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Theory and Applications of an Inverse Source Problem for Anomalous Diffusion Processes

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Abstract: This work investigates an inverse problem associated with fractional diffusion equations in the setting of Banach spaces, with particular emphasis on identifying a time-dependent source term from overspecified data. We rigorously establish existence, uniqueness, and regularity results for both classical and strong solutions by employing perturbation techniques and the theory of strongly continuous semigroups. The proposed theoretical framework accommodates a broad class of linear operators, including sectorial and elliptic types, thereby ensuring wide applicability to models in physics and engineering. To demonstrate the practical relevance of our approach, we apply it to fractional heat transfer problems and the time-fractional Boltzmann equation, which models anomalous neutron transport. The findings advance the theoretical foundation of inverse source problems in fractional systems and offer robust tools for the unique recovery of source terms in anomalously diffusive media.

Keywords: Inverse source problem, identification problem, semigroup theory, fractional differential equations.

1 Introduction

Over the past two decades, fractional differential equations (FDEs) have emerged as powerful mathematical tools for modeling complex systems exhibiting memory effects and nonlocal behavior, phenomena that classical integer-order models fail to capture adequately. Unlike standard differential equations, fractional models naturally incorporate history-dependent dynamics, where the system's current state depends not only on present inputs but also on its entire past evolution. This intrinsic nonlocality renders fractional models particularly well suited for describing anomalous diffusion processes frequently encountered in diverse fields such as viscoelastic materials, porous media, fractal geometries, biological systems, and financial markets.

Classical diffusion models typically assume Gaussian-based Brownian motion and Markovian dynamics, which neglect memory effects. In contrast, fractional differential equations introduce non-Markovian characteristics that capture subdiffusive or superdiffusive behavior arising in heterogeneous or fractal media. The accuracy of fractional models in representing these complex phenomena has been extensively demonstrated in the literature, fueling their growing adoption in both theoretical and applied research (see, e.g., [16,20,28]).

In this paper, we consider a fractional-anomalous diffusion process governed by the time-fractional differential equation

$$\partial_t^\alpha u(x,t) = \mathscr{L}u(x,t) + f(x,t), \qquad x \in \Omega,\ 0 \le t \le T, \tag{1}$$

where $\mathscr{L}$ is a differential operator defined on an open domain $\Omega \subset \mathbb{R}^d$, and ∂_t^α denotes a fractional derivative of order $\alpha \in (0,1)$ in the Caputo sense. Significant progress has been made in both the analysis and numerical treatment of such fractional diffusion equations.

Our focus is on inverse source problems, which involve recovering an unknown source term within a differential equation based on indirect or incomplete information about its solution. These problems are inherently ill-posed, as they require reconstructing the source from limited measurements, such as spatial or temporal observations. Inverse source

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problems arise in a wide range of applications, including heat conduction, environmental science, and medical imaging, where accurately identifying the origin and evolution of heat, pollutants, or signals is essential for effective monitoring, diagnosis, and control.

Inverse source problems for fractional diffusion equations are particularly challenging due to the nonlocal character of fractional derivatives, which encode long-range temporal dependencies. The problem studied here aims to identify an unknown source term $f(x,t)$ using overspecified data related to the solution, which may include integral measurements over the domain, pointwise observations, or boundary data. This topic has been widely investigated in the literature (see, e.g., [23,25,34]).

While inverse problems for FDEs have attracted substantial attention, a comprehensive theoretical framework addressing inverse source problems in Banach spaces remains underdeveloped. This study fills this gap by establishing a rigorous analytical foundation for the unique identification of time-dependent source terms in fractional diffusion equations formulated within Banach spaces.

Specifically, let X be a Banach space, $A : D(A) \subset X \to X$ a linear operator, $z \in X$, $g \in C^1([0,T];\mathbb{R})$, and $\phi \in X^*$ a bounded linear functional. We consider the inverse problem

$$(D_t^\alpha u)(t) = Au(t) + p(t)z, \qquad 0 \le t \le T, \tag{2}$$

$$u(0) = u_0, \tag{3}$$

$$\phi[u(t)] = g(t), \qquad 0 \le t \le T. \tag{4}$$

where $D_t^\alpha u$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$. The goal is to determine the pair (u,p) with $u \in C^1([0,T];X)$ and $p \in C([0,T];\mathbb{R})$ satisfying the system (2)-(4).

Such mathematical models have significant applications in physical and engineering contexts. For instance, they aid in locating groundwater contamination sources via subsurface data, estimating pollutant emissions for environmental monitoring, and detecting heat sources for thermal management in materials. In particular, determining the temporal function $p(t)$ is critical in scenarios like nuclear power plant incidents, where the source location is known, but the time-dependent decay of radiative intensity must be precisely estimated.

Our main contributions include the rigorous establishment of existence, uniqueness, and regularity results for both classical and strong solutions to the inverse problem. This is achieved by combining perturbation theory of linear operators with semigroup methods. We reformulate the inverse problem as a direct Cauchy problem governed by a perturbed operator, thereby enabling the application of well-developed results on fractional resolvent operator families. The developed analytical framework is broad enough to incorporate a wide class of linear operators, notably sectorial operators, thereby covering numerous physically significant systems.

To illustrate the practical impact of our theoretical framework, we apply it to fractional heat conduction models and the time-fractional Boltzmann equation describing anomalous neutron transport. These examples demonstrate the capability of our approach to uniquely identify time-dependent sources in complex anomalous diffusion processes.

The paper is organized as follows. Section 2 reviews key existence and regularity results for the direct problem, introducing Caputo fractional differentiation and the α-resolvent family. Section 3 addresses the inverse problem, presenting existence, uniqueness, and regularity results based on perturbation theory and deriving strong solutions along with Lipschitz stability estimates. The generality of the differential operator A facilitates the application to diverse models. Section 4 discusses several physical examples, including fractional models of heat

2 Preliminary material

Consider a Banach space X, and denote by $\mathscr{B}(X)$ the Banach algebra consisting of all bounded linear operators on X. For a linear operator A, we denote its resolvent set by $\rho(A)$.

For $\alpha > 0$, let $m = \lceil \alpha \rceil$. The fractional integral of order α acting on a function u is expressed through convolution with the kernel

$$g_\alpha(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

that is,

$$(J^\alpha u)(t) := (g_\alpha * u)(t), \qquad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function. From this definition, two commonly used fractional derivatives can be described.

The Riemann–Liouville derivative of order α is given by

$$(I_t^\alpha u)(t) := \left(\frac{d}{dt}\right)^m (J^{m-\alpha} u)(t)$$

while the Caputo derivative of the same order modifies the initial data by subtracting a polynomial in t

$$(D_t^\alpha u)(t) := I_t^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(0) t^k}{k!} \right).$$

The fractional-order Cauchy problem

$$D_t^\alpha u(t) = Au(t) + f(t), \qquad t > 0, \tag{5}$$

$$u(0) = u_0, \tag{6}$$

serves as a prototype for many applications. Its solution framework is naturally linked to the notion of α-resolvent families generated by A.

Let $\alpha > 0$. A closed linear operator A on X is said to generate an (α,α)-resolvent family when there exist a nonnegative constant ω and a strongly continuous function $P_\alpha : [0,\infty) \to \mathscr{B}(X)$ (or $P_\alpha : (0,\infty) \to \mathscr{B}(X)$ for $0 < \alpha < 1$) such that

$$(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} P_\alpha(t) x dt, \qquad \mathrm{Re}(\lambda) > \omega,\ x \in X.$$

Here, $P_\alpha(t)$ is referred to as the (α,α)-resolvent family generated by A (see [1]).

Similarly, A generates an $(\alpha,1)$-resolvent family if there is a strongly continuous function $S_\alpha : [0,\infty) \to \mathscr{B}(X)$ and a number $\omega \geq 0$ such that

$$\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$$

and

$$\lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \qquad \mathrm{Re}(\lambda) > \omega,\ x \in X.$$

In this context, $S_\alpha(t)$ is known as the $(\alpha,1)$-resolvent family generated by A (see [3]).

It is worth noting that any closed, densely defined linear operator A that generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Banach space X also generates an $(\alpha,1)$-resolvent family $(S_\alpha(t))$ for all $0 < \alpha < 1$. The connection between these resolvent families is expressed by

$$S_\alpha(t)x = \int_0^\infty t^{-\alpha} \Phi_\alpha\left(st^{-\alpha}\right) T(s) x ds, \qquad t \geq 0,\ x \in X, \tag{7}$$

where Φ_α is a Wright-type function defined as [27]:

$$\Phi_\alpha(z) := \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}.$$

We now turn to a key result concerning (α,α)-resolvent families. When $0 < \alpha < 1$ and A generates a C_0-semigroup $(T(t))_{t\geq 0}$, it also generates an (α,α)-resolvent family $(P_\alpha(t))$ given by [21]

$$P_\alpha(t)x = \alpha \int_0^\infty \frac{s}{t^{\alpha+1}} \Phi_\alpha\left(st^{-\alpha}\right) T(s) x ds. \tag{8}$$

Moreover, if $x \in D(A)$, then $P_\alpha(t)x \in D(A)$ and

$$S'_\alpha(t)x = AP_\alpha(t)x, \qquad t > 0,\ x \in X.$$

If $S_\alpha(t)$ and $P_\alpha(t)$ are defined by the above formulas, then the problem (5)-(6) admits a unique classical solution

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(s) ds, \tag{9}$$

provided that $f \in W^{1,1}(\mathbb{R}_+;X)$ and $u_0 \in D(A)$. Additionally, for a sectorial operator A, the formula (9) yields a unique strong solution for any $u_0 \in D(A)$ and $f(\cdot) \in C^\beta([0,T];X)$ where $0 < \beta \leq 1$ (see [4,21,29]). For a more comprehensive study of α-resolvent families, we recommend the works of [1,12].

Finally, if A is the infinitesimal generator of a C_0-semigroup on a Banach space $(X, \|\cdot\|)$, the A is closed. Endowing its domain $D(A)$ with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|,$$

turns $D(A)$ into a Banach space, denoted by X_A. We recall the following perturbation theorem from the theory of linear operators [7,8]:

Theorem 2.1 *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $B \in \mathscr{B}(X_A)$. Then $A + B$ is also the generator of a C_0-semigroup on X.*

For background on C_0-semigroup theory, please consult [31].

3 Main results

3.1 Classical solution in Banach spaces

We now present sufficient conditions guaranteeing the existence and uniqueness of a classical solution for the inverse problem associated with equations (2)-(4) in Banach spaces. The result is stated below.

Theorem 3.1 *Let A be the infinitesimal generator of a C_0-semigroup on a Banach space X, and suppose that*

$$u_0 \in D(A),\ \phi[z] \neq 0,\ g \in C^1([0,T];\mathbb{R}),\ z \in D(A).$$

Then the inverse problem (2)-(4) admits a unique solution such that

$$u \in C^1([0,T];X), \quad p \in C([0,T];\mathbb{R}).$$

Proof. Equation (2), after applying the functional ϕ and making use of (4), takes the form

$$(D_t^\alpha g)(t) - \phi[Au(t)] = p(t)\phi[z].$$

This allows us to express the source term $p(t)$ as

$$p(t) = \frac{1}{\phi[z]}((D_t^\alpha g)(t) - \phi[Au(t)]). \tag{10}$$

By substituting (10) into (2), one arrives at

$$(D_t^\alpha u)(t) = Au(t) + \frac{1}{\phi[z]}((D_t^\alpha g)(t) - \phi[Au(t)])z. \tag{11}$$

Rewriting this expression, we obtain:

$$(D_t^\alpha u)(t) - (\ Au(t) + \frac{-1}{\phi[z]}(\phi[Au(t)])z\ ) = \frac{1}{\phi[z]}(D_t^\alpha g)(t)z. \tag{12}$$

Thanks to the operator

$$Bx = \frac{-1}{\phi[z]}(\phi[Ax])z, \tag{13}$$

we can simplify equation (12) to

$$(D_t^\alpha u)(t) - (A+B)u(t) = \frac{1}{\phi[z]}(D_t^\alpha g)(t)z. \tag{14}$$

Direct computations show that

$$\begin{aligned} \|B\|_A &= \sup_{\|x\|_A=1} \|Bx\| \\ &= \sup_{\|x\|_A=1} \left\| \frac{-1}{\phi[z]}(\phi[A(x)])z \right\| \\ &\leq \sup_{\|x\|_A=1} \frac{1}{|\phi[z]|} \|z\|\,\|\phi\|\,\|Ax\| \\ &\leq \frac{1}{|\phi[z]|}\|z\|\,\|\phi\|, \end{aligned}$$

hence B is bounded on X_A. According to Theorem 2.1, $A+B$ generates a C_0-semigroup $(T(t))_{t\geq 0}$. Consequently, the Cauchy problem defined by equations (2)-(3) admits a unique solution

$$u(t) = S_\alpha(t)u_0 + \frac{1}{\phi[z]}\int_0^t P_\alpha(t-s)(D_s^\alpha g)(s)zds. \tag{15}$$

Using (10) together with (15) uniquely determines $p(t)$, which completes the proof.

3.2 Strong solution in Banach spaces

Analytic semigroups generated by sectorial operators form a significant class of semigroups. Sectorial operators with dense domains are infinitesimal generators of semigroups that exhibit notable smoothing properties. The next result gives conditions under which the inverse problem (2)-(4) has a strong solution.

Theorem 3.2 *Let A be a sectorial operator with a dense domain in a complex Banach space X, and assume $\phi[z] \neq 0$, $g \in C^1([0,T];\mathbb{R})$, and $z \in X$. Then the inverse problem (2)-(4) has a unique strong solution with*

$$u \in C^1([0,T];X), \quad p \in C([0,T];\mathbb{R}).$$

Our next result presents a Lipschitz-type stability bound for the inverse problem (2)-(4).

Theorem 3.3 *Under the same conditions as those in Theorem 3.2, there exist constants C_1 and C_2 such that every solution (u,p) of (2)-(4) satisfies the estimates*

$$\|p\|_{C[0,T]} \leq \frac{1}{|\phi[z]|}(\|D_t^\alpha g\|_{C[0,T]} + \|\phi\|),$$
$$\|u\|_{C([0,T];X)} \leq C_1 \|u_0\|_X + C_2 \|D_t^\alpha g\|_{C[0,T]}.$$

Proof. The representation given by (10) justifies the estimate for the function p. On the other hand, the estimate for u is a direct consequence of (15).

4 Applications

4.1 Equations of heat transfer

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class $\mathscr{C}^2$, and let $T > 0$ be a fixed constant. We consider an inverse problem for a time-fractional diffusion model described by

$$\partial_t^\alpha u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right) + b(x)u + p(t)f(x), \qquad (x,t) \in \Omega \times [0,T], \tag{16}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{17}$$
$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad 0 \leq t \leq T. \tag{18}$$

Here, the functions $a_{i,j}$, b, u_0, and f are given. The fractional derivative $\partial_t^\alpha u$ for $0 < \alpha < 1$ is the Caputo derivative of order α, expressed as

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} u(\tau)\, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function. The goal is to determine the source function $p(t)$ in the model defined by equations (16)-(18) using supplementary data that provide additional information over a part of the spatial domain:

$$\int_\Omega u(x,t)w(x)dx = \psi(t), \qquad 0 \leq t \leq T. \tag{19}$$

Let $X = L_2(\Omega)$ and define the strongly elliptic operator [10]

$$Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right) + b(x)u$$

with domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

where

$$a_{i,j} \in C^1(\bar{\Omega}), \quad a_{i,j}(x) = a_{j,i}(x), \tag{20}$$

$$\sum_{i,j=1}^{n} a_{j,i}(x)\,\xi_i \xi_j \geq \delta \sum_{i=1}^{n} \xi_i^2, \quad \text{for some } \delta > 0 \text{ and all } x \in \bar{\Omega}, \tag{21}$$

$$b \in C(\bar{\Omega}). \tag{22}$$

The operator A is both negative definite and self-adjoint, with its inverse A^{-1} being compact, as noted in [32]. These characteristics ensure that A generates a strongly continuous compact semigroup $(T(t))_{t\geq 0}$. Consequently, the inverse problem described by equations (16)-(19) can be reformulated in the abstract framework defined by relations (2)-(4). An application of Theorem 3.1, yields the following result.

Corollary 4.1 *Assume that conditions (20)-(22) are satisfied, with $f \in L_2(\Omega)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\psi \in C^1([0,T];\mathbb{R})$, and $\int_\Omega f(x)w(x)dx \neq 0$. Then, there exists a unique solution (u,p) to the inverse problem (16)-(19) in the function spaces*

$$u \in C^1([0,T];L_2(\Omega)), \quad p \in C([0,T];\mathbb{R}).$$

4.2 Neutron transport equations

The Boltzmann equation [6,18]

$$\frac{\partial u}{\partial t} + (\upsilon, \operatorname{grad}_x u) + \sigma(x,\upsilon)u(x,\upsilon,t) - \int_V k(x,\upsilon,\upsilon')u(x,\upsilon',t)d\mu(\upsilon') = q(x,v,t),$$

describes the evolution of the neutron distribution function in a nuclear reactor, accounting for particle velocities, spatial gradients, and interactions with the medium. The equation is accompanied by initial and boundary conditions that define the system's state at a specific time and its behavior at the domain boundaries:

$$u(x,\upsilon,0) = u_0(x,\upsilon), \quad u(\cdot,\cdot,t)_{|\Gamma_-} = 0,$$

where $(x,\upsilon) \in \Omega \times V$. Here, Ω denotes an open set in $\mathbb{R}^n$ with a smooth boundary, V represents the velocity space, and μ is a Radon measure on $\mathbb{R}^n$ with support V. In most cases, μ is selected as a weighted Lebesgue measure on $\mathbb{R}^n$. The boundary conditions are particularly significant, as they depend on the geometry and orientation of the reactor's boundary, influenced by the unit normal vector $n(x)$ at each boundary point. In this context, the boundary conditions are defined on

$$\Gamma_- = \{(x,\upsilon) \in \partial\Omega \times V : (\upsilon, n(x)) < 0\}.$$

These conditions rely on the physical properties of the system, such as cross-sections, extinction coefficients, and collision frequencies, which are represented by the function $\sigma(\cdot,\cdot)$. Additionally, internal sources of particles or radiation are modeled by a source term $q(x,v,t)$, while the scattering of particles is captured by a kernel function $k(\cdot,\cdot,\cdot)$. The functions σ and k are assumed to measurable. The equation's complexity reflects the diverse phenomena it can model, making it an essential tool in fields like neutron transport, radiative transfer, and gas dynamics. For a more comprehensive understanding of the Boltzmann equation and its applications, see [5].

It has been shown [11, Chapter 12] that if $\sigma(\cdot,\cdot) \in L^\infty(\Omega \times V)$ then the operator

$$A : \varphi \in D(A) \to -(\upsilon, \operatorname{grad}_x \varphi) - \sigma\varphi$$

with the domain

$$D(A) = \left\{\varphi \in L^p(\Omega \times V) : (\upsilon, \operatorname{grad}_x \varphi) \in L^p(\Omega \times V),\ \varphi_{|\Gamma_-} = 0\right\}$$

generates the so-called streaming C_0-semigroup

$$U(t) : \varphi \in L^p(\Omega \times V) \to e^{-\int_0^t \sigma(x-s\upsilon,\upsilon)ds}\varphi(x-t\upsilon,\upsilon)\chi_{\{t\leq\tau(x,\upsilon)\}},$$

where χ_B denotes the characteristic function of the set B, and

$$\tau(x,\upsilon) = \inf\{s > 0 : x - s\upsilon \notin \Omega\}.$$

Particles do not simply stream freely; they undergo collisions and scattering, which are represented by the collision operator

$$K : \varphi \in L^p(\Omega \times V) \to \int_V k(x, \upsilon, \upsilon')\varphi(x, \upsilon', t) d\mu(\upsilon').$$

Provided that K is bounded on the same function space, the perturbed operator $A+K$ remains the generator of a C_0-semigroup $(T(t))_{t\geq 0}$.

Classical neutron transport models, based on the Boltzmann equation, typically describe dynamics through a Markovian process. However, these models are inadequate for capturing certain nonequilibrium phenomena, known as anomalies, which occur in complex structures such as turbulent plasma, interstellar magnetic fields, and disordered semiconductors [28]. These anomalies result from spatial heterogeneities with non-uniform scaling (fractal structures), leading to particle trajectories that exhibit clustering and extended confinement within small regions. It has been observed that distributions of path lengths and waiting times often follow heavy-tailed power-law patterns, indicating an anomalous process. To address this, the conventional partial differential equation is replaced with one that incorporates fractional derivatives in time and/or space. In our case, we use a time-fractional Boltzmann equation with $0 < \alpha \leq 1$, though other generalizations are also possible. For further details, see [35].

This motivates the study of the time-fractional neutron transport equation

$$\partial_t^\alpha u + (\upsilon, \mathrm{grad}_x u) + \sigma(x,\upsilon)u(x,\upsilon,t) - \int_V k(x,\upsilon,\upsilon')u(x,\upsilon',t)d\mu(\upsilon') = p(t)f(x,\upsilon) \tag{23}$$

with

$$u(x,\upsilon,0) = u_0(x,\upsilon), \quad u(\cdot,\cdot,t)_{|\Gamma_-} = 0. \tag{24}$$

The inverse problem is to recover $p(t)$ from the additional measurement condition

$$\int_{\Omega\times V} u(x,\upsilon,t)w(x,\upsilon)dxd\upsilon = \psi(t), \quad 0 \leq t \leq T. \tag{25}$$

The measurement functional ϕ

$$\phi u = \int_{\Omega\times V} u(x,\upsilon)w(x,\upsilon)dxd\upsilon$$

is bounded from the space $X = L^p(\Omega\times V)$ into $\mathbb{R}$, provided that $w \in L^p(\Omega\times V)$. Therefore, $\phi \in X^*$.

Applying Theorem 3.1 in the setting $X = L^p(\Omega\times V)$ yields:

Corollary 4.2 *Let $f, u_0, w \in L^p(\Omega\times V)$, and $\psi \in C^1[0,T]$. Assume that*

$$\int_{\Omega\times V} u_0(x,\upsilon)w(x,\upsilon)dxd\upsilon = \psi(0),$$

and

$$\int_{\Omega\times V} u(x,\upsilon,t)w(x,\upsilon)dxd\upsilon \neq 0, \quad 0 \leq t \leq T.$$

Further, suppose $\sigma(\cdot,\cdot) \in L^\infty(\Omega\times V)$ and that the operator K is bounded on $L^p(\Omega\times V)$. Then, the inverse problem (23)-(25) admits a unique solution (u,p) such that

$$u \in C^1([0,T];X), \quad p \in C[0,T].$$

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