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Towards Proportional Fractional Calculus and Inequalities

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Abstract: Left and right Caputo fractional calculus is developed further with new interesting representation formulae. In that spirit we introduce and develop in great generality the corresponding left and right proportional Caputo style fractional calculus. We establish the analogous proportional fractional representation formulae. Based on all the above we derive left and right fractional integral inequalities of Opial, Poincaré, Sobolev and Hilbert-Pachpatte types.

Keywords: Caputo fractional calculus, proportional fractional calculus, fractional integral inequalities.

1 Background

1.1 From Left Fractional calculus and more

Let $\alpha \in (0,1)$ and $f \in C^1([a,b])$, $[a,b] \subset \mathbb{R}$.

The left Riemann-Liouville fractional integral of f is given by

$$\left(J_a^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1}$$

for $a \leq x \leq b$, where Γ is the gamma function.

We set $J_a^0 := I$, the identity operator.

By Theorem 1.5, p. 3, [1], $J_a^{\alpha} f \in C([a,b])$.

Here $\lceil \cdot \rceil$ denotes the ceiling of the number, i.e. $\lceil \alpha \rceil = 1$.

The left Caputo fractional derivative of f is given by

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f'(t)\, dt, \tag{2}$$

$\forall\, x \in [a,b]$. Clearly $D_{*a}^{\alpha} f \in C([a,b])$.

We set $D_{*a}^1 f := f'$ and $D_{*a}^0 f := f$.

By [1], p. 11, we get that

$$\left(J_a^{\alpha} D_{*a}^{\alpha} f\right)(x) = \left(J_a^1 f'\right)(x) \tag{3}$$

$$= \int_a^x f'(t)\, dt = f(x) - f(a), \;\; \forall\, x \in [a,b].$$

When $f(a) = 0$, we obtain

$$\left(J_a^{\alpha} D_{*a}^{\alpha} f\right)(x) = f(x), \;\; \forall\, x \in [a,b]. \tag{4}$$

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The left Riemann-Liouville fractional derivative of f is defined by

$$ {}_{L}D_{a}^{\alpha} f(x) = DJ_{a}^{1-\alpha} f(x) $$
$$ = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x} (x-t)^{-\alpha} f(t)\, dt, \ \ \forall\, x \in [a,b]. \tag{5} $$

By Lemma 3.5, p. 53, [2], we derive that

$$ {}_{L}D_{a}^{\alpha} f = D_{*a}^{\alpha} f, $$

iff

$$ f(a) = 0. $$

And, by Theorem 3.7, p. 53, [2], we have that

$$ D_{*a}^{\alpha} J_{a}^{\alpha} f = f, \tag{6} $$

given f is continuous over $[a,b]$.

1.2 From Right Fractional Calculus and more

Let $\alpha \in (0,1)$ and $f \in C^{1}([a,b])$, $[a,b] \subset \mathbb{R}$.

The right Riemann-Liouville fractional integral of f is given by

$$ I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (J-x)^{\alpha-1} f(J)\, dJ, \tag{7} $$

$\forall\, x \in [a,b]$. We set $I_{b-}^{0} := I$.

By Theorem 2.5, p. 37, [1], $I_{b-}^{\alpha} f \in C([a,b])$.

The right Caputo fractional derivative of f of order α, is given by

$$ D_{b-}^{\alpha} f(x) := -I_{b-}^{1-\alpha} f'(x) = $$
$$ \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{b} (J-x)^{-\alpha} f'(J)\, dJ, \tag{8} $$

$\forall\, x \in [a,b]$. Clearly $D_{b-}^{\alpha} f \in C([a,b])$.

The right Riemann-Liouville fractional derivative of f is defined by

$$ {}_{R}D_{b-}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{b} (t-x)^{-\alpha} f(t)\, dt, \tag{9} $$

$\forall\, x \in [a,b]$.

We define ${}_{R}D_{b-}^{0} f := I$.

That is

$$ {}_{R}D_{b-}^{\alpha} f = -DI_{b-}^{1-\alpha} f. \tag{10} $$

Given $f(b) = 0$, by Theorem 23.9, p. 339, [3], we obtain that

$$ {}_{R}D_{b-}^{\alpha} f = \left(D_{b-}^{\alpha} f \right), \tag{11} $$

over $[a.b]$.

We have that

$$ D_{b-}^{\alpha} I_{b-}^{\alpha} f =_{R} D_{b-}^{\alpha} I_{b-}^{\alpha} f = $$
$$ (-1)\, DI_{b-}^{1-\alpha} I_{b-}^{\alpha} f = (-1)\, DI_{b-}^{1} f = (-1)^{2} f = f. $$

So when $f(b) = 0$, we have

$$ D_{b-}^{\alpha} I_{b-}^{\alpha} f = f. \tag{12} $$

Also, by [1], p. 43, we have

$$ \left(I_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x) = (-1) \left(I_{b-}^{1} f' \right)(x) = $$
$$ -\int_{x}^{b} f'(J)\, dJ = \int_{b}^{x} f'(J)\, dJ = f(x) - f(b). \tag{13} $$

Consequently, given $f(b) = 0$, we obtain

$$ I_{b-}^{\alpha} D_{b-}^{\alpha} f = f. \tag{14} $$

1.3 Conclusion

Let $\alpha \in (0,1)$, $f \in C^1([a,b])$.

(i) Given $f(a) = 0$, we derive over $[a,b]$ that

$$J_a^\alpha D_{*a}^\alpha f = D_{*a}^\alpha J_a^\alpha f = f. \tag{15}$$

Hence J_a^α, D_{*a}^α are functional inverses.

(ii) Given $f(b) = 0$, we get over $[a,b]$ that

$$I_{b-}^\alpha D_{b-}^\alpha f = D_{b-}^\alpha I_{b-}^\alpha f = f. \tag{16}$$

Hence I_{b-}^α, D_{b-}^α are functional inverses.

So that

$$(J_a^\alpha D_{*a}^\alpha)(J_a^\alpha D_{*a}^\alpha f) = (J_a^\alpha D_{*a}^\alpha)(f) = f, \tag{17}$$

and

$$J_a^\alpha (D_{*a}^\alpha \circ J_a^\alpha) D_{*a}^\alpha f = f, \tag{18}$$

and

$$J_a^\alpha (J_a^\alpha \circ D_{*a}^\alpha) D_{*a}^\alpha f = f. \tag{19}$$

That is

$$(J_a^\alpha)^2 (D_{*a}^\alpha)^2 f = f. \tag{20}$$

Hence for any $N \in \mathbb{N}$, we obtain

$$(J_a^\alpha)^N (D_{*a}^\alpha)^N f = f, \tag{21}$$

given that $f(a) = 0$.

By similar reasoning we derive that

$$\left(I_{b-}^\alpha\right)^N \left(D_{b-}^\alpha\right)^N f = f, \tag{22}$$

$\forall N \in \mathbb{N}$, given that $f(b) = 0$.

Consequently, if $(D_{*a}^\alpha)^{(i)} f \in C([a,b])$, $i = 1, ..., N$, we get that

$$J_a^\alpha (D_{*a}^\alpha)^N f = (D_{*a}^\alpha)^{N-1} f =: D_{*a}^{(N-1)\alpha} f, \tag{23}$$

given that $f(a) = 0$.

Furthermore, if $\left(D_{b-}^\alpha\right)^{(i)} f \in C([a,b])$, $i = 1, ..., N$, we obtain

$$I_{b-}^\alpha \left(D_{b-}^\alpha\right)^N f = \left(D_{b-}^\alpha\right)^{N-1} f =: D_{b-}^{(N-1)\alpha} f, \tag{24}$$

given that $f(b) = 0$.

So both cases are true for some $N \in \mathbb{N}$; $\alpha \in (0,1)$ and $f \in C^1([a,b])$.

1.4 About New Conformable Calculus

Here all come from [4].

Definition 1.*(A class of New Conformable Derivatives) ([4]) Let $\alpha \in [0,1]$, and let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that*

$$\begin{gathered}
\lim_{\alpha \to 0+} k_1(\alpha,t) = 1, \quad \lim_{\alpha \to 0+} k_0(\alpha,t) = 0, \ \forall\, t \in \mathbb{R}, \\
\lim_{\alpha \to 1-} k_1(\alpha,t) = 0, \quad \lim_{\alpha \to 1-} k_0(\alpha,t) = 1, \ \forall\, t \in \mathbb{R}, \\
k_1(\alpha,t) \neq 0,\ \alpha \in [0,1),\ k_0(\alpha,t) \neq 0, \quad \alpha \in (0,1],\ \forall\, t \in \mathbb{R}.
\end{gathered} \tag{25}$$

Then, the following differentiable operator D^α, defined via

$$D^\alpha f(t) = k_1(\alpha,t) f(t) + k_0(\alpha,t) f'(t) \tag{26}$$

is the New Conformable derivative given that $f'(t)$ exists for $t \in \mathbb{R}$.

Definition 2.([4]) (Conformable Exponential Function). *Let $\alpha \in (0,1]$, the points $s,t \in \mathbb{R}$ with $s \leq t$, and let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (25) with $\frac{p}{k_0}$ and $\frac{k_1}{k_0}$ Riemann integrable on $[s,t]$. Then the exponential function with respect to D^α in (26) is defined to be*

$$e_p(t,s) := e^{\int_s^t \frac{p(\tau)-k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau}, \quad e_0(t,s) := e^{-\int_s^t \frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau}. \tag{27}$$

Using (26) and (27) we have the following basic results.

Lemma 1.([4]) (Basic Derivatives). *Let the conformable differential operator D^α be given as in (26), where $\alpha \in [0,1]$. Let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (25), with $\frac{p}{k_0}$ and $\frac{k_1}{k_0}$ Riemann integrable on $[s,t]$. Assume the functions f and g are differentiable as needed. Then*

(i) $D^\alpha [af + bg] = aD^\alpha [f] + bD^\alpha [g]$ for all $a,b \in \mathbb{R}$;

(ii) $D^\alpha c = ck_1(\alpha,\cdot)$ for all constants $c \in \mathbb{R}$;

(iii) $D^\alpha [fg] = fD^\alpha [g] + gD^\alpha [f] - fgk_1(\alpha,\cdot)$;

(iv) $D^\alpha \left[\frac{f}{g}\right] = \frac{gD^\alpha[f] - fD^\alpha[g]}{g^2} + \frac{f}{g} k_1(\alpha,\cdot)$;

(v) for $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$D_t^\alpha [e_p(t,s)] = p(t)\, e_p(t,s) \tag{28}$$

for $e_p(t,s)$ given in (27);

(vi) for $\alpha \in (0,1]$ and for the exponential function e_0 given in (27), we have

$$D^\alpha \left[\int_a^t \frac{f(s)\, e_0(t,s)}{k_0(\alpha,s)} ds\right] = f(t). \tag{29}$$

Definition 3.([4]) (Integrals). *Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. In the light of (27) and Lemma 1 (v) & (vi), define the antiderivative via*

$$\int D^\alpha f(t)\, d_\alpha t = f(t) + ce_0(t,t_0), \quad c \in \mathbb{R}.$$

Similarly, define the integral of f over $[a,b]$ as

$$\int_a^t f(s)\, e_0(t,s)\, d_\alpha s := \int_a^t \frac{f(s)\, e_0(t,s)}{k_0(t,s)} ds, \quad d_\alpha s := \frac{1}{k_0(\alpha,s)} ds; \tag{30}$$

recall that

$$e_0(t,s) := e^{-\int_s^t \frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau} = e^{-\int_s^t k_1(\alpha,\tau) d_\alpha \tau}$$

from (27).

Lemma 2.([4]) (Basic Integrals). *Let the conformable differential operator D^α be given as in (26), the integral be given as in (30) with $\alpha \in (0,1]$. Let the functions k_0, k_1 be continuous and satisfy (25), and let f and g be differentiable as needed. Then*

(i) the derivative of the definite integral of f is given by

$$D^\alpha \left[\int_a^t f(s)\, e_0(t,s)\, d_\alpha s\right] = f(t); \tag{31}$$

(ii) the definite integral of the derivative of f is given by

$$\int_a^t D^\alpha [f(s)]\, e_0(t,s)\, d_\alpha s = f(s)\, e_0(t,s)\, |_{s=a}^t := f(t) - f(a)\, e_0(t,a); \tag{32}$$

(iii) an integration by parts formula is given by

$$\int_a^b f(t)\, D^\alpha [g(t)]\, e_0(b,t)\, d_\alpha t = f(t)\, g(t)\, e_0(b,t)\, |_{t=a}^b -$$

$$\int_\alpha^b g(t)\left(D^\alpha [f(t)] - k_1(\alpha,t)\, f(t)\right) e_0(b,t)\, d_\alpha t; \tag{33}$$

Remark.(to Lemma 2 (iii))
When $f = g$ we get

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t = f^2(b) - f^2(a) e_0(b,a) -$$

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t + \int_a^b k_1(\alpha,t) f^2(t) e_0(b,t) d_\alpha t. \tag{34}$$

Therefore it holds

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t = \left(\frac{f^2(b) - f^2(a) e_0(b,a)}{2} \right) + \tag{35}$$

$$\frac{1}{2} \int_a^b f^2(t) k_1(\alpha,t) e_0(b,t) d_\alpha t.$$

1.5 Proportional left Caputo Fractional Calculus

Proportional left Caputo Fractional Calculus was first introduced in [5], $\alpha \in (0,1)$.
Here we go more generally.
We define for $f \in C^1([a,b])$ in the sense of D^α:

$$\left({}^P D_{*a}^\alpha f\right)(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} D^\alpha f(t) dt, \tag{36}$$

$\forall x \in [a,b]$. Clearly ${}^P D_{*a}^\alpha f \in C([a,b])$.
The function ${}^P D_{*a}^\alpha f$ is called the proportional left Caputo fractional derivative of f.
That is

$$\left({}^P D_{*a}^\alpha f\right)(x) = J_a^{1-\alpha} (D^\alpha f)(x). \tag{37}$$

The left proportional integral is given by (30):

$$\left({}^P I_a f\right)(t) := \int_a^t f(s) e_0(t,s) d_\alpha s, \tag{38}$$

and ${}^P I_a f \in C([a,b])$.
By (32) we get

$$\left({}^P I_a D^\alpha f\right)(t) = f(t) - f(a) e_0(t,a), \tag{39}$$

and ${}^P I_a D^\alpha f \in C([a,b])$.
We define the left Riemann-Liouville proportional integral as follows:

$$\left({}_L^P I_a^{RL} f\right)(t) := \int_a^t {}_L D_a^{1-\alpha} f(s) e_0(t,s) d_\alpha s. \tag{40}$$

Clearly ${}_L^P I_a^{RL} f \in C([a,b])$ when $f(a) = 0$.
We have that

$$\left({}_L^P I_a^{RL} \left({}^P D_{*a}^\alpha f\right)\right)(t) = \int_a^t \left({}_L D_a^{1-\alpha} \left({}^P D_{*a}^\alpha f\right)\right)(s) e_0(t,s) d_\alpha s, \tag{41}$$

with $\left({}_L^P I_a^{RL} \left({}^P D_{*a}^\alpha f\right)\right) \in C([a,b])$.
We see that

$${}^P D_{*a}^\alpha \, {}_L^P I_a^{RL} f(t) = \left(J_a^{1-\alpha} \circ D^\alpha\right) \circ \left({}^P I_a \circ {}_L D_a^{1-\alpha}\right) f(t)$$

$$\overset{(31)}{=} \left(J_a^{1-\alpha} \circ {}_L D_a^{1-\alpha}\right) f(t) \tag{42}$$

(by [5], p. 39)

$$= f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(\alpha)} \lim_{z \to a+} J_a^\alpha f(z) = f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(\alpha)} \cdot 0 = f(t), \; t \neq a.$$

We have proved that
$$ {}^{P}D^{\alpha}_{*a}\, {}^{P}_{L}I^{RL}_{a} f = f,\ t \neq a. \tag{43} $$

Next we study
$$ \left({}^{P}_{L}I^{RL}_{a}\left({}^{P}D^{\alpha}_{*a} f\right)\right)(t) = \left({}^{P}I_{a} \circ {}_{L}D^{1-\alpha}_{a}\right) \circ \left(J^{1-\alpha}_{a} \circ D^{\alpha}\right) f(t) \tag{44} $$
(by Theorem 2.14, p. 30, [2])
$$ = \left({}^{P}I_{a} \circ D^{\alpha}\right) f(t) \overset{(39)}{=} f(t) - f(a)\, e_0(t,a). $$
Assuming $f(a) = 0$, we have
$$ \left({}^{P}_{L}I^{RL}_{a}\, {}^{P}D^{\alpha}_{*a}\right) f = f. \tag{45} $$

Conclusion: When $f(a) = 0$, we have
$$ \left({}^{P}_{L}I^{RL}_{a}\, {}^{P}D^{\alpha}_{*a}\right) f = \left({}^{P}D^{\alpha}_{*a}\, {}^{P}_{L}I^{RL}_{a}\right) f = f,\ \ t \neq a. \tag{46} $$
That is ${}^{P}_{L}I^{RL}_{a}$, ${}^{P}D^{\alpha}_{*a}$ are functional inverses over (a,b].

Clearly for any $N \in \mathbb{N}$ we get
$$ \left({}^{P}_{L}I^{RL}_{a}\right)^{N}\ \left({}^{P}D^{\alpha}_{*a}\right)^{N} f = f,\ t \neq a. \tag{47} $$
given that $f(a) = 0$.

Thus, it holds
$$ {}^{P}_{L}I^{RL}_{a}\left({}^{P}D^{\alpha}_{*a}\right)^{N} f = \left({}^{P}D^{\alpha}_{*a}\right)^{N-1} f =: \left({}^{P}D^{(N-1)\alpha}_{*a}\right) f,\ t \neq a. \tag{48} $$
under the assumptions: $f(a) = 0$;
and $\left({}^{P}D^{\alpha}_{*a}\right)^{(i)} f \in C([a,b])$, for $i = 1,\dots,N$; where $f \in C^1([a,b])$, $\alpha \in (0,1)$.

1.6 Proportional right Caputo Fractional Calculus

We define for $f \in C^1([a,b])$ in the sense of D^{α}, $\alpha \in (0,1)$:
$$ \left({}^{P}D^{\alpha}_{b-} f\right)(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} D^{\alpha} f(t)\, dt, \tag{49} $$
$\forall\, x \in [a,b]$. Clearly ${}^{P}D^{\alpha}_{b-} f \in C([a,b])$.

That is
$$ \left({}^{P}D^{\alpha}_{b-} f\right)(x) = -I^{1-\alpha}_{b-}\left(D^{\alpha} f\right)(x). \tag{50} $$

We call ${}^{P}D^{\alpha}_{b-} f$ the proportional right Caputo fractional derivative of f.

The right proportional integral of f is given by
$$ \left({}^{P}I_{b} f\right)(t) := \int_t^b f(s)\, e_0(b,s)\, d_{\alpha}s, \tag{51} $$
$\forall\, t \in [a,b]$; with ${}^{P}I_{b} f \in C([a,b])$.

We have that
$$ \left({}^{P}I_{b}\left(D^{\alpha} f\right)\right)(t) = \int_t^b \left(D^{\alpha} f\right)(s)\, e_0(b,s)\, d_{\alpha}s \overset{(32)}{=} f(b) - f(t)\, e_0(b,t); \tag{52} $$
with $\left({}^{P}I_{b}\left(D^{\alpha} f\right)\right) \in C([a,b])$.

If $f(b) = 0$, we derive that
$$ \left({}^{P}I_{b} \circ D^{\alpha}\right)(f(t)) = -f(t)\, e_0(b,t),\ \ \forall t \in [a,b]. \tag{53} $$

Next we define
$$ \left({}^{P}_{R}I^{RL}_{b} f\right)(t) := \int_t^b \left({}_{R}D^{1-\alpha}_{b-} f(s)\right) e_0(b,s)\, d_{\alpha}s, \tag{54} $$
$\forall\ t \in [a,b]$, where ${}_{R}D^{1-\alpha}_{b-} f$ the right Riemann-Liouville fractional derivative of f of order $1-\alpha$, $\alpha \in (0,1)$. When $f(b) = 0$, it is ${}^{P}_{R}I^{RL}_{b} f \in C([a,b])$.

Furthermore we consider

$$\left({}_{R}^{P}I_{b}^{RL}\left({}^{P}D_{b-}^{\alpha}f\right)\right)(t)=\int_{t}^{b}\left({}_{R}D_{b-}^{1-\alpha}\left({}^{P}D_{b-}^{\alpha}f\right)\right)(s)\,e_{0}(b,s)\,d_{\alpha}s, \tag{55}$$

$\forall\, t\in[a,b]$, where $\left({}_{R}^{P}I_{b}^{RL}\left({}^{P}D_{b-}^{\alpha}f\right)\right)\in C([a,b])$.

Finally we observe that

$$\left({}_{R}^{P}I_{b}^{RL}\circ{}^{P}D_{b-}^{\alpha}\right)f(t)\overset{(\text{by }(50),(53))}{=}-\left({}^{P}I_{b}\circ{}_{R}D_{b-}^{1-\alpha}\right)\circ\left(I_{b-}^{1-\alpha}\circ D^{\alpha}\right)f(t) \tag{56}$$

$$=-\left({}^{P}I_{b}\circ D^{\alpha}\right)f(t)=f(t)\,e_{0}(b,t),$$

by assuming $f(b)=0$.

Above we used that (see (11) and (12))

$$D_{b-}^{1-\alpha}\circ I_{b-}^{1-\alpha}=I,\ \text{ the identity operator.}$$

Conclusion: When $f(b)=0$, we get that

$$\left({}_{R}^{P}I_{b}^{RL}\circ{}^{P}D_{b-}^{\alpha}\right)f(t)=f(t)\,e_{0}(b,t), \tag{57}$$

$\forall\, t\in[a,b]$.

2 Main Results

We start with a collection of Opial type inequalities.

Here $\alpha\in(0,1)$, $f\in C^{1}([a,b])$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $N\in\mathbb{N}$.

Theorem 1. *Assume that $\alpha>\frac{1}{q}$; $f(a)=0$, $\left(D_{*a}^{i\alpha}f\right)\in C([a,b])$, $i=1,...,N$. Then*

$$\int_{a}^{x}\left|\left(D_{*a}^{(N-1)\alpha}f\right)(w)\right|\left|\left(D_{*a}^{N\alpha}f\right)(w)\right|dw\leq$$

$$\frac{(x-a)^{\alpha+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_{a}^{x}\left|\left(D_{*a}^{N\alpha}f\right)(w)\right|^{q}dw\right)^{\frac{2}{q}}, \tag{58}$$

$\forall\, x\in[a,b]$.

Proof. By (23) we have

$$J_{a}^{\alpha}\left(D_{*a}^{N\alpha}\right)f(x)=D_{*a}^{(N-1)\alpha}f(x), \tag{59}$$

given that $f(a)=0$, under the assumption $\left(D_{*a}^{i\alpha}f\right)\in C([a,b])$, $i=1,...,N$.

That is

$$\left(D_{*a}^{(N-1)\alpha}f\right)(w)=\frac{1}{\Gamma(\alpha)}\int_{a}^{w}(w-t)^{\alpha-1}\left(D_{*a}^{N\alpha}f\right)(t)\,dt, \tag{60}$$

for all $a\leq w\leq x\leq b$.

Hence, by Hölder's inequality we derive

$$\left|\left(D_{*a}^{(N-1)\alpha}f\right)(w)\right|\leq\frac{1}{\Gamma(\alpha)}\int_{a}^{w}(w-t)^{\alpha-1}\left|\left(D_{*a}^{N\alpha}f\right)(t)\right|dt$$

$$\leq\frac{1}{\Gamma(\alpha)}\left(\int_{a}^{w}(w-t)^{p(\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_{a}^{w}\left|\left(D_{*a}^{N\alpha}f\right)(t)\right|^{q}dt\right)^{\frac{1}{q}}= \tag{61}$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{(w-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\right)\left(\int_{a}^{w}\left|\left(D_{*a}^{N\alpha}f\right)(t)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Call
$$z(w) := \int_a^w \left|\left(D_{*a}^{N\alpha} f\right)(t)\right|^q dt, \ \ z(a) = 0, \tag{62}$$
$a \leq w \leq x$.

Then
$$z'(w) := \left|\left(D_{*a}^{N\alpha} f\right)(w)\right|^q, \tag{63}$$
and
$$\left|\left(D_{*a}^{N\alpha} f\right)(w)\right| = \left(z'(w)\right)^{\frac{1}{q}}, \ \ \text{all } a \leq w \leq x. \tag{64}$$
Therefore we have (all $a \leq w \leq x$)
$$\left|\left(D_{*a}^{(N-1)\alpha} f\right)(w)\right| \left|\left(D_{*a}^{N\alpha} f\right)(w)\right| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{(w-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\right) \left(z(w) z'(w)\right)^{\frac{1}{q}}. \tag{65}$$
Hence it holds
$$\int_a^x \left|\left(D_{*a}^{(N-1)\alpha} f\right)(w)\right| \left|\left(D_{*a}^{N\alpha} f\right)(w)\right| dw \leq$$
$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_a^x (w-a)^{\frac{p(\alpha-1)+1}{p}} \left(z(w) z'(w)\right)^{\frac{1}{q}} dw \leq$$
(by Hölder's inequality)
$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x (w-a)^{p(\alpha-1)+1} dw\right)^{\frac{1}{p}} \left(\int_a^x z(w) z'(w) dw\right)^{\frac{1}{q}} =$$
$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \frac{(x-a)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}} \left(\int_a^x z(w) dz(w)\right)^{\frac{1}{q}} =$$
$$\frac{(x-a)^{\alpha+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \frac{z^{\frac{2}{q}}(x)}{2^{\frac{1}{q}}} = \tag{66}$$
$$\frac{(x-a)^{\alpha+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}} \Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^x \left|\left(D_{*a}^{N\alpha} f\right)(t)\right|^q dt\right)^{\frac{2}{q}}.$$
The claim is proved.

We continue with

Theorem 2. *Assume that* $\alpha > \frac{1}{q}$; $f(b) = 0$, $\left(D_{b-}^{i\alpha} f\right) \in C([a,b])$, $i = 1, \ldots, N$. *Then*
$$\int_x^b \left|\left(D_{b-}^{(N-1)\alpha} f\right)(w)\right| \left|\left(D_{b-}^{N\alpha} f\right)(w)\right| dw \leq$$
$$\frac{(b-x)^{\alpha+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}} \Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_x^b \left|\left(D_{b-}^{N\alpha} f\right)(w)\right|^q dw\right)^{\frac{2}{q}}, \tag{67}$$
$\forall\, x \in [a,b]$.

Proof. As similar to Theorem 1 is omitted. It is based on (24).

Next comes a left proportional Caputo fractional Opial type inequality.

Theorem 3. *Here $f(a)=0$ and ${}^{P}D^{i\alpha}_{*a}f \in C([a,b])$, for $i=1,\dots,N$. Then, for every $x\in[a,b]$, we have*

$$\int_a^x \left|\left({}^{P}D^{(N-1)\alpha}_{*a}f\right)(w)\right| \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(w)\right| e_0(x,w)\,d_\alpha w \le$$

$$\int_a^x \left(\left(\int_a^w e_0(w,s)\,d_\alpha s\right) e_0(x,w)\,d_\alpha w\right)^{\frac{1}{p}}$$

$$\left\{\frac{\left(\int_a^x \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(s)\right|^q e_0(x,s)\,d_\alpha s\right)^2}{2}+\right.$$

$$\left.\frac{1}{2}\left[\int_a^x \left(\int_a^w \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(s)\right|^q e_0(w,s)\,d_\alpha s\right) k_1(\alpha,w)\,e_0(x,w)\,d_\alpha w\right]\right\}^{\frac{1}{q}}. \quad (68)$$

Proof. By (48) we have

$${}^{P}_{L}I^{RL}_{a}\left({}^{P}D^{N\alpha}_{*a}\right)f = \left({}^{P}D^{(N-1)\alpha}_{*a}\right)f, \text{ on } (a,b]. \quad (69)$$

given that $f(a)=0$, under the assumption $\left(\left({}^{P}D^{i\alpha}_{*a}\right)f\right)\in C([a,b])$, $i=1,\dots,N$.

That is

$$\left({}^{P}D^{(N-1)\alpha}_{*a}f\right)(w) = \int_a^w \left({}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)\right)(s)\,e_0(w,s)\,d_\alpha s, \quad (70)$$

for all $a\lneqq w\le x\le b$.

Hence by Hölder's inequality we derive

$$\left|\left({}^{P}D^{(N-1)\alpha}_{*a}f\right)(w)\right| \le \int_a^w e_0(w,s)^{\frac{1}{p}} \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(s)\right| e_0(w,s)^{\frac{1}{q}}\,d_\alpha s \quad (71)$$

$$\le \left(\int_a^w e_0(w,s)\,d_\alpha s\right)^{\frac{1}{p}} \left(\int_a^w \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(s)\right|^q e_0(w,s)\,d_\alpha s\right)^{\frac{1}{q}}.$$

Call

$$z(w) := \int_a^w \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(s)\right|^q e_0(w,s)\,d_\alpha s, \quad z(a)=0, \quad (72)$$

$a\le w\le x$.

By (31) we have

$$D^\alpha z(w) = \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(w)\right|^q, \quad (73)$$

and

$$\left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(w)\right| = \left(D^\alpha z(w)\right)^{\frac{1}{q}}, \quad (74)$$

all $a\le w\le x$.

Therefore we have (all $a\lneqq w\le x$)

$$\left|\left({}^{P}D^{(N-1)\alpha}_{*a}f\right)(w)\right| \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(w)\right| e_0(x,w) \le$$

$$\left(\int_a^w e_0(w,s)\,d_\alpha s\right)^{\frac{1}{p}} \left(z(w)\,D^\alpha z(w)\right)^{\frac{1}{q}} e_0(x,w) =$$

$$\left(\left(\int_a^w e_0(w,s)\,d_\alpha s\right) e_0(x,w)\right)^{\frac{1}{p}} \left(z(w)\,D^\alpha z(w)\,e_0(x,w)\right)^{\frac{1}{q}}. \quad (75)$$

Next, we apply again Hölder's inequality and finally we use (35) to get that

$$\int_a^x \left|\left({}^{P}D^{(N-1)\alpha}_{*a}f\right)(w)\right| \left|{}_{L}D^{1-\alpha}_{a}\left({}^{P}D^{N\alpha}_{*a}f\right)(w)\right| e_0(x,w)\,d_\alpha w \le$$

$$\int_a^x \left(\left(\int_a^w e_0(w,s)\,d_\alpha s\right) e_0(x,w)\right)^{\frac{1}{p}} \left(z(w)\,D^\alpha z(w)\,e_0(x,w)\right)^{\frac{1}{q}} d_\alpha w \le$$

$$\left(\int_a^x \left(\left(\int_a^w e_0(w,s)\, d_\alpha s\right) e_0(x,w)\right) d_\alpha w\right)^{\frac{1}{p}}$$

$$\left(\int_a^x z(w)\, D^\alpha z(w)\, e_0(x,w)\, d_\alpha w\right)^{\frac{1}{q}} =: (\xi). \tag{76}$$

By (35), we derive that

$$\int_a^x z(w)\, D^\alpha z(w)\, e_0(x,w)\, d_\alpha w =$$

$$\frac{z^2(x)}{2} + \frac{1}{2}\int_a^x z^2(w)\, k_1(\alpha,w)\, e_0(x,w)\, d_\alpha w = \tag{77}$$

$$\frac{\left(\int_a^x \left|{}_L D_a^{1-\alpha}\left({}^P D_{*a}^{N\alpha} f\right)(s)\right|^q e_0(x,s)\, d_\alpha s\right)^2}{2} +$$

$$\frac{1}{2}\left[\int_a^x \left(\int_a^w \left|{}_L D_a^{1-\alpha}\left({}^P D_{*a}^{N\alpha} f\right)(s)\right|^q e_0(w,s)\, d_\alpha s\right) k_1(\alpha,w)\, e_0(x,w)\, d_\alpha w\right].$$

Consequently, we obtain that

$$(\xi) = \left(\int_a^x \left(\int_a^w e_0(w,s)\, d_\alpha s\right) e_0(x,w)\, d_\alpha w\right)^{\frac{1}{p}}$$

$$\left\{\frac{\left(\int_a^x \left|{}_L D_a^{1-\alpha}\left({}^P D_{*a}^{N\alpha} f\right)(s)\right|^q e_0(x,s)\, d_\alpha s\right)^2}{2} +\right.$$

$$\left.\frac{1}{2}\left[\int_a^x \left(\int_a^w \left|{}_L D_a^{1-\alpha}\left({}^P D_{*a}^{N\alpha} f\right)(s)\right|^q e_0(w,s)\, d_\alpha s\right) k_1(\alpha,w)\, e_0(x,w)\, d_\alpha w\right]\right\}^{\frac{1}{q}}. \tag{78}$$

The claim is proved.

We continue with a left Poincaré type inequality.

Theorem 4. *Here $\alpha \in (0,1)$, $f \in C^1([a,b])$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $N \in \mathbb{N}$. Assume that $\alpha > \frac{1}{q}$; $f(a) = 0$, $\left(D_{*a}^{i\alpha} f\right) \in C([a,b])$, $i = 1,\dots,N$. Then*

$$\left\|D_{*a}^{(N-1)\alpha} f\right\|_{L_q([a,b])} \le \frac{(b-a)^\alpha}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}\,(q\alpha)^{\frac{1}{q}}} \left\|D_{*a}^{N\alpha} f\right\|_{L_q([a,b])}. \tag{79}$$

Proof. As in (61) we have

$$\left|\left(D_{*a}^{(N-1)\alpha} f\right)(x)\right| \le \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x \left|\left(D_{*a}^{N\alpha} f\right)(t)\right|^q dt\right)^{\frac{1}{q}} \le \tag{80}$$

$$\frac{(x-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{*a}^{N\alpha} f\right\|_{L_q([a,b])},$$

$\forall\, x \in [a,b]$.

Hence it holds

$$\left|\left(D_{*a}^{(N-1)\alpha} f\right)(x)\right|^q \le \frac{(x-a)^{q\alpha-1}}{\Gamma(\alpha)^q\,(p(\alpha-1)+1)^{\frac{q}{p}}} \left\|D_{*a}^{N\alpha} f\right\|_{L_q([a,b])}^q, \tag{81}$$

and

$$\int_a^b \left|\left(D_{*a}^{(N-1)\alpha} f\right)(x)\right|^q dx \le \frac{(b-a)^{q\alpha}}{\Gamma(\alpha)^q\,(p(\alpha-1)+1)^{\frac{q}{p}}\, q\alpha} \left\|D_{*a}^{N\alpha} f\right\|_{L_q([a,b])}^q. \tag{82}$$

The last results into

$$\left(\int_a^b \left|\left(D_{*a}^{(N-1)\alpha} f\right)(x)\right|^q dx\right)^{\frac{1}{q}} \le \frac{(b-a)^\alpha}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}\,(q\alpha)^{\frac{1}{q}}} \left\|D_{*a}^{N\alpha} f\right\|_{L_q([a,b])}. \tag{83}$$

The claim is proved.

Next comes a right Poincaré type inequality.

Theorem 5. *Again* $\alpha \in (0,1)$, $f \in C^1([a,b])$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $N \in \mathbb{N}$. *Assume that* $\alpha > \frac{1}{q}$; $f(b) = 0$, $\left(D_{b-}^{i\alpha} f\right) \in C([a,b])$, $i = 1,\ldots,N$. *Then*

$$\left\| D_{b-}^{(N-1)\alpha} f \right\|_{L_q([a,b])} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}\,(q\alpha)^{\frac{1}{q}}} \left\| D_{b-}^{N\alpha} f \right\|_{L_q([a,b])}. \tag{84}$$

Proof. As similar to Theorem 4 is omittted. It is based on (24).

Next comes a left Sobolev type inequality.

Theorem 6. *Here all as in Theorem 4*, $r > 0$. *Then*

$$\left\| D_{*a}^{(N-1)\alpha} f \right\|_{L_r([a,b])} \leq \frac{(b-a)^{\alpha-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left\| D_{*a}^{N\alpha} f \right\|_{L_q([a,b])}. \tag{85}$$

Proof. From (80) we get ($r > 0$)

$$\left| \left(D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r \leq \frac{(x-a)^{r\left(\alpha-\frac{1}{q}\right)}}{\Gamma(\alpha)^r\,(p(\alpha-1)+1)^{\frac{r}{p}}} \left\| D_{*a}^{N\alpha} f \right\|_{L_q([a,b])}^r, \tag{86}$$

$\forall\, x \in [a,b]$.

Hence it holds

$$\int_a^b \left| \left(D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r dx \leq \frac{(b-a)^{r\left(\alpha-\frac{1}{q}\right)+1} \left\| D_{*a}^{N\alpha} f \right\|_{L_q([a,b])}^r}{\Gamma(\alpha)^r\,(p(\alpha-1)+1)^{\frac{r}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)}. \tag{87}$$

That is

$$\left(\int_a^b \left| \left(D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r dx \right)^{\frac{1}{r}} \leq$$

$$\frac{(b-a)^{\alpha-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left\| D_{*a}^{N\alpha} f \right\|_{L_q([a,b])}. \tag{88}$$

The claim is proved.

It follows the right Sobolev inequality.

Theorem 7. *Here all as in Theorem 5*, $r > 0$. *Then*

$$\left\| D_{b-}^{(N-1)\alpha} f \right\|_{L_r([a,b])} \leq \frac{(b-a)^{\alpha-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left\| D_{b-}^{N\alpha} f \right\|_{L_q([a,b])}. \tag{89}$$

Proof. As similar to Theorem 6 is omitted.

Next comes a left proportional Caputo fractional Poincaré type inequality.

Theorem 8. *Here* $\alpha \in (0,1)$, $f \in C^1([a,b])$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $N \in \mathbb{N}$. *We assume* $f(a) = 0$, ${}^P D_{*a}^{i\alpha} f \in C([a,b])$, $i = 1,\ldots,N$. *Then*

$$\left\| {}^P D_{*a}^{(N-1)\alpha} f \right\|_{L_q([a,b],d_\alpha x)} \leq$$

$$\left(\int_a^b \left(\int_a^x e_0(x,s)\, d_\alpha s \right)^{\frac{q}{p}} d_\alpha x \right)^{\frac{1}{q}} \left\| {}_L D_a^{1-\alpha} \left({}^P D_{*a}^{N\alpha} f \right) \right\|_{L_q([a,b],d_\alpha x)}. \tag{90}$$

*Proof.*As in (71) we have

$$\left|\left({}^{P}D_{*a}^{(N-1)\alpha}f\right)(x)\right| \leq$$

$$\left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{1}{p}}\left(\int_a^x \left|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)(s)\right|^q e_0(x,s)\,d_\alpha s\right)^{\frac{1}{q}} \leq$$

$$\left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{1}{p}}\left(\int_a^x \left|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)(s)\right|^q d_\alpha s\right)^{\frac{1}{q}} \leq$$

$$\left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \left|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)(s)\right|^q d_\alpha s\right)^{\frac{1}{q}} =$$

$$\left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{1}{p}}\left\|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)\right\|_{L_q([a,b],d_\alpha s)}. \tag{91}$$

That is

$$\left|\left({}^{P}D_{*a}^{(N-1)\alpha}f\right)(x)\right| \leq \left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{1}{p}}\left\|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)\right\|_{L_q([a,b],d_\alpha s)}, \tag{92}$$

$\forall\, x \in (a,b]$.

Hence it holds

$$\left|\left({}^{P}D_{*a}^{(N-1)\alpha}f\right)(x)\right|^q \leq \left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{q}{p}}\left\|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)\right\|_{L_q([a,b],d_\alpha s)}^q, \tag{93}$$

and

$$\int_a^b \left|\left({}^{P}D_{*a}^{(N-1)\alpha}f\right)(x)\right|^q d_\alpha x \leq$$

$$\int_a^b \left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{q}{p}} d_\alpha x \left\|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)\right\|_{L_q([a,b],d_\alpha s)}^q. \tag{94}$$

We derive

$$\left(\int_a^b \left|\left({}^{P}D_{*a}^{(N-1)\alpha}f\right)(x)\right|^q d_\alpha x\right)^{\frac{1}{q}} \leq$$

$$\left(\int_a^b \left(\int_a^x e_0(x,s)\,d_\alpha s\right)^{\frac{q}{p}} d_\alpha x\right)^{\frac{1}{q}} \left\|{}_LD_a^{1-\alpha}\left({}^PD_{*a}^{N\alpha}f\right)\right\|_{L_q([a,b],d_\alpha s)},$$

proving the claim.

It follows a right proportional Caputo fractional Poincaré type inequality.

Theorem 9.*Here* $\alpha \in (0,1)$, $f \in C^1([a,b])$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $f(b) = 0$. *Then*

$$\left\|f e_0(b,\cdot)\right\|_{L_q([a,b],d_\alpha t)} \leq$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s)\,d_\alpha s\right)^{\frac{q}{p}} d_\alpha x\right)^{\frac{1}{q}} \left\|{}_RD_{b-}^{1-\alpha}\left({}^PD_{b-}^{\alpha}f\right)\right\|_{L_q([a,b],e_0(b,\cdot)d_\alpha t)}. \tag{95}$$

*Proof.*By (57), when $f(b) = 0$, we have

$$f(t)\,e_0(b,t) = {}_R^PI_b^{RL}\left({}^PD_{b-}^{\alpha}(f(t))\right) = \tag{96}$$

$$\int_t^b \left({}_RD_{b-}^{1-\alpha}\left({}^PD_{b-}^{\alpha}f\right)\right)(s)\,e_0(b,s)\,d_\alpha s,$$

$\forall\, t \in [a,b]$.

Hence by Hölder's inequality:

$$|f(t)|e_0(b,t) \le \int_t^b e_0(b,s)^{\frac{1}{p}} \left| \left({}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right)(s) \right| e_0(b,s)^{\frac{1}{q}} d_\alpha s \le \tag{97}$$

$$\left(\int_t^b e_0(b,s) d_\alpha s \right)^{\frac{1}{p}} \left(\int_t^b \left| \left({}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right)(s) \right|^q e_0(b,s) d_\alpha s \right)^{\frac{1}{q}} \le$$

$$\left(\int_t^b e_0(b,s) d_\alpha s \right)^{\frac{1}{p}} \left\| {}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right\|_{L_q([a,b], e_0(b,\cdot) d_\alpha t)}, \tag{98}$$

$\forall\, t \in [a,b]$.

Hence

$$(|f(t)|e_0(b,t))^q \le \left(\int_t^b e_0(b,s) d_\alpha s \right)^{\frac{q}{p}} \left\| {}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right\|_{L_q([a,b], e_0(b,\cdot) d_\alpha s)}^q, \tag{99}$$

and

$$\int_a^b (|f(t)|e_0(b,t))^q d_\alpha t \le$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s) d_\alpha s \right)^{\frac{q}{p}} d_\alpha t \right) \left\| {}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right\|_{L_q([a,b], e_0(b,\cdot) d_\alpha t)}^q. \tag{100}$$

We obtain

$$\left(\int_a^b (|f(t)|e_0(b,t))^q d_\alpha t \right)^{\frac{1}{q}} \le$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s) d_\alpha s \right)^{\frac{q}{p}} d_\alpha t \right)^{\frac{1}{q}} \left\| {}_R D_{b-}^{1-\alpha} \left({}^P D_{b-}^{\alpha} f \right) \right\|_{L_q([a,b], e_0(b,\cdot) d_\alpha t)}. \tag{101}$$

The claim is proved.

A left proportional fractional Sobolev type inequality follows:

Theorem 10. *All as in Theorem 8, $r > 0$. Then*

$$\left\| {}^P D_{*a}^{(N-1)\alpha} f \right\|_{L_r([a,b])} \le$$

$$\left(\int_a^b \left(\int_a^x e_0(x,s) d_\alpha s \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \left\| {}_L D_a^{1-\alpha} \left({}^P D_{*a}^{N\alpha} f \right) \right\|_{L_q([a,b], d_\alpha s)}. \tag{102}$$

Proof. By (92) we get

$$\left| \left({}^P D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r \le \left(\int_a^x e_0(x,s) d_\alpha s \right)^{\frac{r}{p}} \left\| {}_L D_a^{1-\alpha} \left({}^P D_{*a}^{N\alpha} f \right) \right\|_{L_q([a,b], d_\alpha s)}^r, \tag{103}$$

$\forall\, x \in (a,b]$.

Hence it holds

$$\int_a^b \left| \left({}^P D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r dx \le$$

$$\left(\int_a^b \left(\int_a^x e_0(x,s) d_\alpha s \right)^{\frac{r}{p}} dx \right) \left\| {}_L D_a^{1-\alpha} \left({}^P D_{*a}^{N\alpha} f \right) \right\|_{L_q([a,b], d_\alpha s)}^r. \tag{104}$$

That is

$$\left(\int_a^b \left| \left({}^P D_{*a}^{(N-1)\alpha} f \right)(x) \right|^r dx \right)^{\frac{1}{r}} \le$$

$$\left(\int_a^b \left(\int_a^x e_0(x,s)\, d_\alpha s\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}} \left\| {}_L D_a^{1-\alpha}\left({}^P D_{*a}^{N\alpha} f\right)\right\|_{L_q([a,b],d_\alpha s)}. \tag{105}$$

The claim is established.

A right proportional fractional Sobolev type inequality follows:

Theorem 11. *All as in Theorem 9, $r > 0$. Then*

$$\|f e_0(b,\cdot)\|_{L_r([a,b])} \leq$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s)\, d_\alpha s\right)^{\frac{r}{p}} dt\right)^{\frac{1}{r}} \left\| {}_R D_{b-}^{1-\alpha}\left({}^P D_{b-}^{\alpha} f\right)\right\|_{L_q([a,b],e_0(b,\cdot)d_\alpha s)}. \tag{106}$$

Proof. From (98) we have

$$\left(|f(t)|\, e_0(b,t)\right)^r \leq \left(\int_t^b e_0(b,s)\, d_\alpha s\right)^{\frac{r}{p}} \left\| {}_R D_{b-}^{1-\alpha}\left({}^P D_{b-}^{\alpha} f\right)\right\|^r_{L_q([a,b],e_0(b,\cdot)d_\alpha s)}, \tag{107}$$

$\forall\, t \in [a,b]$.

Hence it holds

$$\int_a^b \left(|f(t)|\, e_0(b,t)\right)^r dt \leq$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s)\, d_\alpha s\right)^{\frac{r}{p}} dt\right) \left\| {}_R D_{b-}^{1-\alpha}\left({}^P D_{b-}^{\alpha} f\right)\right\|^r_{L_q([a,b],e_0(b,\cdot)d_\alpha s)}. \tag{108}$$

That is

$$\left(\int_a^b \left(|f(t)|\, e_0(b,t)\right)^r dt\right)^{\frac{1}{r}} \leq$$

$$\left(\int_a^b \left(\int_t^b e_0(b,s)\, d_\alpha s\right)^{\frac{r}{p}} dt\right)^{\frac{1}{r}} \left\| {}_R D_{b-}^{1-\alpha}\left({}^P D_{b-}^{\alpha} f\right)\right\|_{L_q([a,b],e_0(b,\cdot)d_\alpha s)}. \tag{109}$$

Next we proceed with a left fractional Hilbert-Pachpatte type inequality.

Theorem 12. *Here $j = 1,2$; $\alpha_j \in (0,1)$, $f_j \in C^1([a_j,b_j])$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $N_j \in \mathbb{N}$. Assume $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$; $f_j(a_j) = 0$, $\left(D_{*a_j}^{i_j\alpha_j} f_j\right) \in C([a_j,b_j])$, $i_j = 1,\dots,N_j$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\left|\left(D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right|\left|\left(D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| dx_1 dx_2}{\left[\frac{(x_1-a_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(x_2-a_2)^{q(x_2-1)+1}}{q(q(\alpha_2-1)+1)}\right]} \leq \tag{110}$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\|D_{*a_1}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])} \left\|D_{*a_2}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])}.$$

Proof. We have by (23) that $(j = 1,2)$

$$\left(D_{*a_j}^{(N_j-1)\alpha_j} f_j\right)(x_j) = \frac{1}{\Gamma(\alpha_j)}\int_{a_j}^{x_j} (x_j - t_j)^{\alpha_j - 1}\left(D_{*a_j}^{N_j\alpha_j} f_j\right)(t_j)\, dt_j, \tag{111}$$

$\forall\, x_j \in [a_j,b_j]$.

Then

$$\left|\left(D_{*a_j}^{(N_j-1)\alpha_j} f_j\right)(x_j)\right| \leq \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} (x_j - t_j)^{\alpha_j - 1} \left|\left(D_{*a_j}^{N_j\alpha_j} f_j\right)(t_j)\right| dt_j, \tag{112}$$

$j = 1,2, \forall x_j \in [a_j, b_j]$.

We get as before

$$\left|\left(D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \leq \frac{1}{\Gamma(\alpha_1)} \frac{(x_1 - a_1)^{\frac{p(\alpha_1-1)+1}{p}}}{(p(\alpha_1 - 1) + 1)^{\frac{1}{p}}} \left\|D_{*a_1}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])}, \tag{113}$$

and

$$\left|\left(D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| \leq \frac{1}{\Gamma(\alpha_2)} \frac{(x_2 - a_2)^{\frac{q(\alpha_2-1)+1}{q}}}{(q(\alpha_2 - 1) + 1)^{\frac{1}{q}}} \left\|D_{*a_2}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])}. \tag{114}$$

Hence we have

$$\left|\left(D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left(D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| \leq$$

$$\frac{(x_1 - a_1)^{\frac{p(\alpha_1-1)+1}{p}} (x_2 - a_2)^{\frac{q(\alpha_2-1)+1}{q}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)(p(\alpha_1 - 1) + 1)^{\frac{1}{p}} (q(\alpha_2 - 1) + 1)^{\frac{1}{q}}}$$

$$\left\|D_{*a_1}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])} \left\|D_{*a_2}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])} \tag{115}$$

(using Young's inequality for $a^*, b^* \geq 0$, $a^{*\frac{1}{p}} b^{*\frac{1}{q}} \leq \frac{a^*}{p} + \frac{b^*}{q}$)

$$\leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left(\frac{(x_1 - a_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1 - 1) + 1)} + \frac{(x_2 - a_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2 - 1) + 1)}\right)$$

$$\left\|D_{*a_1}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])} \left\|D_{*a_2}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])}, \tag{116}$$

$\forall x_j \in [a_j, b_j]$; $j = 1,2$.

So far we have

$$\frac{\left|\left(D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left(D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right|}{\left[\frac{(x_1-a_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(x_2-a_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right]} \leq$$

$$\frac{\left\|D_{*a_1}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])} \left\|D_{*a_2}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}, \tag{117}$$

$\forall x_j \in [a_j, b_j]$; $j = 1,2$.

The denominator in (117) can be zero only when both $x_1 = a_1$ and $x_2 = a_2$. Therefore we obtain (110), by integrating (117) over $[a_1, b_1] \times [a_2, b_2]$.

The right fractional Hilbert-Pachpatte type inequality follows:

Theorem 13. *Here* $j = 1,2$; $\alpha_j \in (0,1)$, $f_j \in C^1([a_j, b_j])$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $N_j \in \mathbb{N}$. *Assume* $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$; $f_j(b_j) = 0$, $\left(D_{b_j-}^{i_j\alpha_j} f_j\right) \in C([a_j, b_j])$, $i_j = 1, ..., N_j$. *Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left|\left(D_{b_1-}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left(D_{b_2-}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| dx_1 dx_2}{\left[\frac{(b_1-x_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(b_2-x_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right]} \leq \tag{118}$$

$$\frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\|D_{b_1-}^{N_1\alpha_1} f_1\right\|_{L_q([a_1,b_1])} \left\|D_{b_2-}^{N_2\alpha_2} f_2\right\|_{L_p([a_2,b_2])}.$$

Proof. As similar to Theorem 12 is omitted. It is based on (24).

Next we continue with a left proportional fractional Hilbert-Pachpatte inequality.

Theorem 14. *Here it is $j = 1,2$. Let $f_j \in C^1([a_j, b_j])$, $\alpha_j \in (0,1)$, $f_j(a_j) = 0$; and $\left({}^P D_{*a_j}^{i_j \alpha_j} f_j\right) \in C([a_j, b_j])$, for $i_j = 1, \ldots, N_j \in \mathbb{N}$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left|\left({}^P D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left({}^P D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} e_0(x_1,s_1) d_{\alpha_1} s_1}{p} + \frac{\int_{a_2}^{x_2} e_0(x_2,s_2) d_{\alpha_2} s_2}{q}\right]} \leq$$
$$(b_1 - a_1)(b_2 - a_2) \left\| {}_L D_{a_1}^{1-\alpha_1} \left({}^P D_{*a_1}^{N_1 \alpha_1} f_1\right) \right\|_{L_q\left([a_1,b_1], d_{\alpha_1} s_1\right)}$$
$$\left\| {}_L D_{a_2}^{1-\alpha_2} \left({}^P D_{*a_2}^{N_2 \alpha_2} f_2\right) \right\|_{L_p\left([a_2,b_2], d_{\alpha_2} s_2\right)}. \tag{119}$$

Proof. Based on (48) ($j = 1,2$) and as in (92) we have

$$\left|\left({}^P D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \leq$$
$$\left(\int_{a_1}^{x_1} e_0(x_1,s_1) d_{\alpha_1} s_1\right)^{\frac{1}{p}} \left\| {}_L D_{a_1}^{1-\alpha_1} \left({}^P D_{*a_1}^{N_1 \alpha_1} f_1\right) \right\|_{L_q\left([a_1,b_1], d_{\alpha_1} s_1\right)}, \tag{120}$$

$\forall\, x_1 \in (a_1, b_1]$,

and

$$\left|\left({}^P D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| \leq$$
$$\left(\int_{a_2}^{x_2} e_0(x_2,s_2) d_{\alpha_2} s_2\right)^{\frac{1}{q}} \left\| {}_L D_{a_2}^{1-\alpha_2} \left({}^P D_{*a_2}^{N_2 \alpha_2} f_2\right) \right\|_{L_p\left([a_2,b_2], d_{\alpha_2} s_2\right)}, \tag{121}$$

$\forall\, x_2 \in (a_2, b_2]$.

Therefore we have

$$\left|\left({}^P D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left({}^P D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right| \leq$$
$$\left[\left(\int_{a_1}^{x_1} e_0(x_1,s_1) d_{\alpha_1} s_1\right)^{\frac{1}{p}} \left(\int_{a_2}^{x_2} e_0(x_2,s_2) d_{\alpha_2} s_2\right)^{\frac{1}{q}}\right]$$
$$\left\| {}_L D_{a_1}^{1-\alpha_1} \left({}^P D_{*a_1}^{N_1 \alpha_1} f_1\right) \right\|_{L_q\left([a_1,b_1], d_{\alpha_1} s_1\right)} \left\| {}_L D_{a_2}^{1-\alpha_2} \left({}^P D_{*a_2}^{N_2 \alpha_2} f_2\right) \right\|_{L_p\left([a_2,b_2], d_{\alpha_2} s_2\right)} \leq \tag{122}$$
$$\left[\frac{\int_{a_1}^{x_1} e_0(x_1,s_1) d_{\alpha_1} s_1}{p} + \frac{\int_{a_2}^{x_2} e_0(x_2,s_2) d_{\alpha_2} s_2}{q}\right]$$
$$\left\| {}_L D_{a_1}^{1-\alpha_1} \left({}^P D_{*a_1}^{N_1 \alpha_1} f_1\right) \right\|_{L_q\left([a_1,b_1], d_{\alpha_1} s_1\right)} \left\| {}_L D_{a_2}^{1-\alpha_2} \left({}^P D_{*a_2}^{N_2 \alpha_2} f_2\right) \right\|_{L_p\left([a_2,b_2], d_{\alpha_2} s_2\right)}.$$

Consequently it holds

$$\frac{\left|\left({}^P D_{*a_1}^{(N_1-1)\alpha_1} f_1\right)(x_1)\right| \left|\left({}^P D_{*a_2}^{(N_2-1)\alpha_2} f_2\right)(x_2)\right|}{\left[\frac{\int_{a_1}^{x_1} e_0(x_1,s_1) d_{\alpha_1} s_1}{p} + \frac{\int_{a_2}^{x_2} e_0(x_2,s_2) d_{\alpha_2} s_2}{q}\right]} \leq$$
$$\left\| {}_L D_{a_1}^{1-\alpha_1} \left({}^P D_{*a_1}^{N_1 \alpha_1} f_1\right) \right\|_{L_q\left([a_1,b_1], d_{\alpha_1} s_1\right)} \left\| {}_L D_{a_2}^{1-\alpha_2} \left({}^P D_{*a_2}^{N_2 \alpha_2} f_2\right) \right\|_{L_p\left([a_2,b_2], d_{\alpha_2} s_2\right)}. \tag{123}$$

The denominator of (123) is zero only when both $a_1 = x_1$ and $a_2 = x_2$.

Inequality (119) comes after integrating (123) over $[a_1, b_1] \times [a_2, b_2]$.

We finish with the right proportional fractional Hilbert-Pachpatte inequality.

Theorem 15. *Let $j = 1,2$, and $\alpha_j \in (0,1)$, $f_j \in C^1([a_j,b_j])$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $f_j(b_j) = 0$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|e_0(b_1,t_1)e_0(b_2,t_2)dt_1dt_2}{\left[\frac{\int_{t_1}^{b_1} e_0(b_1,s_1)d_{\alpha_1}s_1}{p}+\frac{\int_{t_2}^{b_2} e_0(b_2,s_2)d_{\alpha_2}s_2}{q}\right]} \leq \tag{124}$$

$$(b_1-a_1)(b_2-a_2)\left\|{}_RD_{b_1-}^{1-\alpha_1}\left({}^PD_{b_1-}^{\alpha_1}f_1\right)\right\|_{L_q\left([a_1,b_1],e_0(b_1,\cdot)d_{\alpha_1}s_1\right)}$$

$$\left\|{}_RD_{b_2-}^{1-\alpha_2}\left({}^PD_{b_2-}^{\alpha_2}f_2\right)\right\|_{L_q\left([a_2,b_2],e_0(b_2,\cdot)d_{\alpha_2}s_2\right)}.$$

Proof. Similar to (98), based on (57), we derive

$$|f_1(t_1)|e_0(b_1,t_1) \leq$$

$$\left(\int_{t_1}^{b_1} e_0(b_1,s_1)d_{\alpha_1}s_1\right)^{\frac{1}{p}}\left\|{}_RD_{b_1-}^{1-\alpha_1}\left({}^PD_{b_1-}^{\alpha_1}f_1\right)\right\|_{L_q\left([a_1,b_1],e_0(b_1,\cdot)d_{\alpha_1}s_1\right)}, \tag{125}$$

and

$$|f_2(t_2)|e_0(b_2,t_2) \leq$$

$$\left(\int_{t_2}^{b_2} e_0(b_2,s_2)d_{\alpha_2}s_2\right)^{\frac{1}{q}}\left\|{}_RD_{b_2-}^{1-\alpha_2}\left({}^PD_{b_2-}^{\alpha_2}f_2\right)\right\|_{L_q\left([a_2,b_2],e_0(b_2,\cdot)d_{\alpha_2}s_2\right)}. \tag{126}$$

Hence it holds

$$|f_1(t_1)||f_2(t_2)|e_0(b_1,t_1)e_0(b_2,t_2) \leq$$

$$\left(\int_{t_1}^{b_1} e_0(b_1,s_1)d_{\alpha_1}s_1\right)^{\frac{1}{p}}\left(\int_{t_2}^{b_2} e_0(b_2,s_2)d_{\alpha_2}s_2\right)^{\frac{1}{q}}$$

$$\left\|{}_RD_{b_1-}^{1-\alpha_1}\left({}^PD_{b_1-}^{\alpha_1}f_1\right)\right\|_{L_q\left([a_1,b_1],e_0(b_1,\cdot)d_{\alpha_1}s_1\right)}$$

$$\left\|{}_RD_{b_2-}^{1-\alpha_2}\left({}^PD_{b_2-}^{\alpha_2}f_2\right)\right\|_{L_q\left([a_2,b_2],e_0(b_2,\cdot)d_{\alpha_2}s_2\right)} \leq \tag{127}$$

$$\left[\frac{\int_{t_1}^{b_1} e_0(b_1,s_1)d_{\alpha_1}s_1}{p}+\frac{\int_{t_2}^{b_2} e_0(b_2,s_2)d_{\alpha_2}s_2}{q}\right]$$

$$\left\|{}_RD_{b_1-}^{1-\alpha_1}\left({}^PD_{b_1-}^{\alpha_1}f_1\right)\right\|_{L_q\left([a_1,b_1],e_0(b_1,\cdot)d_{\alpha_1}s_1\right)}$$

$$\left\|{}_RD_{b_2-}^{1-\alpha_2}\left({}^PD_{b_2-}^{\alpha_2}f_2\right)\right\|_{L_q\left([a_2,b_2],e_0(b_2,\cdot)d_{\alpha_2}s_2\right)}. \tag{128}$$

Consequently we get

$$\frac{|f_1(t_1)||f_2(t_2)|e_0(b_1,t_1)e_0(b_2,t_2)}{\left[\frac{\int_{t_1}^{b_1} e_0(b_1,s_1)d_{\alpha_1}s_1}{p}+\frac{\int_{t_2}^{b_2} e_0(b_2,s_2)d_{\alpha_2}s_2}{q}\right]} \leq$$

$$\left\|{}_RD_{b_1-}^{1-\alpha_1}\left({}^PD_{b_1-}^{\alpha_1}f_1\right)\right\|_{L_q\left([a_1,b_1],e_0(b_1,\cdot)d_{\alpha_1}s_1\right)}$$

$$\left\|{}_RD_{b_2-}^{1-\alpha_2}\left({}^PD_{b_2-}^{\alpha_2}f_2\right)\right\|_{L_q\left([a_2,b_2],e_0(b_2,\cdot)d_{\alpha_2}s_2\right)}. \tag{129}$$

The denominator in (129) is zero only when both $t_1 = b_1$ and $t_2 = b_2$.

Inequality (124) is implied by integration of (129) over $[a_1,b_1]\times[a_2,b_2]$.

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