Simple Sum of VaR}} \times 100\% \tag{9}$$

$$\text{Diversification Benefit}_{\text{ES}} = \frac{\text{Simple Sum of ES} - \text{Aggregate ES}}{\text{Simple Sum of ES}} \times 100\% \tag{10}$$

Where:

– **Simple Sum of VaR:** The sum of the individual VaR values of the two risky factors.
– **Simple Sum of ES:** The sum of the individual ES values of the two risky factors.
– **Aggregate VaR:** The portfolio VaR of the two risky factors.
– **Aggregate ES:** The portfolio ES of the two risky factors.

These formulas are used to quantify the diversification benefits in the bivariate case of an equally weighted portfolio. However, the concept can be extended to higher-dimensional models.

3.8 Tests for Stationarity, Heteroscedasticity, and Autocorrelation

To apply the Archimedean Gumbel copula appropriately, the marginal distributions of the financial data must be stationary and continuous. These assumptions are verified using statistical tests, which are summarized in Table 2.

Table 2: Summary of statistical tests used to verify stationarity, heteroscedasticity, and autocorrelation.

Test	Purpose	Hypothesis Tested
Augmented Dickey-Fuller (ADF)	Stationarity	H_0: Unit root (non-stationary)
ARCH LM Test	Heteroscedasticity	H_0: No ARCH effect
Ljung-Box Q Test	Autocorrelation	H_0: No autocorrelation

Table 3: Tests for stationarity, heteroscedasticity, and autocorrelation.

Test	Method
Stationarity	The Augmented Dickey-Fuller (ADF) test, also known as the unit root or non-stationarity test, is used to test for stationarity in the South African Industrial and Financial Index growth rate series.
Heteroscedasticity	To test for the presence of ARCH effects, the Lagrange Multiplier (LM) test is used. It assesses the presence of heteroscedasticity in the returns of the Industrial and Financial Index growth rate series.
Autocorrelation	The Ljung-Box test is used to test for autocorrelation in each of the South African Industrial and Financial Index growth rate series.

In Table 3, the statistical tests applied to verify stationarity, heteroscedasticity, and autocorrelation assumptions are described.

4 Empirical Results

This section applies the Archimedean Gumbel copula to the monthly South African Industrial and Financial Index growth rates over the period 1995 to 2018 to estimate portfolio risk and diversification effects.

4.1 Software Used and Research Data

This study utilizes secondary data for the South African Industrial and Financial Indices, extracted from the IRESS Expert platform: https://expert.inetbfa.com (with permission). The analysis involves the growth rates of the South African Industrial (J520) and Financial (J580) Indices spanning the years 1995 to 2018.

These indices represent the performance of the industrial and financial sectors listed on the Johannesburg Stock Exchange (JSE). According to [24], although the South African All Share Index (ALSI) is weakly efficient, its sub-indices may not be informationally efficient. This inefficiency could create opportunities for excess returns or losses.

The financial growth rates of the indices are assumed to follow a Log-Normal distribution, which is positively skewed. A bivariate portfolio is constructed from the growth rates of the two indices and used to estimate portfolio VaR and ES using the Archimedean Gumbel copula with Log-Normal marginals. Diversification benefits are then calculated.

The monthly financial growth rate for both indices is estimated using the following equation:

$$x_t = \frac{M_t}{M_{t-1}} = 1 + r_t, \tag{11}$$

where x_t is the monthly growth rate at time t, M_t is the index value at month t, M_{t-1} is the index value at month $t-1$, and r_t is the return in month t.

4.2 Exploratory Data Analysis

This section provides a preliminary analysis of the financial growth rate data, including summary statistics and distributional features.

4.2.1 Descriptive Statistics

Table 4 presents the summary descriptive statistics for the monthly South African Industrial and Financial Index growth rates.

Table 4: Summary descriptive statistics of financial growth rates.

Statistic	Industrial Index	Financial Index
Number of Observations	271	271
Minimum	0.7200	0.5993
Maximum	1.1506	1.2417
Mean	1.0110	1.0102
Median	1.0105	1.0102
Variance	0.0032	0.0034
Skewness	-0.6018	-0.9605
Kurtosis	2.5080	9.2496

In Table 4, the descriptive statistics for the monthly South African Industrial and Financial Indices growth rate series are presented. The results for the kurtosis indicate that the South African Industrial and Financial growth rate series exhibit excess kurtosis of 2.51 and 9.25, implying the distributions are heavy-tailed and exhibit leptokurtosis. This implies that the stylised fact of heavy-tailedness of the growth rates/financial returns is satisfied.

4.2.2 Histogram of financial growth rates

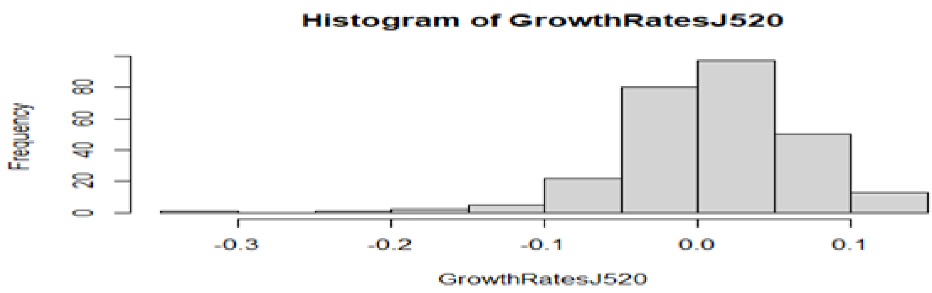

Fig. 1: Histogram for the South African Industrial Index growth rates.

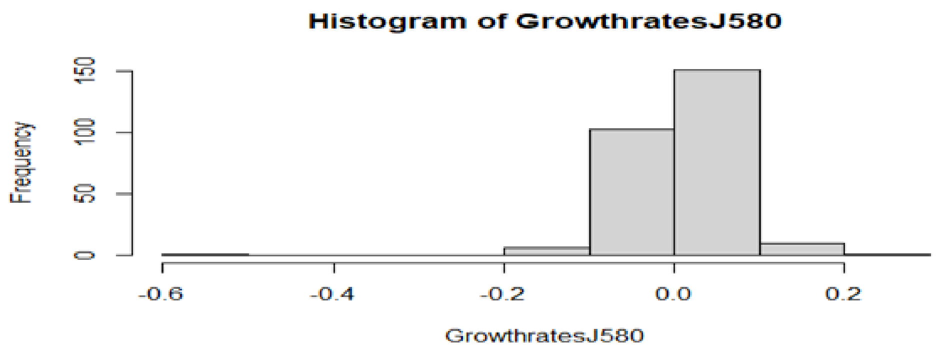

Fig. 2: Histogram for the South African Financial Index growth rates.

In Figure 1 and Figure 2, the histograms show that observations are relatively asymmetrical about the mean. This is consistent with the large kurtosis value. Therefore, based on the other findings above, analysing the growth rates and forecasting financial risk using the heavy-tailed distributions would be justified. This satisfies the stylised fact of **Asymmetry**.

4.2.3 Tests for Stationarity, Heteroscedasticity and Auto-correlation

The financial growth rate series are checked for stationarity, heteroscedasticity and auto-correlation.

Testing for stationarity

The Augmented Dickey Fuller (ADF) test is used to investigate whether the monthly South African Industrial and Financial Indices growth rates are a stationary series. The results show that a p-value < 0.05 was obtained: therefore, it is concluded that the returns data is stationary.

Testing for heteroscedasticity

The Arch (LM) test is applied to test for heteroscedasticity in the South African Industrial and Financial Indices growth rate series. The Arch (LM) test checks for the presence of ARCH effects. The p-values for the South African Industrial Index growth rates are p-value = 0.3633 and South African Financial Index growth rates the p-value = 0.05047. The p-values are greater than 0.05 for both the financial growth rate series data, confirming that there are no Arch effects in both datasets.

Testing for autocorrelation

The Ljung-Box test is used to test for autocorrelation in the monthly South African Industrial and Financial Indices growth rate series. The results reveal a p-value > 0.05 for each of the growth rate series data, indicating weak evidence against the null hypothesis, so we fail to reject the null hypothesis of no autocorrelation. This means that the growth rate series are each independent and identically distributed. Therefore, applying the EVT to the financial growth rate series is appropriate as each series is independently and identically distributed or there is no autocorrelation. The two series may however be dependent to each other.

4.3 Characterisation of Tails of the Data Using Statistical Parent Distributions

The tails of the South African Industrial and Financial Indices growth rates are characterised using statistical parent distributions (Weibull, Log-Normal and Pareto distributions). The Generalised QQ plot, QQ plot, derivative plot, AIC and BIC are used to determine the best-fit distribution for the growth rates of the two Indices.

a) South African Industrial Index growth rates

In Figure 3 3, the Generalised QQ plot of the South African Industrial Index growth rates is presented.

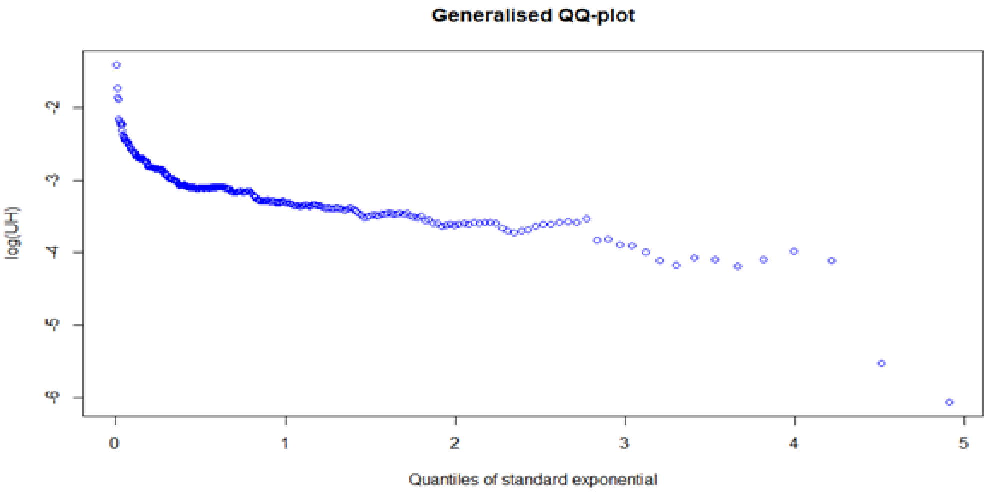

Fig. 3: Generalised QQ plot for the South African Industrial Index growth rates.

In Figure 3, the Generalised QQ plot is nearly a constant in the upper tail implying that the distribution belongs to the Gumbel domain, which includes the Weibull type tail with $\tau > 1$, Exponential type tail, Weibull type tail with $\tau < 1$, and the Log-Normal type tail distributions [**?**].

In Figure 4, the Weibull QQ plot and the Derivative plots for the South African Industrial Index growth rates are presented.

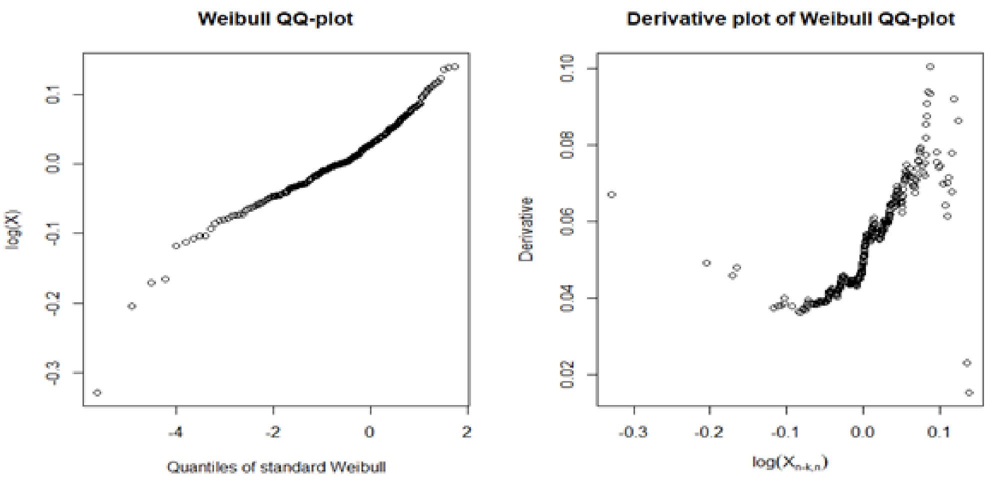

Fig. 4: Weibull QQ plot and Derivative plots for the South African Industrial Index growth rates.

In Figure 4, the Weibull QQ plot is convex, implying that the data exhibit heavier tails than any Weibull distribution. This observation is further supported by the derivative plot of the data points, which shows an upward-increasing trend, indicating that the data are indeed heavier-tailed than what a Weibull distribution can capture.

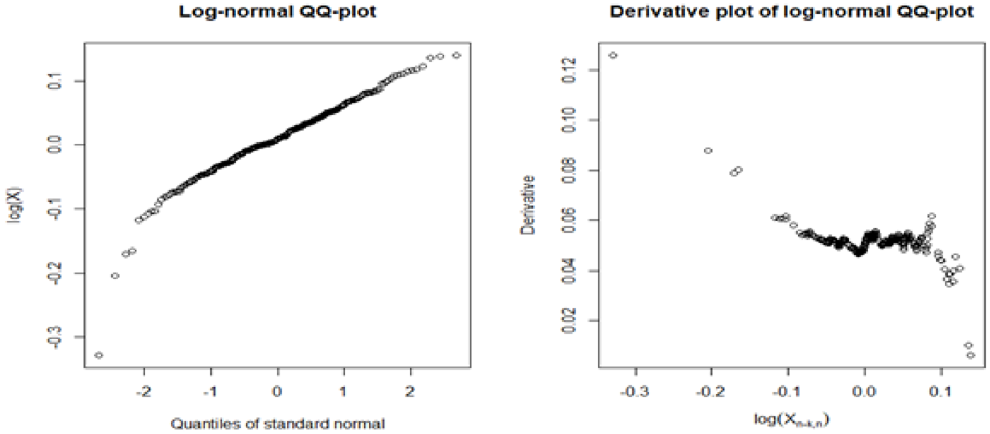

Fig. 5: The Log-Normal QQ plot and the Derivative plots for the South African Industrial Index growth rates.

In Figure 5, the Log-Normal QQ plot is approximately linear, particularly in the upper tail, suggesting that the data closely follow a Log-Normal distribution. This is further supported by the derivative plot, which remains nearly constant in the upper tail (except for the last few points), indicating that the tail behavior of the data aligns well with that of a Log-Normal distribution.

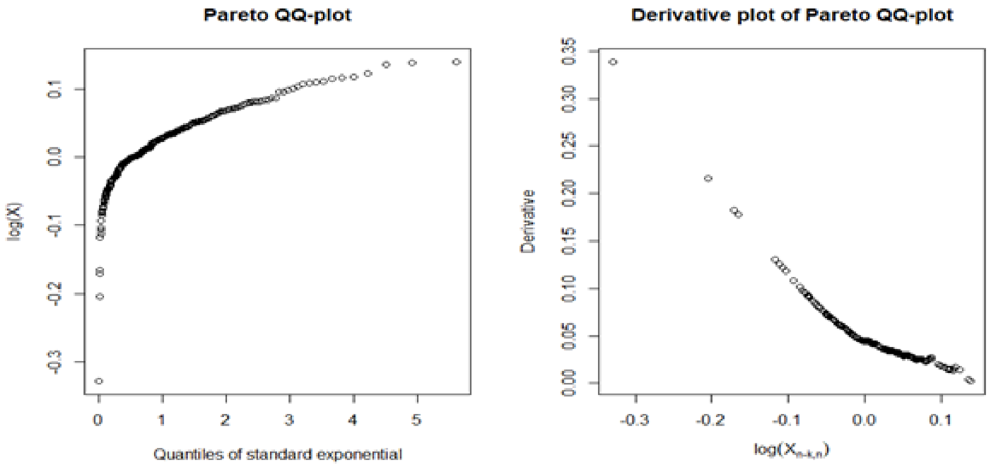

Fig. 6: The Pareto QQ plot and the Derivative plots for the South African Industrial Index growth rates.

In Figure 6, the Pareto QQ plot is concave, indicating that the data are lighter-tailed than the Pareto distribution. This is corroborated by the derivative plot, which shows a decreasing trend with a negative gradient, further implying that the data exhibit lighter tails than any Pareto-type distribution.

Therefore, it can be deduced that the South African Industrial Index growth rates best fit the Log-Normal distribution which is in the Gumbel domain.

Table 5: AIC and BIC for South African Industrial Index growth rates

Distribution	AIC	BIC
Weibull	-776.5104	-769.3062
Log-Normal	-771.1123	-763.9081
Pareto	-551.9519	-559.1561

In Table 5, the AIC and BIC suggest the Weibull distribution as the best fit for the financial growth rates since it has the lowest values for both goodness of fit tests. However, for the Q-Q and derivative plots, the Log-Normal distribution is best fit for financial growth rates, especially in the upper tail. In this case, the Log-Normal distribution is chosen as the Q-Q and derivative plots allow us to consider extreme values unlike the AIC and BIC, which considers the full data distribution. Our risk measures, VaR and ES are associated with the upper tail.

b) South African Financial Index growth rates

In Figure7 , the Generalised QQ plot of the South African Financial Index growth rates is presented.

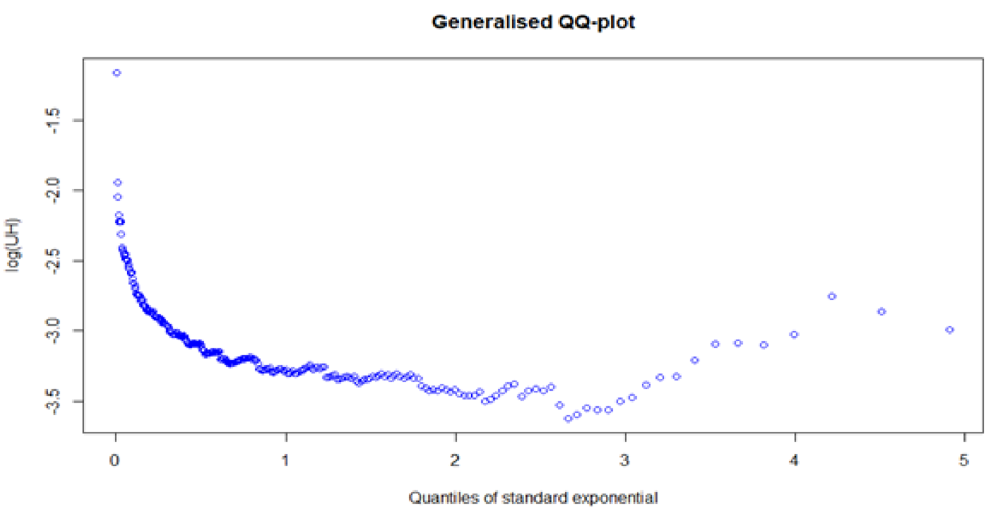

Fig. 7: The Generalised QQ plot of the South African Financial Index growth rates.

In Figure 7, the Generalised QQ plot is constant in the upper tail suggesting that the distribution belongs to the Gumbel domain.

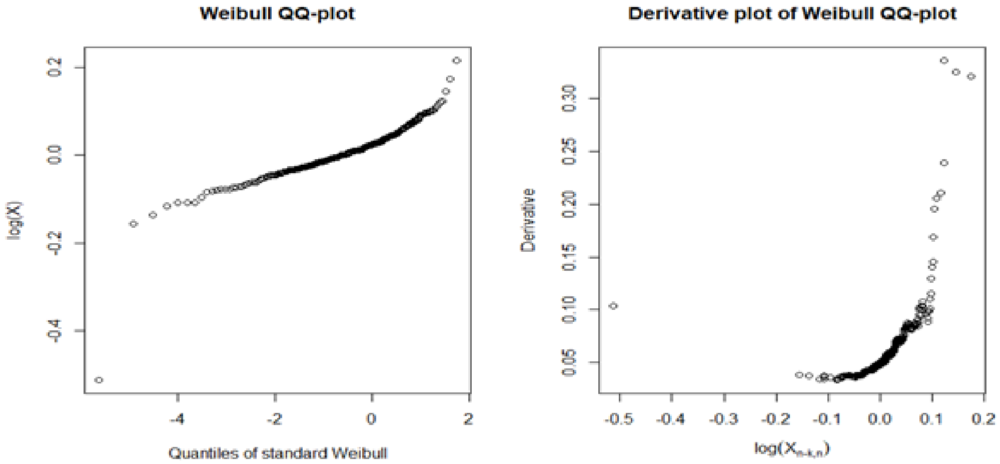

Fig. 8: The Weibull QQ plot and the Derivative plots for the South African Financial Index growth rates.

In Figure 8, the Weibull QQ plot is convex, implying that the data are heavier-tailed than the Weibull distribution. This is further supported by the derivative plot, which shows an upward-increasing trend, indicating that the data points exhibit heavier tails than any Weibull distribution can adequately capture.

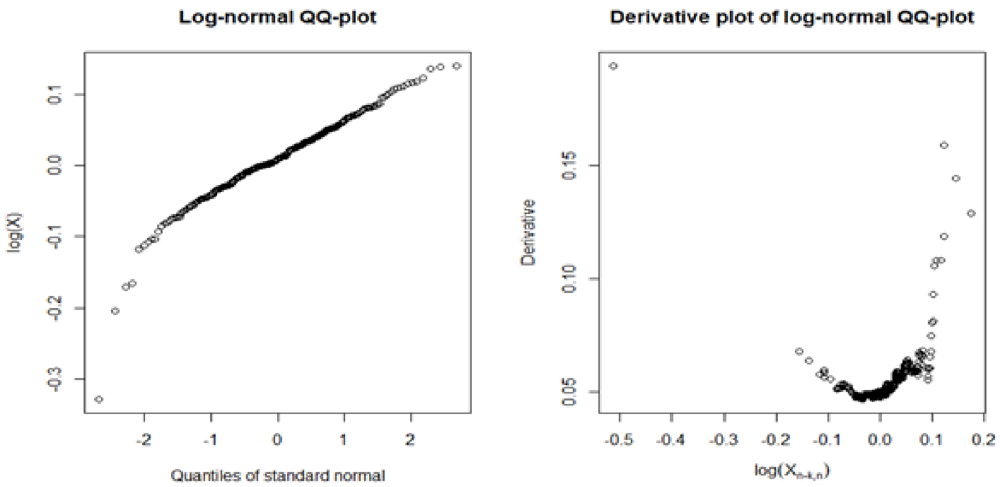

Fig. 9: The Log-Normal QQ plot and the Derivative plots for the South African Financial Index growth rates.

In Figure 9, the Log-Normal QQ plot is approximately linear, suggesting that the data may follow a Log-Normal distribution. This is further supported by the derivative plot, where the data points are relatively constant in the upper tail, indicating consistency with Log-Normal tail behavior. However, the presence of a few outliers suggests minor deviations that should be considered in the overall model fit.

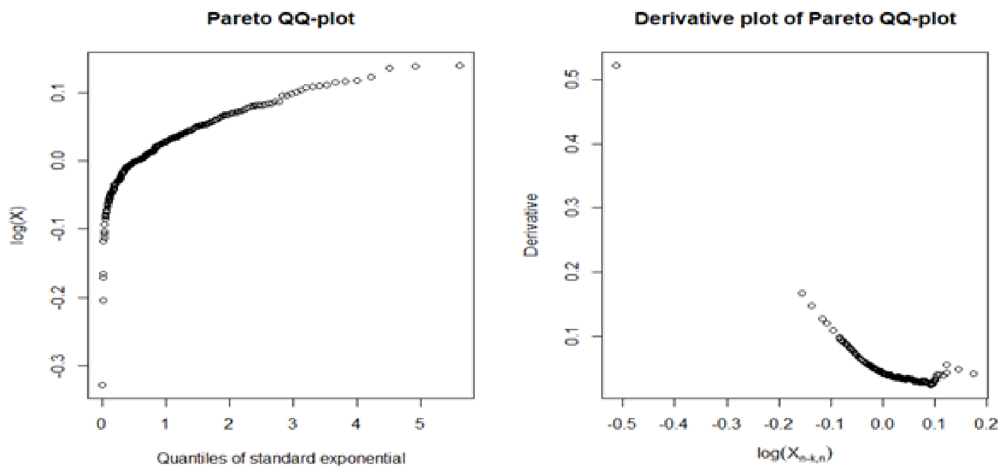

Fig. 10: The Pareto QQ plot and the Derivative plots for the South African Financial Index growth rates.

In Figure 10, the Pareto QQ plot is concave, indicating that the data are lighter-tailed than the Pareto distribution. This observation is reinforced by the derivative plot, which shows a decreasing trend with a negative gradient, further suggesting that the data points exhibit lighter tails than the Pareto distribution can represent. Therefore, it can be deduced that the South African Financial Index growth rates best fit the Log-Normal distribution which is in the Gumbel domain. The Log-Normal distribution will be used as the marginal distribution with Archimedean Gumbel copula to estimate portfolio risk.

Table 6: AIC and BIC South African Financial Index growth rates

Distribution	AIC	BIC
Weibull	-735.0885	-727.8842
Log-Normal	-744.4538	-737.2495
Pareto	-551.4748	-558.6790

In Table 6, the AIC and BIC confirms that Log-Normal distribution is the best fit for the financial growth since it has the lowest values for both model selection criteria.

4.4 Parameter estimates

Table 7: Estimates Log-Normal distribution parameters of South African Industrial and Financial Indices growth rates

Financial Asset	μ-location parameter	σ - scale parameter
South African Industrial Index	0.0094	0.0574
South African Financial Index	0.0084	0.0603
Average	0.0089	0.0588

In Table 7, the parameters are obtained for the Log-Normal distribution fit of the two financial asset growth rates distributions. The parameters are used to estimate the VaR and ES of the two financial assets for the univariate distributions. The average parameters are used to estimate VaR and ES of the portfolio. The inverse quantile function of the joint distribution will use the average of the input parameters as the new Log-Normal distribution parameters.

4.5 Selection of appropriate copula function for bivariate analysis

The AIC/BIC and scatterplot method in Table 8 are used for model selection.

4.5.1 Model selection using the AIC and BIC methods

Table 8: Estimation of AIC and BIC goodness of fit criterion.

Copula	AIC	BIC
Gumbel	-229.0522	-225.4500
Clayton	-148.9424	-145.3403
Frank	-217.8444	-214.2423

Table 8, shows that the Archimedean Gumbel copula has the lowest values using both the AIC and the BIC, and therefore it has the best fit to the data. For these models, Archimedean Gumbel copula gave a better fit to describe the dependence and model the upper dependence structure of the two assets.

4.5.1 Scatterplot Method

The scatterplots are used as one of the methods for choosing and confirming the right copula function for fitting to the financial growth rate data. The uniform marginal plots in Figure 11, were plotted and used to determine or confirm the type of copula function to fit to the bivariate growth rates.

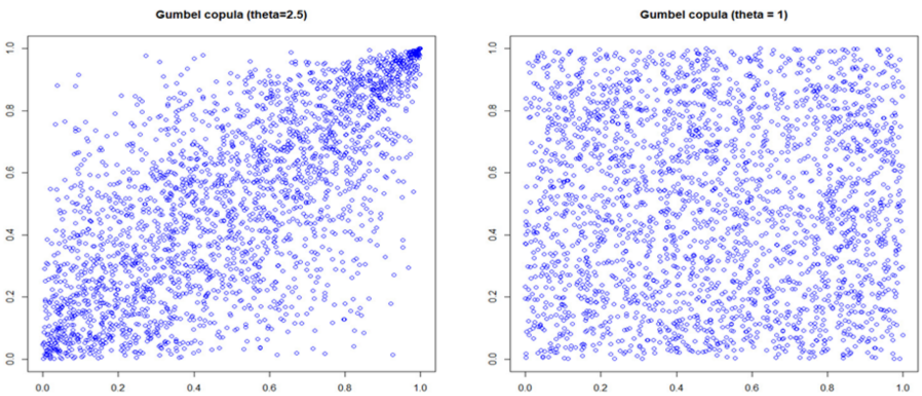

Fig. 11: Archimedean Gumbel Copula simulation with $\theta = 2.5$ (left) and $\theta = 1$ (right).

In Figure 11, the bivariate Gumbel Copula simulation with $\theta = 2.5$ shows marginal uniform distributions have upper tail dependence, and Copula simulation with $\theta = 1$ shows that the joint marginal distributions are independent. These uniform marginal plots are going to be compared with uniform marginal plots of the growth rates and to determine whether they fit the Archimedean Gumbel copula. The uniform marginal plots of the growth rates are compared with the Archimedean Gumbel Copula simulation when $\theta = 2.5$ (left) and $\theta = 1$ (right), and if the shape similar to the shape when $\theta = 2.5$ then the Gumbel copula is confirmed and vice versa.

In Figure 12, the uniform marginals plot seem to have an increasing pattern towards an upper tail dependence so this dataset is modelled using the Archimedean Gumbel copula. This is comparable to the copula simulation of uniform marginals with $\theta = 2.5$ in Figure 12, showing the data fits an Archimedean Gumbel copula with an extreme upper tail dependence. The uniform marginals scatterplot indicates that the financial growth rates can be fitted to the Archimedean Gumbel copula. Therefore, the Archimedean Gumbel copula is of particular interest in this study as it is the best fitting copula.

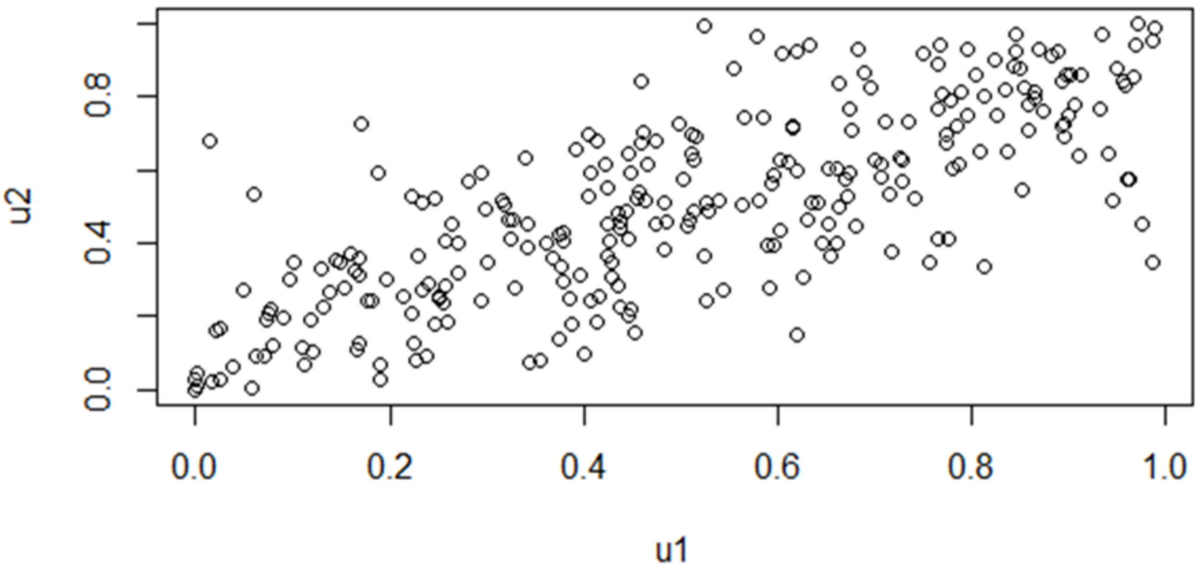

Fig. 12: The uniform marginals u_1 and u_2 marginal for the bivariate growth rates transformed uniform variables, indicating that it is a Gumbel copula.

Kendall's tau, copula parameter, upper and lower tail dependence measures

Table 9: The Kendall's tau, copula parameter and tail dependency estimates.

Kendall's tau $\hat{\tau}$	Copula Parameter $\hat{\theta}$	Upper tail $\hat{\lambda}_U$	Lower tail $\hat{\lambda}_L$
0.5591089	2.268134	0.6425575	0

In Table 9, the Kendall's tau, copula parameter and tail dependency estimates are presented. Kendall's tau is a measure of association between two variables, measuring how much one variable tends to increase as the other increases. It ranges from -1 to 1. A Kendall's tau value of 0.56 suggests a moderately strong positive dependence between the two assets. This indicates that as one asset's value increases, the other asset's value also is likely to increase.

The Archimedean Gumbel copula parameter implies the presence of upper tail dependence. The parameter $\theta = 2.27$ means that two stock indices have extreme upper tail dependence. The growth rates of the two financial assets have greater probability to co-move together concurrently. Alternatively, this implies that the two stock markets indices may tend to rise together during periods of economic booms.

An upper tail dependence of 0.64 indicates that there is a considerable level of dependence between the extreme values of the two assets. This means that when both assets experience extreme positive returns, they are likely to occur together more frequently.

The tail dependence measures in Table 8 indicate the degree of extreme co-movements of financial growth rates which allows investors and practitioners to quantify portfolio risk and quantify diversification effects. This information is useful for risk management and portfolio optimisation, as it indicates how extreme events in one asset may be associated with extreme events in another.

Estimation of univariate risk measures

The univariate risk measures presented are used to estimate diversification benefits.

Table 10: Univariate risk measures for South African Industrial and Financial Indices

Confidence level	J520 VaR	J580 VaR	Simple Sum	J520 ES	J580 ES	Simple Sum
0.90	0.540	0.538	1.078	0.553	0.554	1.108
0.95	0.551	0.550	1.102	0.562	0.565	1.127

In Table 10, the simple sum of VaR and ES of the constituent Indices growth rates are presented.

Estimation of portfolio risk using the Archimedean Gumbel copula model

Table 11: Estimates of portfolio risks of Archimedean Gumbel copula with the Log-Normal distributed marginals

Confidence level	Portfolio VaR	Portfolio ES
90%	1.068	1.101
95%	1.094	1.123

In Table 10, portfolio VaR and ES are estimated. To estimate portfolio VaR/ES the Archimedean Gumbel copula parameter is used to simulate 10,000 uniform random numbers u_1 and u_2 which are used to construct the joint uniform distribution u_3 and transform them to the original scales of the growth rates using the inverse quantile function of the joint distribution. Using the average of the input parameters as the new Log-Normal distribution parameters on the output u_3 to estimate portfolio risk. In Table 10, the portfolio VaR are 1.068 and 1.094 at the 90% and 95% confidence levels respectively and they reflect the potential loss associated with more extreme scenarios within a period of one month. The interpretation is the same for portfolio ES. The estimated portfolio risk measures are used in the calculation of the diversification effects/benefits.

Estimation of diversification benefits

In Table 11 and Table 12 the diversification effects are estimated for the Archimedean Gumbel copula model for VaR and ES.

Table 12: Estimates of Archimedean Gumbel copula diversification benefits for VaR with Log-Normal distributed marginals

Confidence Interval	ES J520	ES J580	Simple Sum	Portfolio ES	Diversification benefits %
90%	0.540	0.538	1.078	1.068	1.070
95%	0.551	0.550	1.102	1.094	0.758

In Table 12, the diversification benefit reflects the reduction in risk or the improvement in expected growth rates resulting from combining assets in a portfolio. A 1.070% benefit at the 90% confidence level and a 0.758% benefit at the 95% confidence level suggest that the gains from diversification are minimal. This implies that the assets in the portfolio are not sufficiently uncorrelated to produce a significant reduction in overall portfolio risk.

In Table 13, a 1.070% benefit at the 90% confidence level and a 0.758% benefit at the 95% confidence level indicate that diversification has only a small impact in reducing portfolio risk. These low percentages suggest that the two assets in the portfolio (despite being combined) do not significantly offset each other's risk. In other words, the dependence between the assets is such that the overall risk reduction from diversification is minimal, and the portfolio remains relatively exposed to market fluctuations.

Table 13: Estimates of Archimedean Gumbel copula diversification benefits for ES with Log-Normal distributed marginals

Alpha	ES J520	ES J580	Simple Sum	Portfolio ES	Diversification benefits %
90%	0.553	0.554	1.108	1.101	0.641
95%	0.562	0.565	1.127	1.123	0.377

5 Conclusion

This study estimated portfolio risk and diversification effects using the Archimedean Gumbel copula with Log-Normal distributed marginals.

The Archimedean Gumbel copula model is preferable to the Gaussian copula as it better captures non-linear dependencies between assets. Portfolio diversification results indicate only a minimal decrease in overall portfolio risk compared to the combined risk of the individual assets. This suggests that the portfolio offers limited diversification benefits, with little reduction in risk for investors.

These minimal diversification benefits have important implications. They may signal a degree of stability within the South African economy, suggesting limited contagion between the Industrial and Financial sectors. In other words, the economic fundamentals of one sector appear relatively independent of the other, reducing the likelihood that a crisis in one sector would spill over into the other. For both local and international investors, this insight is crucial for shaping investment strategies, diversification planning, and hedging decisions.

For future possible research, the authors will be interested in estimating diversification effects of a portfolio with more than two risky factors in the South African context.

Data Analysis R Packages used in this study

Data was analysed using R programming statistical packages: actuar [26] , Copula [27], fCopulae [28], QRM [29], Mass [30], evir [31], fitdistrplus [32], fExtremes [33] and extremes [34].

Conflicts of Interest Statement

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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