Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110209

Probabilistic Interpretation of the L and Prabhakar Integrals of Fractional Order

Marc Jornet[1] *and Juan J. Nieto*[2,*]

[1] Departament de Matemàtiques, Universitat de València, 46100 Burjassot, Spain
[2] CITMAga, Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

Received: 12 Sep. 2024, Revised: 12 Nov. 2024, Accepted: 13 Dec. 2024
Published online: 1 Apr. 2025

Abstract: We present a probabilistic interpretation of the L-fractional integration. This integral is the inverse of the known L-fractional derivative. We prove that the fractional integral can be expressed as an expected value of a random variable, which describes dilation or scaling and is related to the beta distribution. The proposed explanation gives the possibility of a generalization of non-integer-order integration and differentiation, by using continuous probability densities. In fact, the general Prabhakar integral operator can be given a probabilistic interpretation as well, in terms of an average, and thus obtain the Riemann-Liouville and L integrals as particular cases.

Keywords: Fractional calculus, L-fractional integral, dilation operator, probability density function, beta distribution, Prabhakar integral.

1 Introduction

Fractional calculus is a relevant area of research and applications. See, for example, the classical monograph [1] or [2,3,4]. To cite just an example, the classical fractional Riemann-Liouville operator was considered by Hardy and Littlewood [5] or Riesz [6].

The L-fractional derivative of an absolutely continuous function $x:[0,T]\to\mathbb{R}$, $x\in \mathrm{AC}[0,T]$, is [7,8]

$$ {}^{L}D^{\alpha}x(t)=\frac{{}^{C}D^{\alpha}x(t)}{{}^{C}D^{\alpha}t}, \tag{1}$$

where $t\in[0,T]$ is the time, $\alpha\in(0,1)$ is the fractional order of differentiation, and

$$ {}^{C}D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(\tau)}{(t-\tau)^{\alpha}}d\tau \tag{2}$$

is the Caputo fractional derivative of x with first-order derivative x', being Γ the gamma function. For a motivation of L-fractional models, we refer the reader to [9].

We will give an explicit definition of the L-fractional integral as an inverse of the L-fractional derivative. We will show that the L-fractional integral can be interpreted as an expected value of a random variable, related to scaling and having a beta distribution. We will present extensions concerning the Prabhakar fractional integral as well [10,11], that include existing fractional operators. Thus, our work provides a relation between fractional integral operators and probability theory. Essentially, we will see that, in general, a fractional integral is related to an expectation with respect to a random variable Ξ of density $\rho(\xi)$, i.e., $\Xi\sim\rho(\xi)d\xi$, having support in $(0,1)$.

* Corresponding author e-mail: juanjose.nieto.roig@usc.es

2 L-fractional integral

Let $\alpha \in (0,1)$, $T > 0$, and $x \in L^1(0,T)$ (i.e., Lebesgue integrable). We recall that the classical Riemann-Liouville fractional integral is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \tag{3}$$

The operator $I^\alpha : L^1(0,T) \to L^1(0,T)$ is linear and bounded with norm $\|I^\alpha\| \leq T^\alpha / \Gamma(\alpha+1)$. Moreover, it is injective and therefore it has a left inverse. For the Caputo fractional derivative (**??**), defined for $x \in \mathrm{AC}[0,T]$, we know that

$$[I^\alpha \circ {}^C D^\alpha] x(t) = x(t) + c, \quad c = -x(0),$$

for all $t \in [0,T]$. Also, if $x \in \mathrm{AC}[0,T]$, then $I^\alpha x \in \mathrm{AC}[0,T]$ (see property (6) of Proposition 3.2 in [12]) and

$$[{}^C D^\alpha \circ I^\alpha] x(t) = x(t),$$

for almost every $t \in [0,T]$. This means that ${}^C D^\alpha$ is a left inverse of I^α, or, in other words, I^α is a right inverse of ${}^C D^\alpha$.

In general, a fractional integral is of the form $I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t k(t,s) x(s) ds$, with k the kernel of the integral operator. Different kernels give different fractional integrals.

We now consider the L-fractional integral case of order $\alpha \in (0,1)$, as a right inverse of ${}^L D^\alpha$. It should be true that

$$[{}^L D^\alpha \circ {}^L I^\alpha] x(t) = x(t).$$

To define ${}^L I^\alpha$, set ${}^L I^\alpha x = y$, or ${}^L D^\alpha y(t) = x(t)$. Then,

$${}^C D^\alpha y(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} x(t) := \hat{x}(t).$$

Therefore,

$$I^\alpha \circ {}^C D^\alpha y(t) = I^\alpha \hat{x}(t),$$

or

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(2-\alpha)} s^{1-\alpha} x(s) ds.$$

This gives us the definition of the L-fractional integral,

$${}^L I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(2-\alpha)} s^{1-\alpha} x(s) ds. \tag{4}$$

The kernel is identified as

$${}^L I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t k_L(t,s) x(s) ds, \quad k_L(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(2-\alpha)} s^{1-\alpha}.$$

This expression defines a linear and bounded operator from $L^1(0,T)$ into itself. The kernel of the L-fractional integral has some properties of a density function [13], as will be seen. Observe that

$$[{}^L I^\alpha \circ {}^L D^\alpha] x(t) = I^\alpha \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \cdot {}^L D^\alpha x(t) \right) = [I^\alpha \circ {}^C D^\alpha] x(t) = x(t) + c.$$

For (**??**), notice that

$${}^C D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha},$$

for $\beta > 0$. In particular, ${}^C D^\alpha t = t^{1-\alpha}/\Gamma(2-\alpha)$, hence the L-fractional derivative (1) can be rewritten as

$${}^L D^\alpha x(t) = \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} {}^C D^\alpha x(t).$$

For power $\beta = 0$, ${}^C D^\alpha 1 = 0$, so that ${}^L D^\alpha 1 = 0$. That is, the fractional derivative of any constant is zero. This property is very important when dealing with initial states and power series [9]. On the other hand, ${}^L D^\alpha t = 1$ and, for a smooth x, ${}^L D^\alpha x(0) = x'(0) \in (-\infty, \infty)$. The units of ${}^L D^\alpha x(t)$ are time^{-1}, in contrast to time$^{-\alpha}$.

A related fractional calculus, of better use in differential geometry and topology, is the Λ-fractional calculus [14]. It is based on normalizing the Riemann-Liouville derivative, but it will not be treated in this contribution. On the other hand, the normalization of other fractional operators has been investigated in the literature, see [15,16].

3 Probabilistic interpretation of the L-fractional integral

Making the change $\xi = s/t$ in (4), one has

$$ {}^{L}I^{\alpha}x(t) = \frac{t}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 t^{\alpha-1}(1-\xi)^{\alpha-1}\xi^{1-\alpha}t^{1-\alpha}x(\xi t)d\xi = \frac{t}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 (1-\xi)^{\alpha-1}\xi^{1-\alpha}x(\xi t)d\xi. $$

Introducing the dilation operator

$$ [S_{\xi}x](t) = x(\xi t) $$

and the function

$$ \Theta(\xi) = \frac{(1-\xi)^{\alpha-1}\xi^{1-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)}, $$

we can write

$$ {}^{L}I^{\alpha}x(t) = \frac{t}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 (1-\xi)^{\alpha-1}\xi^{1-\alpha}[S_{\xi}x](t)d\xi = t\int_0^1 \Theta(\xi)[S_{\xi}x](t)d\xi. $$

We note that

$$ \int_0^1 \Theta(\xi)d\xi = 1 $$

and, in fact,

$$ \Theta(\xi)d\xi \sim B(2-\alpha,\alpha). $$

From these expressions, the operator ${}^{L}I^{\alpha}$ in (4) can be interpreted as an expected value of a random variable $\xi \in (0,1)$ related to the scaling and having a beta distribution:

$$ {}^{L}I^{\alpha}x(t) = t\,\mathbb{E}[x(\xi t)]. \tag{5} $$

When $\alpha \to 1^{-}$, the distribution is uniform on $[0,1]$ and the standard integral is obtained.

4 Prabhakar fractional integral and probabilistic interpretation

We recall some notations in order to define the Prabhakar fractional integral.

The three-parameter Mittag–Leffler function, introduced by Prabhakar in 1971 [10], is defined by

$$ E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \cdot \frac{z^n}{n!}, $$

where $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$ is the Pochhammer symbol and $\alpha,\beta,\gamma,z \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Notice that the classical two-parameter Mittag-Leffler function is retrieved for $\gamma = 1$, as well as the classical exponential function e^{z} for $\alpha = \beta = \gamma = 1$. For $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, the Prabhakar kernel is

$$ e^{\gamma}_{\alpha,\beta}(\omega;t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \cdot \frac{\omega^n t^{\alpha n+\beta-1}}{n!}. \tag{6} $$

We have the following values:

$$ E^{0}_{\alpha,\beta}(z) = 1/\Gamma(\beta), \quad e^{1}_{1,1}(\omega;t) = \mathrm{e}^{\omega t}, \quad e^{0}_{\alpha,\beta}(\omega;t) = e^{\gamma}_{\alpha,\beta}(0;t) = t^{\beta-1}/\Gamma(\beta). \tag{7} $$

For $x \in L^1(0,T)$, the Prabhakar fractional integral is defined as [10, 11]:

$$ \mathbb{P}^{\gamma}_{\alpha,\beta,\omega}x(t) = \int_0^t e^{\gamma}_{\alpha,\beta}(\omega;t-s)x(s)ds, \tag{8} $$

for $t \in [0,T]$. It is a linear and bounded operator from $L^1(0,T)$ into $L^1(0,T)$.

If $x \in L^1(0,T)$ is such that the convolution $x * e^{-\gamma}_{\alpha,1-\beta}(\omega;\cdot) \in \mathrm{AC}[0,T]$, then the Prabhakar fractional derivative in the Riemann-Liouville sense is defined by

$$\mathbb{D}^{\gamma}_{\alpha,\beta,\omega}x(t) = \frac{d}{dt}\mathbb{P}^{-\gamma}_{\alpha,1-\beta,\omega}x(t).$$

If $x \in \mathrm{AC}[0,T]$, the Prabhakar derivative in the sense of Liouville-Caputo is defined as

$$^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,\omega}x(t) = \mathbb{P}^{-\gamma}_{\alpha,1-\beta,\omega}x'(t).$$

By (7), we note that

$$\mathbb{P}^{0}_{\alpha,\alpha,1}x(t) = \int_0^t e^{0}_{\alpha,\alpha}(\omega,t-s)x(s)ds = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds = I^{\alpha}x(t), \tag{9}$$

and also,

$$\mathbb{P}^{\gamma}_{\alpha,\alpha,0}x(t) = \int_0^t e^{\gamma}_{\alpha,\alpha}(0,t-s)x(s)ds = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds = I^{\alpha}x(t).$$

In both cases, the classical Riemann-Liouville integral (3) of order $\alpha > 0$ is a particular case of the Prabhakar operator (8).

This Prabhakar operator also includes the L-fractional integral (4), with $\alpha \in (0,1)$. Indeed, we note that

$$^{L}I^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\mathbb{P}^{\gamma}_{\alpha,\alpha,0}[t^{1-\alpha}x(t)]$$

or

$$^{L}I^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\mathbb{P}^{0}_{\alpha,\alpha,1}[t^{1-\alpha}x(t)],$$

revealing the generality of the Prabhakar definition.

For $s \in [0,t]$, let $\xi = s/t$, so that

$$\mathbb{P}^{\gamma}_{\alpha,\beta,\omega}x(t) = \int_0^1 e^{\gamma}_{\alpha,\beta}(\omega;t(1-\xi))x(t\xi)td\xi = t^{\beta}\int_0^1 (1-\xi)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}(1-\xi)^{\alpha})x(t\xi)d\xi.$$

Now, with (6) and (8),

$$\begin{aligned} t^{\beta}\int_0^1 (1-\xi)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}(1-\xi)^{\alpha})d\xi = \mathbb{P}^{\gamma}_{\alpha,\beta,\omega}1 &= \int_0^t e^{\gamma}_{\alpha,\beta}(\omega;s)ds \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(n\alpha+\beta)}\cdot\frac{\omega^n\int_0^t s^{\alpha n+\beta-1}ds}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(n\alpha+\beta+1)}\cdot\frac{\omega^n t^{\alpha n+\beta}}{n!} \\ &= e^{\gamma}_{\alpha,\beta+1}(\omega;t). \end{aligned}$$

Therefore,

$$\frac{t^{\beta}(1-\xi)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}(1-\xi)^{\alpha})}{e^{\gamma}_{\alpha,\beta+1}(\omega;t)}$$

verifies

$$\int_0^1 \frac{t^{\beta}(1-\xi)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}(1-\xi)^{\alpha})}{e^{\gamma}_{\alpha,\beta+1}(\omega;t)}d\xi = 1,$$

and an analogous probabilistic interpretation is then possible for the Prabhakar fractional integral (8). We can write

$$\mathbb{P}^{\gamma}_{\alpha,\beta,\omega}x(t) = e^{\gamma}_{\alpha,\beta+1}(\omega;t)\mathbb{E}[x(t\,\Xi_t)],$$

where the random variable of support $(0,1)$ is distributed as

$$\Xi_t \sim \frac{t^{\beta}(1-\xi)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}(1-\xi)^{\alpha})}{e^{\gamma}_{\alpha,\beta+1}(\omega;t)}d\xi.$$

When $(\alpha,\alpha,\omega,\gamma)=(\alpha,\alpha,1,0)$ and $\alpha\in(0,\infty)$, that is, we are considering the Riemann-Liouville integral (9), we have the density

$$\Xi_t \sim \frac{t^\alpha(1-\xi)^{\alpha-1}E^0_{\alpha,\alpha}(t^\alpha(1-\xi)^\alpha)}{e^0_{\alpha,\alpha+1}(1;t)}d\xi = \frac{t^\alpha(1-\xi)^{\alpha-1}\frac{1}{\Gamma(\alpha)}}{\frac{t^\alpha}{\Gamma(\alpha+1)}}d\xi = \alpha(1-\xi)^{\alpha-1}d\xi \sim B(1,\alpha),$$

by (7), and

$$I^\alpha x(t) = e^0_{\alpha,\alpha+1}(1;t)\mathbb{E}[x(t\,\Xi_t)] = \frac{t^\alpha}{\Gamma(\alpha+1)}\mathbb{E}[x(t\,\Xi_t)]. \tag{10}$$

In this case, Ξ_t is actually time independent. In particular, for the L-fractional integral (4) and $\alpha\in(0,1)$,

$$^L I^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)}I^\alpha[t^{1-\alpha}x(t)] = \frac{1}{\Gamma(2-\alpha)}\frac{t^\alpha}{\Gamma(\alpha+1)}\mathbb{E}[t^{1-\alpha}\Xi_t^{1-\alpha}x(t\,\Xi_t)] = t\,\mathbb{E}[x(t\,\xi)],$$

where $\xi\sim B(2-\alpha,\alpha)$, and we retrieve (5). Observe that (10) contains the power t^α, whereas (5) only has t, which matches with the units of the corresponding differential operators, time$^{-\alpha}$ and time^{-1}, respectively.

5 Conclusions

We introduce the L-fractional integral.

We give a probabilistic explanation of the L-fractional integral operator, by means of dilation and the beta distribution.

The L-fractional integral is a particular case of the Prabhakar integral, as many other fractional operators. For this general Prabhakar integral, we derive a probabilistic interpretation as well, by means of a certain density function.

Funding

The research of J. J. Nieto was supported by the Agencia Estatal de Investigación (AEI) of Spain Grant PID2020-113275GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”, by the “European Union” and Xunta de Galicia, grant ED431C 2023/12 for Competitive Reference Research Groups (2023–2026).

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