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A Modified Operational Matrix Method for Solving System of Fractional Lotka-Volterra Equations

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Abstract: This paper examines the fractional Lotka-Volterra system, a cornerstone in the mathematical modeling of ecological and chemical processes. The primary objective is to introduce an innovative numerical technique for addressing nonlinear problems characterized by a nonsingular kernel. The proposed method is designed to be straightforward in implementation while minimizing computational effort and cost. To achieve this, the operational matrix method is refined and adapted. The study validates the effectiveness of this approach by applying it to various classes of fractional Lotka-Volterra systems. A thorough comparison is provided between the results obtained through this method and those generated by Mathematica's built-in commands for integer-order derivatives. Additionally, the L_2-truncation error is evaluated to highlight the method's enhanced accuracy and efficiency. The paper also addresses theoretical aspects, including the existence, uniqueness, error bounds, and convergence of the solutions, providing a robust foundation for the proposed approach.

Keywords: Moddified operational matrices, ABC derivative, fractional Lotka-Volterra system.

1 Introduction

The field of fractional calculus extends the classical notions of differentiation and integration to non-integer orders, providing a more general framework for these fundamental operations. The origins of fractional differentiation and integration trace back to 1695 when Leibniz and Euler first proposed these ideas [1]. A key feature of fractional operators is their non-local nature, which offers insights into the historical significance of fractional models. In recent times, fractional differential equations have been extensively applied to model various real-world phenomena. For example, Matušů [2] explored how this mathematical framework could be leveraged in the analysis and design of control systems. Magin [3] highlighted three bioengineering research areas where fractional calculus principles are employed to formulate novel mathematical models. Matlob and Jamali [4] applied fractional differential equations to model the behavior of viscoelastic systems. In image processing, fractional orders have been utilized to enhance denoising techniques [5]. Other applications of fractional calculus include finance [6] and solid mechanics [7]. Furthermore, several applications in mathematical biology are explored in [8]-[29].

A recent advancement in fractional calculus is the Atangana-Baleanu fractional derivative [8]. Introduced by Abdon Atangana and Dumitru Baleanu in 2016, this fractional derivative is defined using non-singular kernels and the Mittag-Leffler function. The concept of fractional differentiation was applied in [8] to develop a mathematical model for heat conduction in materials. This approach has demonstrated success in addressing numerous real-world challenges, as illustrated in [30]-[59].

The present study focuses on the following problem

$$D^{\eta}\Psi(t) = a\Psi(t) - b\Psi(t)\Phi(t), \quad \Psi(0) = \psi_0, \tag{1}$$

$$D^{\eta}\Phi(t) = c\Phi(t)\Psi(t) - d\Phi(t), \quad \Phi(0) = \phi_0, \tag{2}$$

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where D^{η} are in MABC type fractional order derivative, $0 < \eta \leq 1$, and $\psi_0, \phi_0 > 0$.

The Operational Matrix Method (OMM) is a powerful numerical technique for solving various differential equations, particularly those encountered in engineering and applied mathematics [18]-[31]. The method transforms differential equations into algebraic matrix equations by representing the differential operators as operational matrices [19]. This transformation enables the immediate computation of solutions without the need for analytical integration or symbolic manipulation. OMM is particularly effective in problems where solutions can be expressed in terms of polynomials or where linear combinations of polynomial functions can serve as approximations. It has been widely used in control system analysis, structural dynamics, heat transfer problems, and fluid mechanics.

The Operational Matrix Method (OMM) has demonstrated its effectiveness across various applications, particularly where equations are either linear or nonlinear with polynomial forms in terms of the solution. This method excels due to its ability to accurately approximate solutions by transforming differential or integral equations into algebraic equations. OMM achieves this by leveraging operational matrices that efficiently handle the underlying mathematical operations involved. In practical terms, OMM is particularly suited for problems where the solution can be expressed in terms of polynomial functions or where linear combinations of such functions can approximate the solution well. This approach not only simplifies the computational process but also enhances accuracy, making it a preferred choice in many scientific and engineering disciplines. These references provide excellent applications of the Operational Matrix Method (OMM) where OMM has been successfully applied, underscoring its versatility and robustness in solving a wide range of mathematical and physical problems.

The structure of the paper is as follows. In the subsequent section, we will present a set of definitions and lemmas that are crucial for the analysis that follows. In Section 3, we will introduce the modified iteration version of the Operational Matrix Method (OMM). Section 4 will be focused on proving key theoretical results, including the existence and uniqueness of solutions, error estimates, and convergence analysis. Section 5 will provide three practical examples to numerically validate the convergence of the proposed method to the unique solution of Problem (1)-(2). Finally, the paper will conclude with a summary and concluding remarks in the last section.

2 Fundamental Principles

This section outlines the essential concepts and results employed in this study.

Definition 1.[14] *The Atangana-Baleanu fractional derivative of order $0 < n < 1$ in the Caputo sense is defined as*

$$^{ABC}D^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_n(-\mu_\alpha (t-x)^{\alpha}) f'(x) dx, \quad t \geq 0, \tag{3}$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$, and $B(\alpha)$ is a positive function satisfying $B(\alpha) = 1$ at $x = 0, 1$.

The corresponding integral operator is introduced in the following definition:

Definition 2.[15] *The Atangana-Baleanu integral operator for $0 < \alpha < 1$ is given by*

$$^{ABC}I^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(x)(t-x)^{\alpha-1} dx. \tag{4}$$

Building on these definitions, Al-Refai and Baleanu [14] extended the ABC derivative using integration by parts and the derivative of the Mittag-Leffler function in (3). This modified derivative is as follows:

Definition 3.[14] *The modified Atangana-Baleanu fractional derivative of order $0 < n < 1$ in the Caputo sense is expressed as*

$$D^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \left[f(t) - E_\alpha(-\mu_\alpha t^{\alpha}) f(0) - \mu_n \int_0^t (t-x)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-x)^{\alpha}) f(x) dx \right], \tag{5}$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$, and $B(\alpha)$ is a positive function that equals 1 at $\alpha = 0, 1$.

The integral operator corresponding to the modified Atangana-Baleanu derivative is presented in [14]:

Definition 4.[29] *For* $f \in L^1(0,\infty)$ *and* $\alpha \in (0,1)$, *the MAB integral operator is defined as follows:*

$$I^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} (f(t) - f(0)) + \frac{\mu_{\alpha}}{\Gamma(\alpha)} \int_0^t (f(x) - f(0))(t-x)^{\alpha-1} dx, \tag{6}$$

where $\mu_{\alpha} = \frac{\alpha}{1-\alpha}$.

The following lemma highlights key relationships between the operators:

Lemma 1.[29, 33] *For* $\eta > 0 \in (0,1)$, *and* $\Phi(t) \in C[0,T]$, *the following properties hold:*

$$I^{\eta} D^{\eta} \Phi(t) = \Phi(t) - \Phi(0), \tag{7}$$

$$D^{\eta} I^{\eta} \Phi(t) = \Phi(t) - \Phi(0). \tag{8}$$

Next, we introduce the OMM through the following definition:

Definition 5.[30, 31, 32] *Let* $t_s = s\Delta, s \in 0,1,2,\ldots,M-1, \Delta = \frac{T}{M}$, *and* $M \in N$. *The s-block pulse function (BPF) is defined as*

$$\mu_s(t) = \begin{cases} 1, & t_s \le t < t_{s+1}, \\ 0, & otherwise, \end{cases} \quad 0 \le s < M.$$

The following theorem provides product and orthogonality properties of BPFs:

Theorem 1.[32, 33] *Let* $\{t_0 = 0, t_1, \ldots, t_M = T\}$ *be a uniform partition of* $[0,T]$, *and* $\{\mu_0(t), \mu_1(t), \ldots, \mu_{M-1}(t)\}$ *be the corresponding BPFs. Then, for any* $0 \le i, j \le M-1$, *the following holds:*

$$\mu_i(x)\mu_j(x) = \begin{cases} \mu_i(x), & if\ i = j, \\ 0, & otherwise. \end{cases} \tag{9}$$

and

$$\int_0^T \mu_i(x)\mu_j(x) dx = \begin{cases} \Delta, & if\ i = j, \\ 0, & otherwise. \end{cases} \tag{10}$$

Finally, the completeness property is presented in the following lemma:

Lemma 2.[31, 32] *If* $\Phi \in L^2[0,T)$, *then*

$$\Phi(t) = \sum_{s=0}^{\infty} \phi_s \mu_s(t), \tag{11}$$

where

$$\phi_s = \frac{1}{\Delta} \int_{s\Delta}^{(s+1)\Delta} \Phi(x) dx. \tag{12}$$

For practical computations, Φ can be approximated as

$$\Phi(t) \approx \sum_{s=0}^{M-1} \phi_s \mu_s(t), \quad M \gg 1. \tag{13}$$

3 Enhanced Iterative Operational Matrix Approach

In this section, we introduce an enhanced version of the iterative operational matrix approach (E-IOMA) for solving Problem (1)-(2). By applying the fractional integral operator (6) to both sides of (1)-(2), we derive the following system of equations:

$$\Psi(t) - \psi_0 = \frac{1-\alpha}{B(\alpha)} (\Psi(t)(a - b\Phi(t)) - \psi_0(a - b\phi_0)) + \frac{\mu_{\alpha}}{\Gamma(\alpha)} \int_0^t (\Psi(x)(a - b\Phi(x)) - \psi_0(a - b\phi_0)) (t-x)^{\alpha-1} dx \tag{14}$$

$$\Phi(t)-\Phi_0=\frac{1-\alpha}{B(\alpha)}\left(\Phi(t)(c\Psi(t)-d)-\phi_0(c\psi_0-d)\right)+\frac{\mu_\alpha}{\Gamma(\alpha)}\int_0^t\left(\Phi(x)(c\Psi(x)-d)-\phi_0(c\psi_0-d)\right)(t-x)^{\alpha-1}dx. \quad (15)$$

Let $\{t_0=0,t_1,...,t_M=T\}$ denote a uniform partition of $[0,T]$, and $\{\mu_0(t),\mu_1(t),\ldots,\mu_{M-1}(t)\}$ represent the corresponding Barycentric interpolation functions (BPFs). We approximate $\Psi(t)$ and $\Phi(t)$ as linear combinations of these BPFs:

$$\Psi(t)=\sum_{i=0}^{M-1}\psi_i\mu_i(t),\quad \Phi(t)=\sum_{i=0}^{M-1}\phi_i\mu_i(t). \quad (16)$$

Using the collocation method for Equations (14)-(15) at the collocation points t_j, where $1\le j<M$, we obtain:

$$\Psi(t_j)=\sum_{i=0}^{M-1}\psi_i\mu_i(t_j)=\psi_j,\quad \Phi(t_j)=\sum_{i=0}^{M-1}\phi_i\mu_i(t_j)=\phi_j,\quad 0\le j\le M-1, \quad (17)$$

which leads to the following system of equations:

$$\psi_j=\psi_0+\frac{1-\alpha}{B(\alpha)}\left(\psi_j(a-b\phi_j)-\psi_0(a-b\phi_0)\right)+\frac{\mu_\alpha}{\Gamma(\alpha)}\int_0^{t_j}\left(\Psi(x)(a-b\Phi(x))-\psi_0(a-b\phi_0)\right)(t_j-x)^{\alpha-1}dx, \quad (18)$$

$$\phi_j=\phi_0+\frac{1-\alpha}{B(\alpha)}\left(\phi_j(c\psi_j-d)-\phi_0(c\psi_0-d)\right)+\frac{\mu_\alpha}{\Gamma(\alpha)}\int_0^{t_j}\left(\Phi(x)(c\Psi(x)-d)-\phi_0(c\psi_0-d)\right)(t_j-x)^{\alpha-1}dx. \quad (19)$$

It is important to note that Equation (17) holds because:

$$\mu_i(t_j)=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

By using the properties of the Riemann integral, we can rewrite Equations (18) and (19) as:

$$\begin{aligned}&\psi_j=\psi_0+\tfrac{1-\alpha}{B(\alpha)}\left(\psi_j(a-b\phi_j)-\psi_0(a-b\phi_0)\right)\\&+\tfrac{\mu_\alpha}{\Gamma(\alpha)}\textstyle\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\left(\left(\sum_{i=0}^{M-1}\psi_i\mu_i(x)\right)\left(a-b\left(\sum_{i=0}^{M-1}\phi_i\mu_i(x)\right)\right)-\psi_0(a-b\phi_0)\right)(t_j-x)^{\alpha-1}dx,\\&\phi_j=\phi_0+\tfrac{1-\alpha}{B(\alpha)}\left(\phi_j(c\psi_j-d)-\phi_0(c\psi_0-d)\right)\\&+\tfrac{\mu_\alpha}{\Gamma(\alpha)}\textstyle\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\left(\left(\sum_{i=0}^{M-1}\phi_i\mu_i(x)\right)\left(c\left(\sum_{i=0}^{M-1}\psi_i\mu_i(x)\right)-d\right)-\phi_0(c\psi_0-d)\right)(t-x)^{\alpha-1}dx.\end{aligned} \quad (20)$$

For any $1\le k,j<M$, we have:

$$\mu_j(t)=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases},\quad t\in[t_k,t_{k+1})$$

This results in the following expressions:

$$\begin{aligned}&\textstyle\int_{t_k}^{t_{k+1}}\left(\left(\sum_{i=0}^{M-1}\psi_i\mu_i(x)\right)\left(a-b\left(\sum_{i=0}^{M-1}\phi_i\mu_i(x)\right)\right)-\psi_0(a-b\phi_0)\right)(t_j-x)^{\alpha-1}dx\\&=\textstyle\int_{t_k}^{t_{k+1}}\left(\psi_k(a-b\phi_k)-\psi_0(a-b\phi_0)\right)(t_j-x)^{\alpha-1}dx,\\&\textstyle\int_{t_k}^{t_{k+1}}\left(\left(\sum_{i=0}^{M-1}\phi_i\mu_i(x)\right)\left(c\left(\sum_{i=0}^{M-1}\psi_i\mu_i(x)\right)-d\right)-\phi_0(c\psi_0-d)\right)(t-x)^{\alpha-1}dx\\&=\textstyle\int_{t_k}^{t_{k+1}}\left(\phi_k(c\psi_k-d)-\phi_0(c\psi_0-d)\right)(t-x)^{\alpha-1}dx.\end{aligned} \quad (21)$$

Thus, we obtain the following simplified expressions for ψ_j and ϕ_j:

$$\begin{aligned}&\psi_j=\psi_0+\tfrac{1-\alpha}{B(\alpha)}\left(\psi_j(a-b\phi_j)-\psi_0(a-b\phi_0)\right)\\&+\tfrac{\mu_\alpha}{\Gamma(\alpha)}\textstyle\sum_{k=0}^{j-1}\left(\psi_k(a-b\phi_k)-\psi_0(a-b\phi_0)\right)\left((j-k)^\alpha-(j-k-1)^\alpha\right),\\&\phi_j=\phi_0+\tfrac{1-\alpha}{B(\alpha)}\left(\phi_j(c\psi_j-d)-\phi_0(c\psi_0-d)\right)\\&+\tfrac{\mu_\alpha}{\Gamma(\alpha)}\textstyle\sum_{k=0}^{j-1}\left(\phi_k(c\psi_k-d)-\phi_0(c\psi_0-d)\right)\left((j-k)^\alpha-(j-k-1)^\alpha\right).\end{aligned} \quad (22)$$

*Remark.*It is essential to note the following points:

1. To find the coefficients in the approximation (16) using the operational matrix method (OMM), it is necessary to compute operational matrices, which results in a nonlinear algebraic system that must be solved. This system can be extensive, often containing over 100 equations, leading to high computational costs and time demands. Solving the system analytically is generally not feasible, requiring numerical methods that introduce approximation errors. This accounts for why the errors from using the OMM in nonlinear physical and engineering problems can exceed 10^{-5}.
2. In contrast, our modified approach for determining the coefficients of the approximation (16), as described in Equations (22), employs a forward method and does not require solving systems. It is evident that each ψ_k can be determined from $\psi_0, \psi_1, \ldots, \psi_{k-1}$. This makes our modified OMM more practical, cost-effective, and accurate since no nonlinear systems need to be solved.

4 Error Analysis

Let $\Phi \in L^2([0,T))$ be a function, and let its norm be defined by the following expression

$$||\Phi|| = \sqrt{\int_0^T |\Phi(x)|^2 \, dx}. \tag{23}$$

Based on Lemma (2), the function $\Phi(x)$ can be approximated by the expression

$$\Phi_M(x) = \sum_{i=0}^{M-1} \phi_i \mu_i(x). \tag{24}$$

The objective of Theorem (2) is to show that the mean square error achieves its minimum when ϕ_i is determined according to Equation (12).

Theorem 2. *Let $\Phi \in L^2([0,T))$ and define $\Phi_M(x)$ as in Equation (24). Then, the mean square error term*

$$\mathscr{E}(\phi_0, \phi_1, \ldots, \phi_{M-1}) = \int_0^T (\Phi(x) - \Phi_M(x))^2 \, dx$$

attains its minimum when ϕ_i is given by Equation (12) for each $i = 0, 1, \ldots, M-1$.

Proof. For $0 \leq i \leq M-1$, applying Theorem (1), we compute

$$\begin{aligned} \frac{\partial \mathscr{E}}{\partial \phi_i} &= 2\int_0^T (\Phi(x) - \Phi_M(x))\, \mu_i(x)\, dx \\ &= 2\left(\int_0^T \Phi(x)\mu_i(x)\, dx - \Delta \phi_i\right) = 0. \end{aligned}$$

Thus, we obtain

$$\phi_i = \frac{1}{\Delta} \int_0^T \Phi(x)\mu_i(x)\, dx.$$

Furthermore, Theorem (1) gives us the second-order derivatives

$$\frac{\partial^2 \mathscr{E}}{\partial \phi_i \partial \phi_j} = \begin{cases} 2\Delta, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad 0 \leq i, j \leq M-1.$$

For any $0 \leq i \leq M-1$, we compute the determinant of the Hessian matrix:

$$\begin{vmatrix} \frac{\partial^2 \mathscr{E}}{\partial \phi_0^2} & \frac{\partial^2 \mathscr{E}}{\partial \phi_0 \partial \phi_1} & \cdots & \frac{\partial^2 \mathscr{E}}{\partial \phi_0 \partial \phi_i} \\ \frac{\partial^2 \mathscr{E}}{\partial \phi_1 \partial \phi_0} & \frac{\partial^2 \mathscr{E}}{\partial \phi_1^2} & \cdots & \frac{\partial^2 \mathscr{E}}{\partial \phi_1 \partial \phi_i} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathscr{E}}{\partial \phi_i \partial \phi_0} & \frac{\partial^2 \mathscr{E}}{\partial \phi_i \partial \phi_1} & \cdots & \frac{\partial^2 \mathscr{E}}{\partial \phi_i^2} \end{vmatrix} = \begin{vmatrix} 2\Delta & 0 & \ldots & 0 \\ 0 & 2\Delta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2\Delta \end{vmatrix} = (2\Delta)^{i+1} > 0.$$

Therefore, the error $\mathscr{E}$ reaches its minimum when ϕ_i is computed according to Equation (12) for each $i = 0, 1, \ldots, M-1$.

Next, we seek to determine the order of the mean square error in approximating $\Phi(x)$ over the interval $[0,T)$.

Theorem 3. *Let $\Phi(x)$ be a differentiable function on $[0,T)$ such that*

$$|\Phi'(x)| \leq \mathscr{F} \tag{25}$$

for all $x \in [0,T)$, where $\mathscr{F}$ is a positive constant. Then,

$$||\mathscr{E}||^2 \leq C\Delta^2 \tag{26}$$

where $\mathscr{E}(x) = \Phi(x) - \Phi_M(x)$, $x \in [0,T)$, and C is a positive constant, with $\Delta = \frac{T}{M}$.

Proof. Let $x_i = i\Delta$ and $\mathscr{A}_i = [x_i, x_{i+1})$, where $\Delta = \frac{T}{M}$ and $i = 0,1,\ldots,M-1$. By the mean value theorem for integrals and Equation (12), we obtain

$$\begin{aligned}
\Phi_M(x) &= \phi_i, \quad x \in [x_i, x_{i+1}), \\
&= \frac{1}{\Delta}\int_{x_i}^{x_{i+1}} \Phi(t)\,dt, \\
&= \Phi(\nu_i), \quad \nu_i \in [x_i, x_{i+1}), \quad i = 0,1,\ldots,M-1.
\end{aligned}$$

Using the mean value theorem for integrals, we have

$$\begin{aligned}
||\mathscr{E}||^2 &= \int_0^T (\Phi(x) - \Phi_M(x))^2\,dx \\
&= \sum_{i=0}^{M-1}\int_{x_i}^{x_{i+1}} (\Phi(x) - \Phi_M(x))^2\,dx \\
&= \Delta \sum_{i=0}^{M-1} (\Phi(\omega_i) - \Phi(\nu_i))^2
\end{aligned}$$

where $\omega_i, \nu_i \in [x_i, x_{i+1})$. By the mean value theorem and Equation (25), we obtain

$$\begin{aligned}
||\mathscr{E}||^2 &\leq \Delta \mathscr{F}^2 \sum_{i=0}^{M-1} (\omega_i - \nu_i)^2 \\
&\leq \Delta \mathscr{F}^2 \sum_{i=0}^{M-1} \Delta^2 \\
&= M\mathscr{F}^2\Delta^3 = (T\mathscr{F}^2)\Delta^2 = C\Delta^2.
\end{aligned}$$

Thus, $C = \mathscr{F}^2 T$.

The result above establishes that the mean square error in our approximation is of order Δ^2.

5 Numerical Results

In this section, we present three examples to illustrate the efficiency of the proposed method.

Example 1. Consider the following Lotka-Volterra type system:

$$\begin{aligned}
D^\eta \Psi(t) &= \Psi(t)(1 - 0.1\Phi(t)), \quad \Psi(0) = 10, \\
D^\eta \Phi(t) &= \Phi(t)(-1.1 + 0.2\Psi(t)), \quad \Phi(0) = 20,
\end{aligned}$$

where $t \in [0,50]$, and $0 < \eta \leq 1$. We approximate $\Psi(t)$ and $\Phi(t)$ using the following series expansions:

$$\Psi(t) = \sum_{i=0}^{M-1} \psi_i \mu_i(t), \quad \Phi(t) = \sum_{i=0}^{M-1} \phi_i \mu_i(t).$$

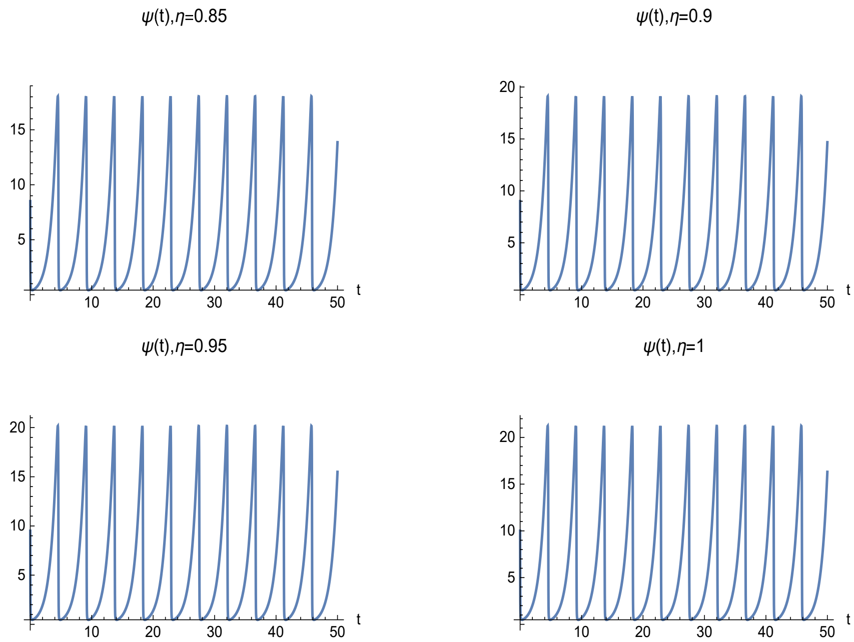

Figure 1. The solution Ψ for $\eta = 0.85, 0.9, 0.95, 1$.

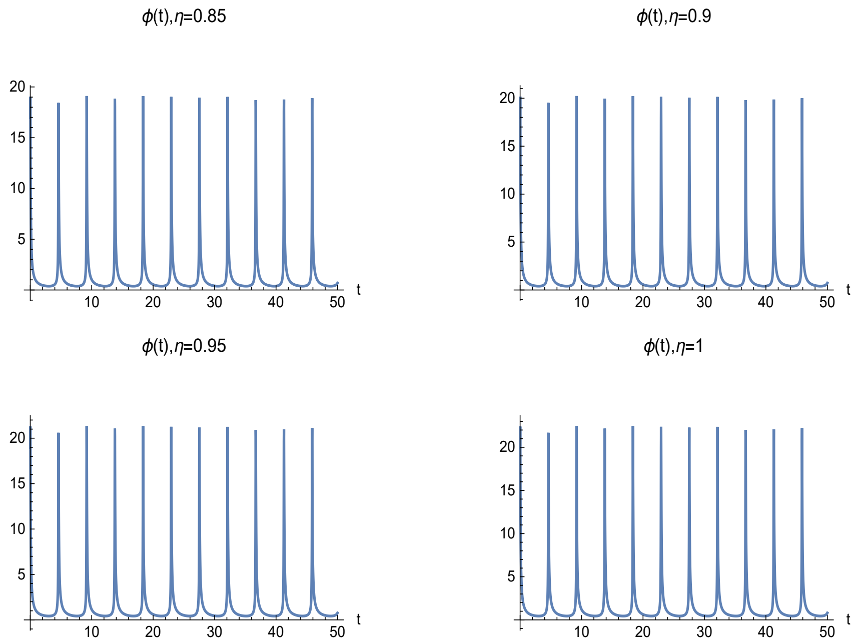

Figure 2. The solution Φ for $\eta = 0.85, 0.9, 0.95, 1$.

Let $\Delta = 0.01$. We begin by investigating the influence of the fractional order derivative on the behavior of the solution. Figures 1 and 2 display the approximate solutions for Ψ and Φ for various values of $\eta = 0.85, 0.9, 0.95, 1$ using the MIOMM method.

Figures 3 and 4 illustrate the impact of the fractional derivative on the solution profile within the interval $[0, 10]$ for different values of $\eta = 0.7, 0.8, 0.9, 0.95, 1$.

Next, we compare the approximate solutions generated by the MIOMM method and the NDSolve function in Mathematica for $\eta = 1$, as shown in Figures 5 and 6.

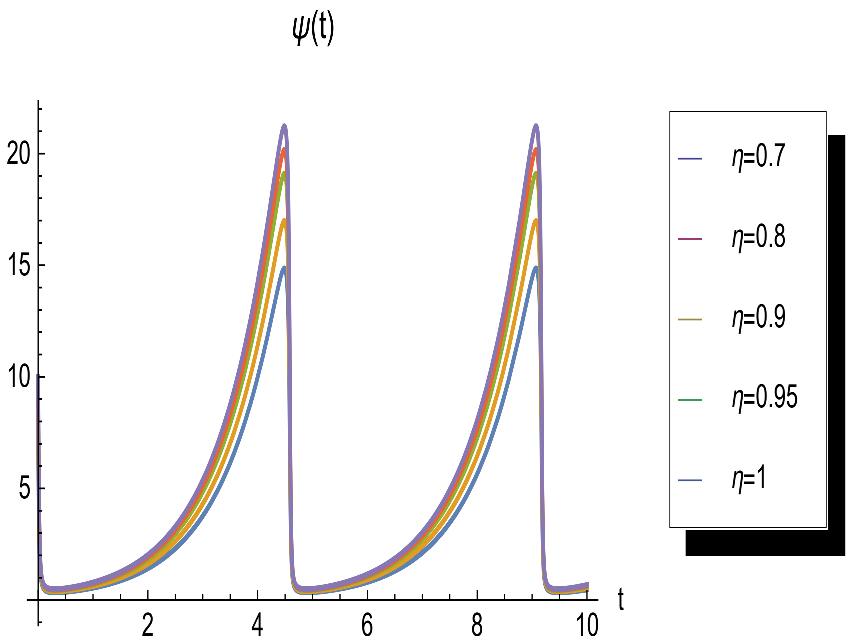

Figure 3. The solution Ψ for $\eta = 0.7, 0.8, 0.9, 0.95, 1$ on $[0,10]$.

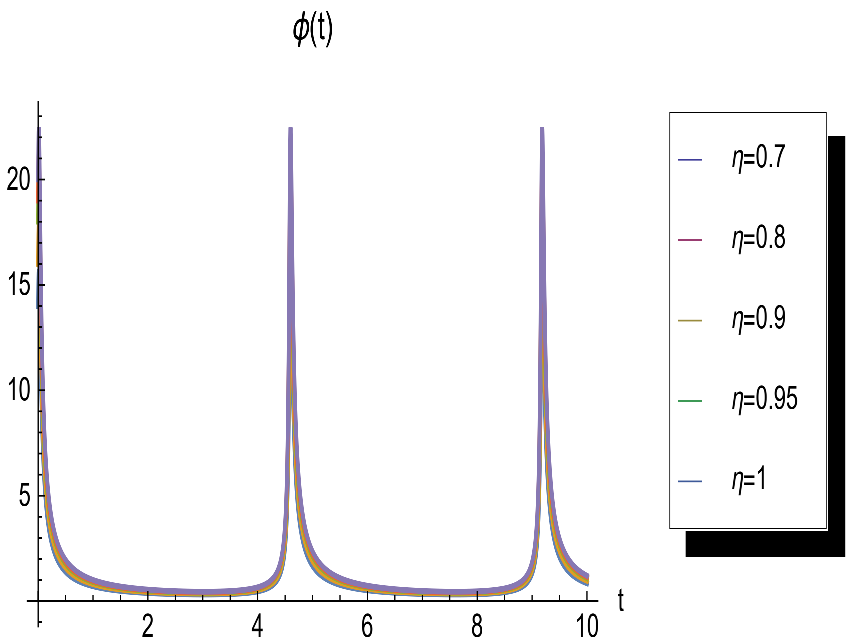

Figure 4. The solution Φ for $\eta = 0.7, 0.8, 0.9, 0.95, 1$ on $[0,10]$.

Finally, to assess the accuracy of the approximation, we compute the L_2-truncation errors, defined as:

$$\varepsilon(\eta) = \left(\int_0^{50} \left\| \begin{pmatrix} D^\eta \Psi_m(t) - \Psi_m(t)(\alpha - \beta \Phi_m^p(t)) \\ D^\eta \Phi_m(t) - \Phi_m^p(t)(-\gamma + \delta \Psi_m(t)) \end{pmatrix} \right\|^2 dt \right)^{\frac{1}{2}}$$

The errors are shown in Table 1.

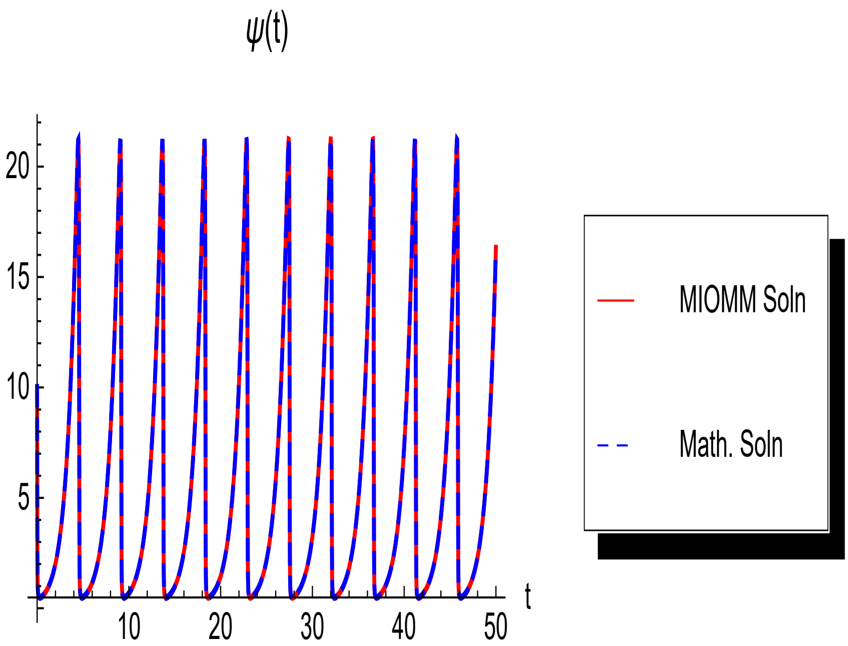

Figure 5. The solutions of Ψ generated by MIOMM and Mathematica for $\eta = 1$.

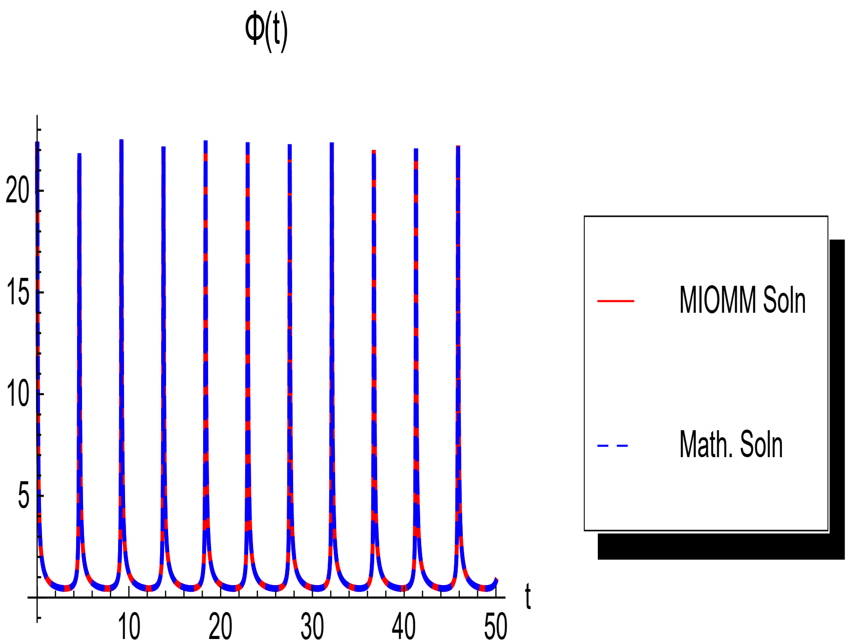

Figure 6. The solutions of Φ generated by MIOMM and Mathematica for $\eta = 1$.

*Example 2.*Consider the following Lotka-Volterra type system:

$$D^{\eta}\Psi(t) = \Psi(t)(2 - 0.3\Phi(t)), \quad \Psi(0) = 10,$$
$$D^{\eta}\Phi(t) = \Phi(t)(-1.3 + 0.4\Psi(t)), \quad \Phi(0) = 20,$$

where $t \in [0, 75]$, and $0 < \eta \leq 1$. The procedure from Example 1 is followed to generate Figures 7-10.

Table 1: The L_2-error for $\eta = 0.7, 0.8, 0.9, 0.99, 1$.

η	$\varepsilon(\eta)$
0.6	3.22×10^{-15}
0.75	2.44×10^{-15}
0.9	2.31×10^{-15}
0.99	1.27×10^{-15}
1	1.11×10^{-15}

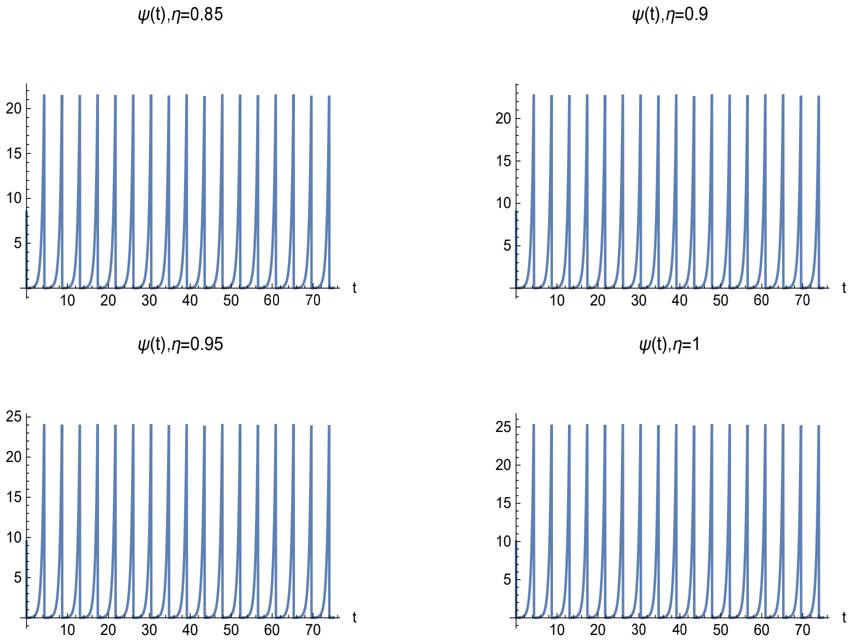

Figure 7. The solution Ψ for $\eta = 0.85, 0.9, 0.95, 1$.

Additionally, the approximate solution generated by MIOMM is compared to that produced by the NDSolve command in Mathematica for $\eta = 1$, as shown in Figures 11 and 12.

Finally, to evaluate the accuracy of the approximation, we compute the L_2-truncation errors. The results are presented in Table 2.

Table 2: The L_2-error for $\eta = 0.7, 0.8, 0.9, 0.99, 1$.

η	$\varepsilon(\eta)$
0.6	1.27×10^{-15}
0.75	1.45×10^{-15}
0.9	1.22×10^{-15}
0.99	0.21×10^{-15}
1	0.11×10^{-15}

6 Conclusion

The operational matrix method is a powerful tool for solving systems of fractional initial value problems. The core of this method lies in approximating the solution of such systems using Block Pulse functions. To compute the coefficients for this approximation, conventional methods require the determination of operational matrices corresponding to integral,

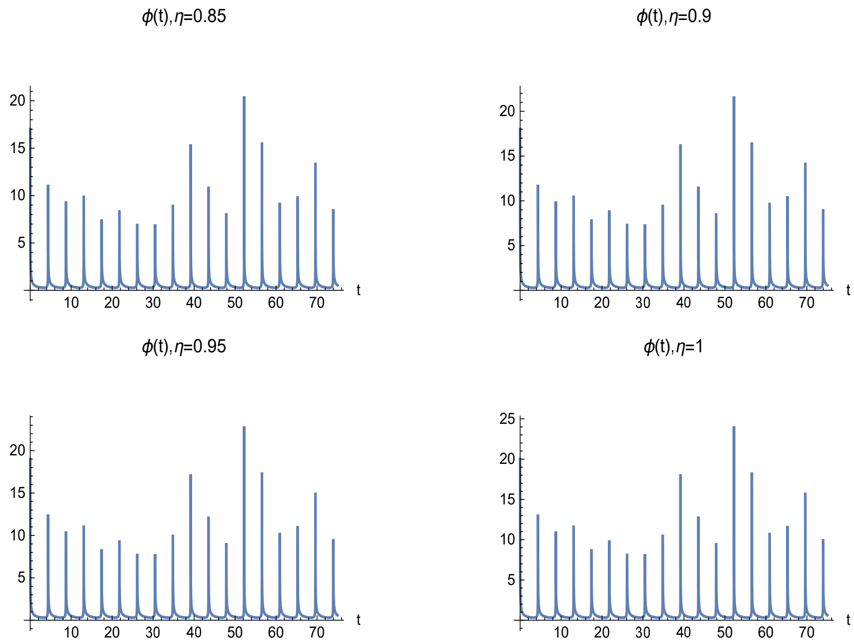

Figure 8. The solution Φ for $\eta = 0.85, 0.9, 0.95, 1$.

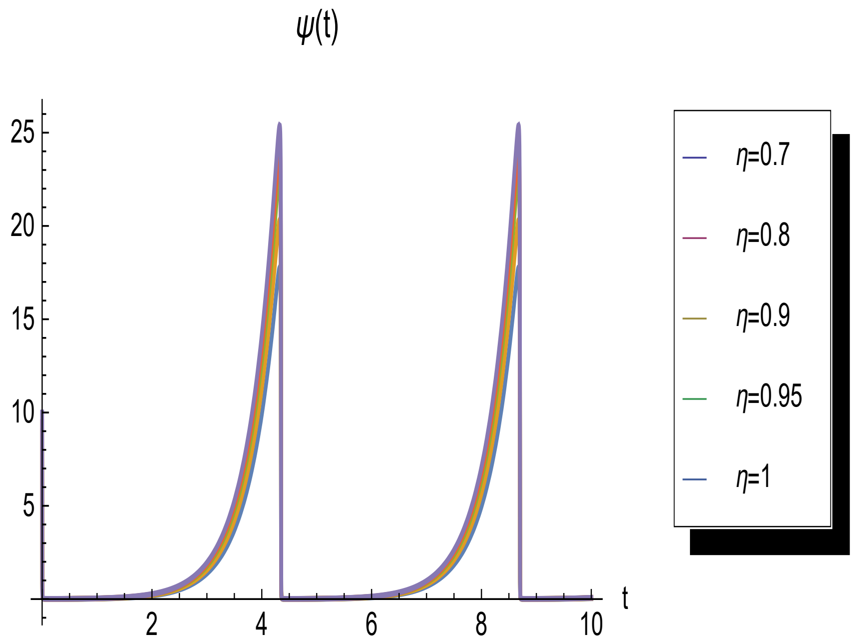

Figure 9. The solution Ψ for $\eta = 0.7, 0.8, 0.9, 0.95, 1$ on $[0, 10]$.

derivative, and product operators. This often leads to the formation of a large system of nonlinear algebraic equations, whose solution can be computationally intensive and time-consuming, thereby limiting accuracy.

In this paper, we have proposed a novel modification of this method that eliminates the need to solve for the coefficients of the solution expansion. Instead, we suggest an alternative approach where the coefficients are determined iteratively and explicitly based on the previously computed coefficients. We have developed the theoretical framework for this method and established its convergence properties, proving that the iterative calculation of the coefficients generates a sequence of functions that converges to the unique solution of the system.

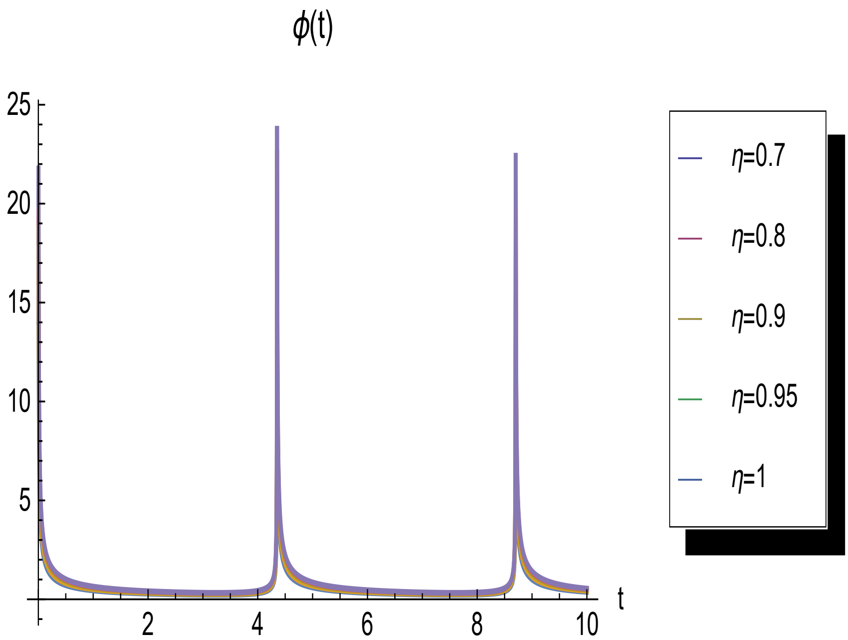

Figure 10. The solution Φ for $\eta = 0.7, 0.8, 0.9, 0.95, 1$ on $[0, 10]$.

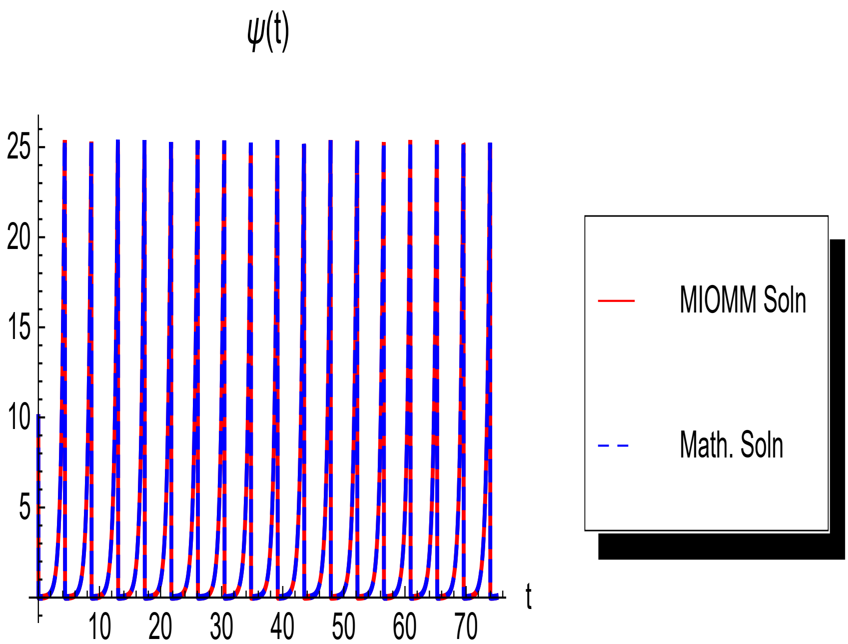

Figure 11. The solutions of Ψ generated by MIOMM and Mathematica for $\eta = 1$.

To demonstrate the effectiveness of our proposed approach, we conducted numerical experiments using a variety of examples. The L_2-norm of the error was used to assess the accuracy of our method.

The following observations can be made based on the results:

1. Example 1 shows that our modified method provides approximate solutions that converge to the exact solution. The L_2-norm remains on the order of 10^{-14} for different values of η, highlighting both the speed and accuracy of our method. Moreover, the solutions obtained using our approach match precisely with those obtained from Mathematica's

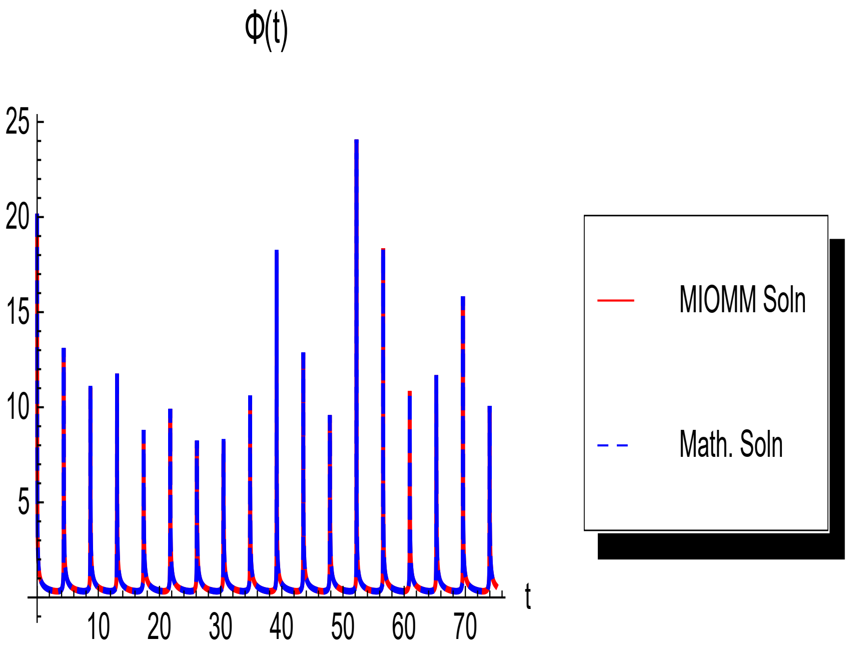

Figure 12. The solutions of Φ generated by MIOMM and Mathematica for $\eta = 1$.

NDSolve command. Additionally, the behavior of the solution is largely unaffected by the fractional derivative, and as η approaches unity, the solution profile converges to that of the integer-order solution.

2.Example 2 reinforces the results from Example 1, confirming that solutions with fractional derivatives exhibit behavior similar to integer derivatives. The L_2-norm error remains around 10^{-14} for various values of η, and the solutions generated by our method are consistent with those produced by Mathematica.

3.The results suggest that our proposed method is highly promising and can be applied to a wide range of models in physics and engineering, even in the presence of significant nonlinearity.

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