Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110306

A Semi-Analytical Technique for Solving the Time-Fractional Kaup-Kupershmidt Equation

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Received: 15 Jan. 2025, Revised: 30 Mar. 2025, Accepted: 19 May 2025
Published online: 1 Jul. 2025

Abstract: In this study, an accurate solution for the time-fractional Kaup-Kupershmidt equation is provided. A semi-analytical method, the ARA- Residual power series method, is employed for creating a series solution to the given equation. The concepts of ARA transform, power series, residual function, and the limit at infinity are the main components of the proposed method. To verify the accuracy of the introduced method, numerical and graphical examples are given to compare our approximate solution with the exact one. We conclude that this approach is simple, effortless to use, and successful in solving fractional differential equations.

Keywords: Caputo's derivative, ARA transform, fractional power series, Kaup-Kupershmidt equation.

1 Introduction

Fractional Calculus (FC) goes back to the beginning of differential calculus theory. In reality, Leibniz had many notes on the fractional derivative in his correspondence with Bernoulli, L'Hopital, and Wallis in 1695. However, a number of mathematicians, including Letnikov, Riemann, Euler, and Liouville, have contributed to the creation of the theory of FC [1, 2, 3]. Nowadays, FC has been a productive area of research in science and engineering. Actually, a wide range of scientific fields are currently focusing on the concepts of FC. Examples of these include signal processing, electronics, heat transfer, system identification, traffic systems, biology, genetic algorithms, percolation, chemistry, irreversibility, physics, control systems, as well as economy and finance [4, 5, 6, 7, 8, 9, 10].

Solving fractional differential equations has received a lot of attention lately, particularly partial differential equations (PDEs) of fractional order, which are used to describe a wide range of physical events. As a result, many scholars have examined a variety of analytical and numerical techniques to determine the exact and approximate solutions of these problems. These techniques include the Adomian decomposition method (ADM) [11, 12], the Laplace transform method [13], the homotopy analysis method (HAM) [14], the variational iteration method (VIM) [15], the homotopy perturbation method (HPM) [16] and other techniques [17, 18, 19, 20].

Certain kinds of linear fractional differential equations can be solved using the power series method (PSM). The basic idea is that the solution to the equation may be expressed as a power series, which yields a closed solution form of the exact solution. This method is utilized to solve several types of linear fractional differential equations and has drawn the attention of multiple researchers [21, 22]. Regretfully, it is a very difficult task to determine the series coefficients needed to get the approximate solutions for nonlinear fractional differential equations. In order to handle those difficulties, the PSM is improved, and the residual power series method (RPSM) is created to consider the coefficients as transformed functions by using the derivatives in the calculation process [23]. In advancement, ARA transform [24] and RPSM are combined to create ARA residual power series method (ARA-RPSM), which is established in [25]. ARA-RPSM requires converting the provided equation into ARA space, creating a series solution for the new equation, and using ARA inverse of the result to get the solution to the main problem. By utilizing the idea of limits, ARA-RPSM establishes the unknown coefficients in a new ARA expansion as opposed to the RPSM, which depends on derivatives. The significant dispersive classical Kaup-Kupershmidt (KK) equation was initially presented by Kaup in 1980 [26], and Kupershmidt modified it in

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1994 [27]. The study of nonlinear dispersive waves and the behavior of capillary gravity waves are investigated using the time-fractional KK equation. The comprehensive fifth-order nonlinear evolution formula is provided by

$$D_t^\alpha u(x,t)+\xi uu_{xxx}+\rho u_x u_{xx}+\sigma u^2 u_x+u_{xxxxx}=0, \tag{1}$$

where ξ, ρ, and σ are real constants, and $0<\alpha\ \leq 1$ is the parameter symbolizing the order of fractional time derivative. Recently, Extensive effort has been focused to the analysis of the classical KK equation. It is known that the classical KK equation has bilinear forms [28] and that it is integrable at $p\ =\ 5/2$ [29]. For general nonlinear evolution equations, solitary and soliton wave solutions can be obtained by importing four different techniques. Ablowitz and Clarkson employed the inverse scattering approach to create soliton solutions for the purpose of examining the nonlinear equations and their physical implications [30]. Tam and Hu employed Hirota's technique, and they utilized Mathematica to get the corresponding solution [32]. Musette and Verhoeven introduced the fifth order KK equation, which was one of the integrable examples of the Henon-Heiles system. This paper uses ARA-RPSM to create power series solution of the time-fractional KK equation which is the fifth-order nonlinear PDE.

This article is structured as follows. Key definitions, properties of fractional derivatives, and the ARA transform are provided in Section 2 to provide the basic knowledge that is needed. Section 3 then introduces the procedure of ARA-RPSM to create a series solution to the time-fractional KK equation, while Section 4 demonstrates the efficacy of the coefficients formula using illustrative examples. Finally, in the last part, Section 5, we give conclusions.

2 Basic Definitions and Theorems

This section contains the definitions and certain theorems needed to construct series solutions of fractional PDEs. Furthermore, we provide some fundamental aspects of ARA transform that will be used in the next section to get the series solution of the nonlinear time-fractional KK equation.

Definition 1. The Caputo fractional derivative of the function $u(x,t)$ of order $\alpha>0$ is given by

$$D_t^\alpha u(x,t)=J_t^{m-\alpha}D_t^m u(x,t)\,,\ \ m-1<\alpha<m,\ \ m\in\mathbb{N},\ \ x\in I,\ \ \ t>0, \tag{2}$$

where

$$J_t^\alpha u(x,t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(x,\tau)\ d\tau, & t>\tau>0\\ u(x,\tau)\,, & \alpha=0\end{cases} \tag{3}$$

is the time- fractional Riemann–Liouville integral operator of order $\alpha>0$.

Definition 2. [24] ARA transform of the continuous function $u(x,t)$ for the variable t of order m, is defined by

$$\mathscr{A}_m[u(x,t)]=s\int_0^\infty t^{m-1}e^{-st}u(x,t)\,dt,\ \ s>0. \tag{4}$$

Some fundamental aspects of ARA transform [25] that are crucial to our study are presented in the arguments that follow.

Properties of ARA-transform:

Let $u(x,t)$ and $v(x,t)$ be continuous functions in which ARA-transform for the variable t exists.

If $0<\alpha\leq 1,\ \ x\in I,\ \ s>0,$ then we have

1. $\mathscr{A}_m[au(x,t)+bv(x,t)]=a\ \mathscr{A}_m[u(x,t)]+b\ \mathscr{A}_m[v(x,t)]$, where a and b are constants and $m\in\mathbb{N}$.
2. $\lim\limits_{s\to\infty}\mathscr{A}_1[u(x,t)]=u(x,0)\,.$
3. $\lim\limits_{s\to\infty}s\mathscr{A}_2[u(x,t)]=u(x,0)\,.$
4. $\mathscr{A}_1[D_t^\alpha u(x,t)]=s^\alpha\mathscr{A}_1[u(x,t)]-s^\alpha u(x,0)\,.$
5. $\mathscr{A}_2[D_t^\alpha u(x,t)]=s^\alpha\mathscr{A}_2[u(x,t)]-\alpha s^{\alpha-1}\mathscr{A}_1[u(x,t)]+(\alpha-1)s^{\alpha-1}u(x,0)\,.$
6. $\mathscr{A}_2[t^\alpha]=\frac{\Gamma(\alpha+2)}{s^{\alpha+1}}.$

Theorem 1. [25] If the continuous function $u(x,t)$ has the fractional power series (FPS) representation at $t=0$ of the form

$$u(x,t)=\sum_{n=0}^{\infty}a_n(x)t^{n\alpha},\ \ m-1<\alpha\leq m,\ \ m=1,\ 2,\ \cdots,\ \ 0\leq t\leq\beta. \tag{5}$$

If $D_t^{n\alpha}u(x,t)$ are continuous for all $n \in \mathbb{N}$, then $a_n(x)$ are on the form

$$a_n(x) = \frac{D_t^{n\alpha}u(x,0)}{\Gamma(n\alpha+1)}, \quad \text{for } n = 0,\ 1,\ 2,\ \ldots \tag{6}$$

where $D_t^{n\alpha} = D_t^{\alpha}\, D_t^{\alpha} \ldots D_t^{\alpha}$ (n- times).

Theorem 2. [25] Let $u(x,t)$ be a continuous function that has the FPS representation

$$\mathscr{A}_2[u(x,t)] = \sum_{n=0}^{\infty} \frac{l_n(x)}{s^{n\alpha+1}}, \quad 0 < \alpha \leq 1,\ x \in I \ \ and \ \ s > 0, \tag{7}$$

where ARA-transform for the variable t exists. Then

$$l_n(x) = (n\alpha+1)\, D_t^{n\alpha}u(x,0). \tag{8}$$

Remark 1.
1. If ARA-transform of order two of the function $u(x,t)$ has the series representation 7, then ARA-transform of order one can be expressed as follows:

$$\mathscr{A}_1[u(x,t)] = \sum_{n=0}^{\infty} \frac{l_n(x)}{(n\alpha+1)\, s^{n\alpha}}, \tag{9}$$

and the kth truncated series is defined as follows:

$$\mathscr{A}_1[u(x,t)]_k = \sum_{n=0}^{k} \frac{l_n(x)}{(n\alpha+1)\, s^{n\alpha}}. \tag{10}$$

2. The inverse of ARA-transform of order two for the FPS 7 is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{D_t^{n\alpha}y(x,0)}{\Gamma(n\alpha+1)} t^{n\alpha}. \tag{11}$$

3 Constructing Series Solution of the Time-Fractional KK Equation

We outline the ARA-RPSM approach in this section for solving the time-fractional KK equation. The basic idea of the introduced method is converting the provided equation to ARA space and suggesting a series solution in a Laurent series expansion form. After that the coefficients are determined using the concept of residual function and by evaluating the limit at infinity. Finally, the series solution of the given initial value problem is the inverse ARA transform of the obtained series. To perform ARA-RPSM, consider the time fractional KK equation

$$D_t^{\alpha}u(x,t) + \xi u u_{xxx} + \rho u_x u_{xx} + \sigma u^2 u_x + u_{xxxxx} = 0, \tag{12}$$

with the initial condition

$$u(x,0) = g(x). \tag{13}$$

Employing ARA transform ($\mathscr{A}_2$) with respect to the variable t, on equation 12, we get

$$\mathscr{A}_2\left[D_t^{\alpha}\ u(x,t)\right] + \mathscr{A}_2[\xi u u_{xxx}] + \mathscr{A}_2[\rho u_x u_{xx}] + \mathscr{A}_2[\sigma u^2 u_x] + \mathscr{A}_2[u_{xxxxx}] = 0. \tag{14}$$

Followed by considering the property (5) and the initial condition in equation 13 and multiplying by $s^{-\alpha}$, equation 14 becomes

$$\mathscr{A}_2[u(x,t)] - \frac{\alpha}{s}\mathscr{A}_1[u(x,t)] + \frac{\alpha-1}{s}g(x) + \frac{\xi}{s^{\alpha}}\mathscr{A}_2[u u_{xxx}] + \frac{\rho}{s^{\alpha}}\mathscr{A}_2[u_x u_{xx}] + \frac{\sigma}{s^{\alpha}}\mathscr{A}_2\left[u^2 u_x\right] + \frac{1}{s^{\alpha}}\mathscr{A}_2[u_{xxxxx}] = 0, \tag{15}$$

which is equivalent to

$$\begin{aligned}
&\mathscr{A}_2[u(x,t)] - \frac{\alpha}{s}\mathscr{A}_1[u(x,t)] + \frac{\alpha-1}{s}g(x) + \frac{\xi}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)\partial_x^3\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)\right] \\
&+\frac{\rho}{s^{\alpha}}\mathscr{A}_2\left[\partial_x\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)\partial_x^2\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)\right] + \frac{\sigma}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)^2\partial_x\left(\mathscr{A}_2^{-1}[\mathscr{A}_2[u(x,t)]]\right)\right] \\
&+\frac{1}{s^{\alpha}}\partial_x^5\left(\mathscr{A}_2[u(x,t)]\right) = 0.
\end{aligned} \tag{16}$$

Assume that ARA residual power series solution of equation 16 has the following series representations

$$\mathscr{A}_1\left[u(x,t)\right]=\sum_{n=0}^{\infty}\frac{l_n(x)}{(n\alpha+1)\,s^{n\alpha}}, \tag{17}$$

$$\mathscr{A}_2\left[u(x,t)\right]=\sum_{n=0}^{\infty}\frac{l_n(x)}{s^{n\alpha+1}}. \tag{18}$$

The kth truncated series of the expansion 17 and 18 are

$$\mathscr{A}_1\left[u(x,t)\right]_k=\sum_{n=0}^{k}\frac{l_n(x)}{(n\alpha+1)\,s^{n\alpha}}, \tag{19}$$

$$\mathscr{A}_2\left[u(x,t)\right]_k=\sum_{n=0}^{k}\frac{l_n(x)}{s^{n\alpha+1}}. \tag{20}$$

Taking the limit as $s\to\infty$ after multiplying equation 20 by s, property (3) yields

$$l_0(x)=u(x,0)=g(x).$$

Hence the series representations 19 and 20 become

$$\mathscr{A}_1\left[u(x,t)\right]_k=g(x)+\sum_{n=1}^{k}\frac{l_n(x)}{(n\alpha+1)\,s^{n\alpha}}, \tag{21}$$

$$\mathscr{A}_2\left[u(x,t)\right]_k=\frac{g(x)}{s}+\sum_{n=1}^{k}\frac{l_n(x)}{s^{n\alpha+1}}. \tag{22}$$

To find the coefficients of the series expansions in equations 21 and 22, we define ARA-residual function of equation 16 as follows:

$$\begin{aligned}\mathscr{A}_2Res(x,s)=&\ \mathscr{A}_2\left[u(x,t)\right]-\frac{\alpha}{s}\mathscr{A}_1\left[u(x,t)\right]+\frac{\alpha-1}{s}g(x)+\frac{\xi}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)\partial_x^3\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)\right]\\&+\frac{\rho}{s^{\alpha}}\mathscr{A}_2\left[\partial_x\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)\partial_x^2\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)\right]\\&+\frac{\sigma}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)^2\partial_x\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]\right]\right)\right]+\frac{1}{s^{\alpha}}\partial_x^5\left(\mathscr{A}_2\left[u(x,t)\right]\right),\end{aligned} \tag{23}$$

and the kth ARA-residual function is

$$\begin{aligned}\mathscr{A}_2Res(x,s)_k=&\ \mathscr{A}_2\left[u(x,t)\right]_k-\frac{\alpha}{s}\mathscr{A}_1\left[u(x,t)\right]_k+\frac{\alpha-1}{s}g(x)+\frac{\xi}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)\partial_x^3\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)\right]\\&+\frac{\rho}{s^{\alpha}}\mathscr{A}_2\left[\partial_x\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)\partial_x^2\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)\right]\\&+\frac{\sigma}{s^{\alpha}}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)^2\partial_x\left(\mathscr{A}_2^{-1}\left[\mathscr{A}_2\left[u(x,t)\right]_k\right]\right)\right]+\frac{1}{s^{\alpha}}\partial_x^5\left(\mathscr{A}_2\left[u(x,t)\right]_k\right),\ k=1,2,\ \ldots.\end{aligned} \tag{24}$$

The following facts are needed to obtain ARA residual power series solution [25],

- $\mathscr{A}_2Res(x,s)=0,\ \ x\in I,\ \ s>0.$
- $\lim_{k\to\infty}\mathscr{A}_2Res(x,s)_k=\mathscr{A}_2Res(x,s),\ \ x\in I,\ \ s>0.$
- $\lim_{s\to\infty}s\mathscr{A}_2Res(x,s)=0$ and $\lim_{s\to\infty}s\mathscr{A}_2Res(x,s)_k=0,\ \ x\in I,\ \ s>0.$
- $\lim_{s\to\infty}s^{k\alpha+1}\mathscr{A}_2Res(x,s)=\lim_{s\to\infty}s^{k\alpha+1}\mathscr{A}_2Res(x,s)_k=0,\ \ x\in I,\ \ s>0.$ (25)

In order to find the coefficient $l_1(x)$ in the series expansion 22, we substitute $\mathscr{A}_1[u(x,t)]_1 = g(x) + \frac{l_1(x)}{(\alpha+1)s^\alpha}, \mathscr{A}_2[u(x,t)]_1 = \frac{g(x)}{s} + \frac{l_1(x)}{s^{\alpha+1}}$ into $\mathscr{A}_2 Res(x,s)_1$ to get

$$\begin{aligned}
\mathscr{A}_2 Res(x,s)_1 =& \frac{l_1(x)}{s^{\alpha+1}} - \frac{\alpha\, l_1(x)}{(\alpha+1)s^{\alpha+1}} + \frac{\xi}{s^\alpha}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)\partial_x^3\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)\right] \\
&+ \frac{\rho}{s^\alpha}\mathscr{A}_2\left[\partial_x\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)\partial_x^2\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)\right] \\
&+ \frac{\sigma}{s^\alpha}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)^2\partial_x\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right]\right)\right] + \frac{1}{s^\alpha}\partial_x^5\left(\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right). \qquad (26)
\end{aligned}$$

Using the property $\mathscr{A}_2[t^\alpha] = \frac{\Gamma(\alpha+2)}{s^{\alpha+1}}$, after simple computations, we obtain

$$\begin{aligned}
\mathscr{A}_2 Res(x,s)_1 =& \frac{l_1(x)}{s^{\alpha+1}} - \frac{\alpha l_1(x)}{(\alpha+1)s^{\alpha+1}} + \frac{\xi}{s^\alpha}\mathscr{A}_2\left[\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)\partial_x^3\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)\right] \\
&+ \frac{\rho}{s^\alpha}\mathscr{A}_2\left[\partial_x\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)\partial_x^2\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)\right] \\
&+ \frac{\sigma}{s^\alpha}\mathscr{A}_2\left[\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)^2\partial_x\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}\right)\right] + \frac{1}{s^\alpha}\partial_x^5\left(\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}\right). \qquad (27)
\end{aligned}$$

Multiply both sides of equation 27 by $s^{\alpha+1}$ and take the limit as $s\to\infty$, then solve the equation $\lim\limits_{s\to\infty} s^{\alpha+1}\, \mathscr{A}_2 Res(x,s)_1 = 0$ for $l_1(x)$, we have

$$l_1(x) = -(\alpha+1)\left(\xi g(x)g'''(x) + \rho g'(x)g''(x) + \sigma g^2(x)g'(x) + g^{(5)}(x)\right). \qquad (28)$$

In the same way, to establish the form of the second unknown coefficient $l_2(x)$, we substitute $\mathscr{A}_1[u(x,t)]_2 = g(x) + \frac{l_1(x)}{(\alpha+1)s^\alpha} + \frac{l_2(x)}{(2\alpha+1)s^{2\alpha}}$ and $\mathscr{A}_2[u(x,t)]_2 = \frac{g(x)}{s} + \frac{l_1(x)}{s^{\alpha+1}} + \frac{l_2(x)}{s^{2\alpha+1}}$ into $\mathscr{A}_2 Res(x,s)_2$ to get

$$\begin{aligned}
\mathscr{A}_2 Res(x,s)_2 =& \frac{l_1(x)}{s^{\alpha+1}} + \frac{l_2(x)}{s^{2\alpha+1}} - \frac{\alpha}{s}\left(\frac{l_1(x)}{(\alpha+1)s^\alpha} + \frac{l_2(x)}{(2\alpha+1)s^{2\alpha}}\right) \\
&+ \frac{\xi}{s^\alpha}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)\partial_x^3\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)\right] \\
&+ \frac{\rho}{s^\alpha}\mathscr{A}_2\left[\partial_x\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)\partial_x^2\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)\right] \qquad (29) \\
&+ \frac{\sigma}{s^\alpha}\mathscr{A}_2\left[\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)^2\partial_x\left(\mathscr{A}_2^{-1}\left[\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right]\right)\right] + \frac{1}{s^\alpha}\partial_x^5\left(\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right)
\end{aligned}$$

Again, using the property $\mathscr{A}_2[t^\alpha] = \frac{\Gamma(\alpha+2)}{s^{\alpha+1}}$, we obtain

$$\begin{aligned}
\mathscr{A}_2 Res(x,s)_2 =& \frac{l_1(x)}{s^{\alpha+1}} + \frac{l_2(x)}{s^{2\alpha+1}} - \frac{\alpha}{s}\left(\frac{l_1(x)}{(\alpha+1)s^\alpha} + \frac{l_2(x)}{(2\alpha+1)s^{2\alpha}}\right) \\
&+ \frac{\xi}{s^\alpha}\mathscr{A}_2\left[\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)\partial_x^3\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)\right] \\
&+ \frac{\rho}{s^\alpha}\mathscr{A}_2\left[\partial_x\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)\partial_x^2\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)\right] \qquad (30) \\
&+ \frac{\sigma}{s^\alpha}\mathscr{A}_2\left[\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)^2\partial_x\left(g(x)+\frac{l_1(x)t^\alpha}{\Gamma(\alpha+2)}+\frac{l_2(x)t^{2\alpha}}{\Gamma(2\alpha+2)}\right)\right] + \frac{1}{s^\alpha}\partial_x^5\left(\frac{g(x)}{s}+\frac{l_1(x)}{s^{\alpha+1}}+\frac{l_2(x)}{s^{2\alpha+1}}\right)
\end{aligned}$$

Multiply the obtained equation by $s^{2\alpha+1}$ and take the limit as $s\to\infty$, then solve the equation $\lim\limits_{s\to\infty} s^{2\alpha+1}\mathscr{A}_2 Res(x,s)_2 = 0$ for $l_2(x)$ to get

$$l_2(x) = -\frac{(2\alpha+1)}{\alpha+1}\left(\xi g'''(x)l_1(x) + \xi g(x)\, l_1'''(x) + \rho\, g''(x)l_1'(x) + \rho g'(x)\, l_1''(x) + \sigma g^2(x)\, l_1'(x) + 2\sigma g(x)g'(x)l_1(x) + l_1^{(5)}(x)\right) \qquad (31)$$

Continuing in a similar way, we get

$$l_3\,(x) = -\frac{3\alpha+1}{(2\alpha+1)\Gamma(\alpha+2)^2}\Big[\xi\Gamma(\alpha+2)^2\, g^{(3)}\,(x)\, l_2\,(x) + \rho\Gamma(\alpha+2)^2\, g''(x)\, l_2'\,(x) + \rho\Gamma(\alpha+2)^2\, g'\,(x)\, l_2''\,(x)$$
$$+2\sigma\Gamma(\alpha+2)^2\, g(x)\, l_2\,(x)\, g'\,(x) + \sigma\Gamma(2\alpha+2)\, l_1\,(x)^2\, g'\,(x) + \xi\Gamma(\alpha+2)^2\, g(x)\, l_2^{(3)}\,(x) + \sigma\Gamma(\alpha+2)^2\, g(x)^2\, l_2'\,(x)$$
$$+2\sigma\Gamma(2\alpha+2) g(x)\, l_1\,(x)\, l_1'\,(x) + \xi\Gamma(2\alpha+2)\, l_1\,(x)\, l_1^{(3)}\,(x) + \rho\Gamma(2\alpha+2)\, l_1'\,(x)\, l_1''\,(x)$$
$$+\Gamma(\alpha+2)^2\, l_2^{(5)}(x)\Big]. \tag{32}$$

$$l_4\,(x) = -\frac{4\alpha+1}{(3\alpha+1)\Gamma(\alpha+2)^3\,\Gamma(2\alpha+2)}\,(\xi\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g^{(3)}\,(x)\, l_3\,(x) + \rho\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g''(x)\, l_3'(x)$$
$$+\rho\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g'(x)\, l_3''(x) + 2\sigma\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g(x)\, l_3\,(x)\, g'(x)$$
$$+2\sigma\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, l_1\,(x)\, l_2(x)\, g'(x) + \xi\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g(x)\, l_3^{(3)}(x) + \sigma\Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, g(x)^2\, l_3''(x)$$
$$+2\sigma\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, g(x)\, l_2\,(x)\, l_1'\,(x) + 2\sigma\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, g(x)\, l_1\,(x)\, l_2'\,(x)$$
$$+\xi\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, l_2\,(x)\, l_1^{(3)}(x) + \xi\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, l_1\,(x)\, l_2^{(3)}(x) + \rho\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, l_2'\,(x)\, l_1''(x)$$
$$+\rho\Gamma(3\alpha+2)\Gamma(\alpha+2)^2\, l_1'(x)\, l_2''(x) + \sigma\Gamma(2\alpha+2)\Gamma(3\alpha+2)\, l_1(x)^2\, l_1'\,(x) + \Gamma(2\alpha+2)\Gamma(\alpha+2)^3\, l_3^{(5)}(x)) \tag{33}$$

After that, the obtained coefficients $l_n\,(x)$ are substituted into equation 18. Therefore, ARA-RPS solution of the equation 14 is

$$\mathscr{A}_2\,[u(x,t)] = \frac{g(x)}{s} + \frac{l_1\,(x)}{s^{\alpha+1}} + \frac{l_2\,(x)}{s^{2\alpha+1}} + \frac{l_3\,(x)}{s^{3\alpha+1}} + \ldots \tag{34}$$

Finally, operate $\mathscr{A}_2^{-1}$ on equation 34, gives the following approximate infinite series solution of the target problem

$$u(x,t) = \mathscr{A}_2^{-1}\,[\frac{g(x)}{s} + \frac{l_1\,(x)}{s^{\alpha+1}} + \frac{l_2\,(x)}{s^{2\alpha+1}} + \frac{l_3\,(x)}{s^{3\alpha+1}} + \ldots]. \tag{35}$$

4 Numerical Examples

The proposed method yields analytical approximation solutions for various free parameters. The analytical findings are extremely useful in understanding the fundamental mechanics of physical problem. Depending on the underlying physical conditions, the explicit responses reflected various types of approximate solutions. In order to verify the accuracy and consistency of the given approach, we investigate two exciting nonlinear fractional partial equations. These examples show how effectively and practically the suggested approach.

Example 4.1: Consider the time-fractional equation

$$D_t^{\alpha} u\,(x,t) - 15uu_{xxx} - 15pu_x u_{xx} + 45\,u^2 u_x + u_{xxxxx} = 0, \tag{36}$$

with the initial condition

$$u\,(x,0) = \frac{1}{4}\omega^2\gamma^2 sech^2\left(\frac{\omega\gamma x}{2}\right) + \frac{\omega^2\gamma^2}{12}. \tag{37}$$

The exact solution of this equation with $p = \frac{5}{2}$ is

$$u\,(x,t) = \frac{\gamma^2\omega^2}{12} + \frac{1}{4}\gamma^2\omega^2\,\mathrm{sech}^2\left(\frac{\gamma}{2}\left(x\omega - \frac{t\,(\gamma^4\omega^5)}{16\Gamma(2)}\right)\right). \tag{38}$$

Comparing equations 36 and 37 with equations 12 and 13, we find that $\xi = -15$, $\rho = -\frac{15\,(5)}{2}$, $\sigma = 45$ and $g\,(x) = \frac{1}{4}\omega^2\gamma^2 sech^2\left(\frac{\omega\gamma x}{2}\right) + \frac{\omega^2\gamma^2}{12}$. Therefore, according to the construction in Section 3, the third approximation of ARA-RPSM solution of the IVP 36-37 is

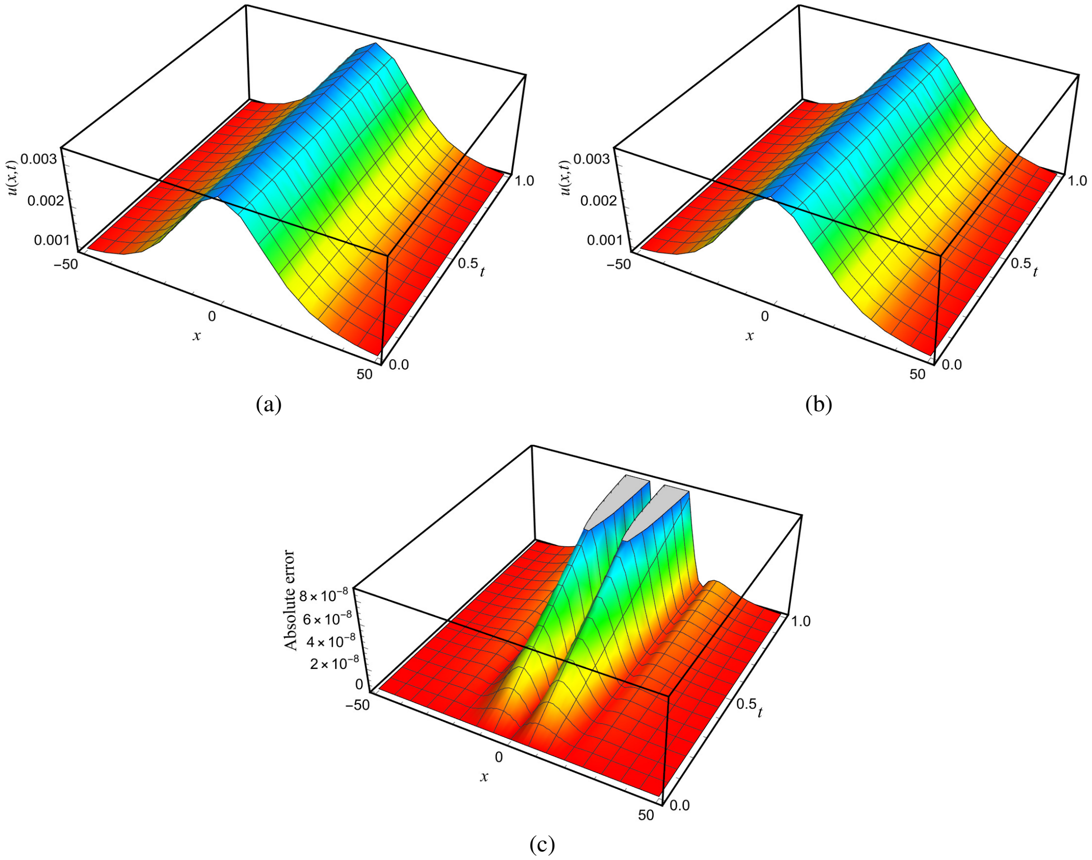

Fig. 1: 3D-Plot of $u(x,\ t)$ for considered Example 4.1when $\omega = 1,\ \ \gamma\ =\ 0.1$, and $\alpha = 1$. (a) Third approximate solution (b) Exact solution (c) Absolute error $|u_{exat} - u_{app}|$.

$$
\begin{aligned}
u_3(x,t) &= \frac{\gamma^2\omega^2}{12} + \frac{1}{4}\gamma^2\omega^2\mathrm{sech}^2\left(\frac{\gamma x\omega}{2}\right) + \frac{(\alpha+1)\,\gamma^7\omega^7 t^\alpha}{512\Gamma(\alpha+2)}\tanh\left(\frac{\gamma x\omega}{2}\right)\mathrm{sech}^6\left(\frac{\gamma x\omega}{2}\right)(-1436\cosh(\gamma x\omega)+\cosh(2\gamma x\omega)+5043)\\
&+\frac{(2\alpha+1)\,\gamma^{12}\omega^{12}t^{2\alpha}\mathrm{sech}^{12}\left(\frac{\gamma x\omega}{2}\right)}{524288\,(\alpha+1)\,\Gamma(2\alpha+1)}\times[7048508994\cosh(\gamma x\omega) - 1404748848\cosh(2\gamma x\omega) + 74191677\cosh(3\gamma x\omega)\\
&- 524156\cosh(4\gamma x\omega) + \cosh(5\gamma x\omega) - 5910503124]\,\frac{(2\alpha+1)\,\gamma^{12}\omega^{12}t^{2\alpha}\mathrm{sech}^{12}\left(\frac{\gamma x\omega}{2}\right)}{524288\,(\alpha+1)\,\Gamma(2\alpha+1)}\,[7048508994\cosh(\gamma x\omega)\\
&- 1404748848\cosh(2\gamma x\omega) + 74191677\cosh(3\gamma x\omega) - 524156\cosh(4\gamma x\omega) + \cosh(5\gamma x\omega) - 5910503124]\,. \quad (39)
\end{aligned}
$$

Figure 1 shows the comparison of the solution obtained by the help of ARA-RPSM in comparison with the absolute error and exact solution for the time fractional KK in equation 36. Figure 2 indicates how the solution for equation (4.1) obtained by ARA-RPSM behaves with time t at constant $x = 5$ for various values of α. On the other hand, Figure 38 shows the behavior of $u(x,t)$ at constant time for various values of α. While Figure 39 is a plot of the approximate solution $u(x,t)$ for $\alpha = 1$ in comparison with the exact solution with respect to x for considered at $t = 0.1$. Tabulated the comparison between the results obtained by ARA-RPSM and the exact solution in tables 1 and 2, the results are very close to those was calculated earlier using q-HATM technique in [32].

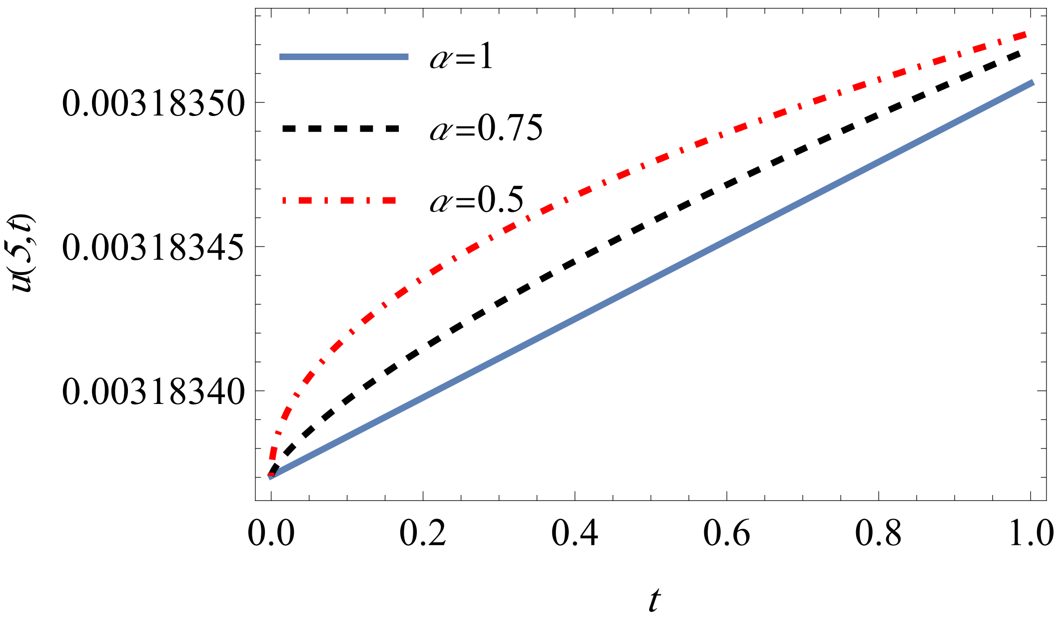

Fig. 2: Plot of the third approximate solution of $u(x,t)$ with respect to t for considered Example 4.1 when $\omega = 1, \gamma = 0.1$, $x = 5$ with distinct α.

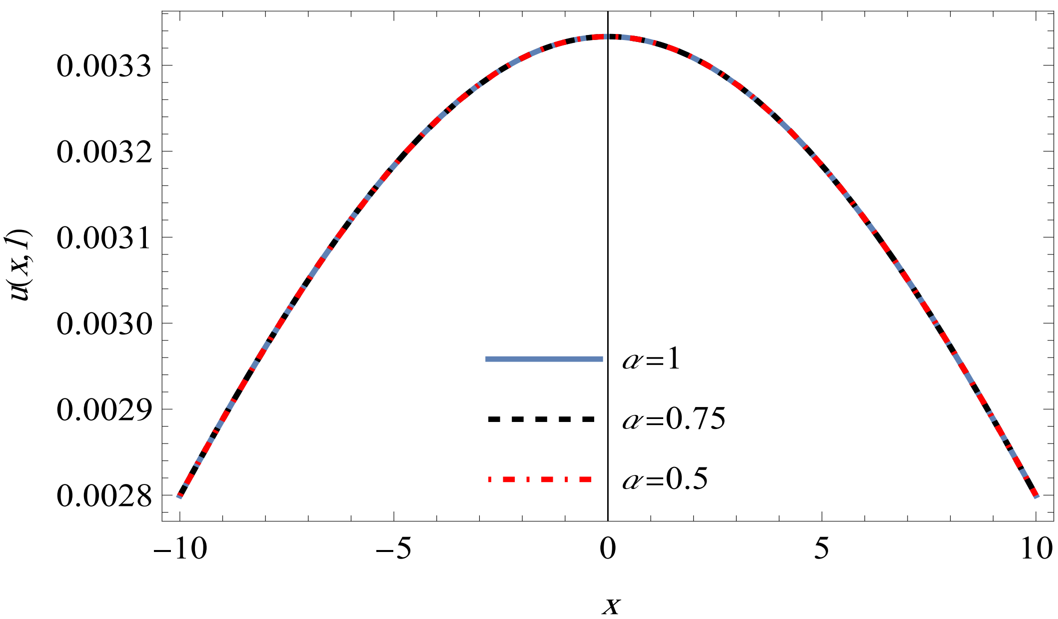

Fig. 3: Plot of the third approximate solution of $u(x,t)$ with respect to x for considered Example 4.1 when $\omega = 1$, $\gamma = 0.1$, $t = 1$ with distinct α.

Table 1: Error analysis for Example 4.2 when $\omega = 1, \gamma = 0.1, t = 0.1$ and $k = 3$ with $\alpha = 1$.

x	Error
0.1	3.48766×10^{-10}
0.2	7.0008×10^{-10}
0.3	1.05096×10^{-9}
0.4	1.40119×10^{-9}
0.5	1.75056×10^{-9}

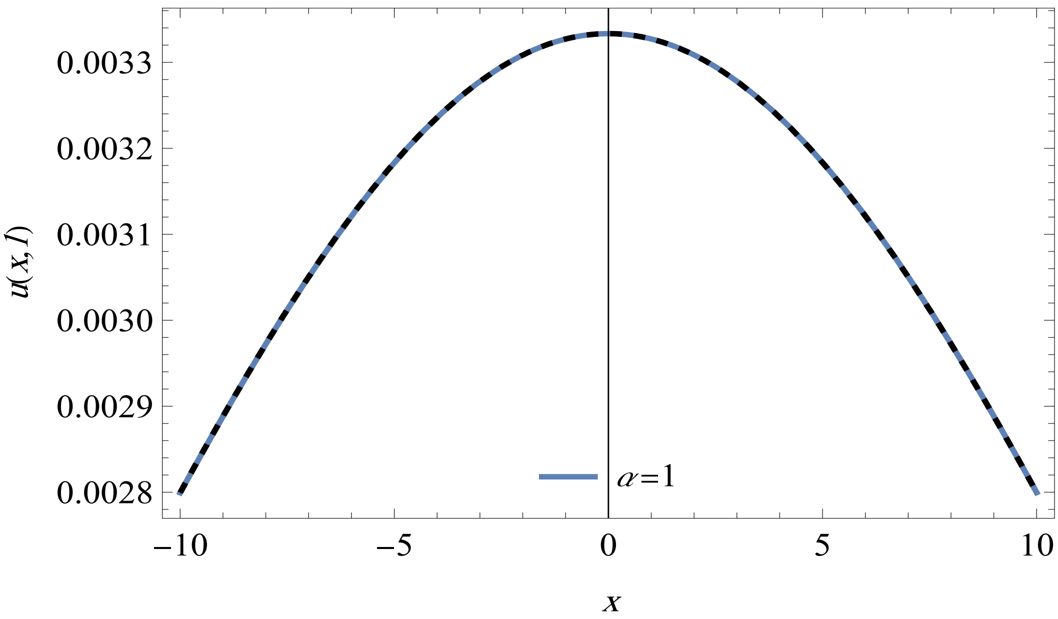

Fig. 4: Plot of the third approximate solution of $u(x,t)$ (blue) and the exact solution (black-dashed line) with respect to x for considered Example 4.1 when $\omega = 1, \gamma = 0.1, t = 0.1$ and α=1.

Table 2: Error analysis for Example 4.1 when $\omega = 1, \gamma = 0.1$, and $k = 3$ with $\alpha = 1$.

t	x	Error	t	x	Error
0.25	1	8.6826×10^{-9}	0.5	1	1.73329×10^{-8}
	2	1.68535×10^{-8}		2	3.36813×10^{-8}
	3	2.40173×10^{-8}		3	4.80185×10^{-8}
	4	2.97947×10^{-8}		4	5.95841×10^{-8}
	5	3.39371×10^{-8}		5	6.78794×10^{-8}
0.75	1	2.33723×10^{-8}	1.00	1	3.45364×10^{-8}
	2	2.59509×10^{-8}		2	6.7259×10^{-8}
	3	5.04832×10^{-8}		3	9.59715×10^{-8}
	4	7.20032×10^{-8}		4	1.19147×10^{-7}
	5	8.93682×10^{-8}		5	1.35779×10^{-7}

Example 4.2: Consider the nonlinear time-fractional equation

$$D_t^{\alpha} u(x,t) - 15uu_{xxx} - 15p\, u_x u_{xx} + 45u^2 u_x + u_{xxxxx} = 0, \tag{40}$$

with initial condition

$$u(x,0) = \frac{4c}{3} - \frac{4\,c}{p}\operatorname{sech}^2\left(\sqrt{c}x\right). \tag{41}$$

The exact solution of this equation is given by

$$u(x,t) = \frac{4c}{3} - \frac{4\,c}{p}\operatorname{sech}^2\left(\sqrt{c}\left(x + 8\left(3c^2 - 5pc\right)t\right)\right). \tag{42}$$

Solution: Comparing equations 40 and 41 with equations 12 and 13, we find that $\xi = -15$, $\rho = -15p$, $\sigma = 45$ and $g(x) = \frac{4c}{3} - \frac{4\,c}{p}\operatorname{sech}^2\left(\sqrt{c}\left(x + 8\left(3c^2 - 5pc\right)t\right)\right)$. Therefore, according to the construction in Section 3, the third approximation of ARA-RPSM solution of the IVP 40-41 is

$$
\begin{aligned}
u_3(x,t) = {} & \frac{4c}{3} - \frac{4c\operatorname{sech}^2(\sqrt{c}x)}{p} + \frac{\left(16(\alpha+1)c^{7/2}t^{\alpha}\right)\tanh(\sqrt{c}x)\operatorname{sech}^6(\sqrt{c}x)}{p^3\Gamma(\alpha+2)}\left(p\left((64p-60)\cosh\left(2\sqrt{c}x\right)+p\cosh\left(4\sqrt{c}x\right)\right)\right.\\
& \left.+21p(3p-20)+360\right) + \frac{1024(2\alpha+1)c^6t^{2\alpha}\operatorname{sech}^2(\sqrt{c}x)}{(\alpha+1)p^5\Gamma(2\alpha+1)}\left[6(40-39p)p^3\operatorname{sech}^2\left(\sqrt{c}x\right)-4p^4\right.\\
& 60(p-1)p^2(95p+333)\operatorname{sech}^4\left(\sqrt{c}x\right)-90(p-1)p(p(250p+1879)-480)\operatorname{sech}^6\left(\sqrt{c}x\right)\\
& \left.675(p-1)(p(p(27p+593)-312)+60)\operatorname{sech}^8\left(\sqrt{c}x\right)-7425(p-1)(p(35p-24)+6)\operatorname{sech}^{10}\left(\sqrt{c}x\right)\right]\\
& -\frac{16(3\alpha+1)c^{17/2}t^{3\alpha}\tanh(\sqrt{c}x)\operatorname{sech}^{16}(\sqrt{c}x)}{(\alpha+1)p^7\Gamma(3\alpha+1)}\left[19782144000+12p(-23307264000+p(139680288000\right.\\
& +p(p(2142287239p-279202529160)+555245547840)-396206841600)))\\
& +3(-4354560000+p(80517888000+p(-567316224000+p(1752334473600+p(-2594696852160\\
& +p(121172647p+1333394104080))))))\cosh(2\sqrt{c}x)-2(10357632000+p(-126593280000\\
& +p(p(10024405571p+453979163340)-838558716480)+490790793600)))\cosh\left(4\sqrt{c}x\right)\\
& +p^2\left((p(p(p(4999411759p+74299080120)-126404893440)+56219875200)-9113472000)\cosh\left(6\sqrt{c}x\right)\right.\\
& +p(-4(p(p(73219933p+437702040)-659832480)+148910400)\cosh\left(8\sqrt{c}x\right)\\
& +\ p\left((p(2825693p+6963480)-9789120)\cosh\left(10\sqrt{c}x\right)+p\left((600-602p)\cosh\left(12\sqrt{c}x\right)-p\cosh\left(14\sqrt{c}x\right)\right)\right)))\\
& -\frac{15\;4^{\alpha+1}(\alpha+1)^3(p-1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}(2\alpha+1)\Gamma(\alpha+2)}(18(3110400+p(-10742400+p(3467520+p(4659840+p(38293p-551232))))\\
& +12(p(p(p(p(48647p-114204)-8361240)-1524960)+13564800)-3628800)\cosh\left(2\sqrt{c}x\right)\\
& -3p(p(p(p(89963p-2386272)-6627840)+7292160)+1555200)\cosh\left(4\sqrt{c}x\right)\\
& 6(p(4513p-4848)-3840)p^3\cosh\left(8\sqrt{c}x\right)+2(p(429840-p(68827p+711072))+388800)p^2\cosh\left(6\sqrt{c}x\right)\\
& \left.\left. p^5\cosh\left(12\sqrt{c}x\right)++6(59p+24)p^4\cosh\left(10\sqrt{c}x\right)\right)\right].
\end{aligned} \tag{43}
$$

Figure 5 shows the comparison of the solution obtained by the help of ARA-RPSM in comparison with the absolute error and exact solution for the time fractional KK in equation 41. In order to track the behavior of the solution as a function of time the solution obtained by ARA-residual power series as a function of time is plotted at $x = 5$ for various values of α as shown in Figure 6. Furthermore, Figure 7 shows $u(x,t)$ at constant time for various values of α for $k = 3$. Figure 8 shows a comparison between the exact solution and ARA approximation for $\alpha = 1$ at $t = 0.1$. Tabulated the comparison between the results obtained by ARA-RPSM and the exact solution in tables 3 and 4, the results are very close to those was calculated earlier using q-HATM technique in [32].

Table 3: Error analysis for Example 4.2 when $c = 0.01$, $p = 2.5$, $t = 0.1$, and $k = 3$ with $\alpha = 1$.

x	Error
0.1	1.60244×10^{-6}
0.2	4.79732×10^{-6}
0.3	7.9883×10^{-6}
0.4	1.11728×10^{-5}
0.5	1.43482×10^{-5}

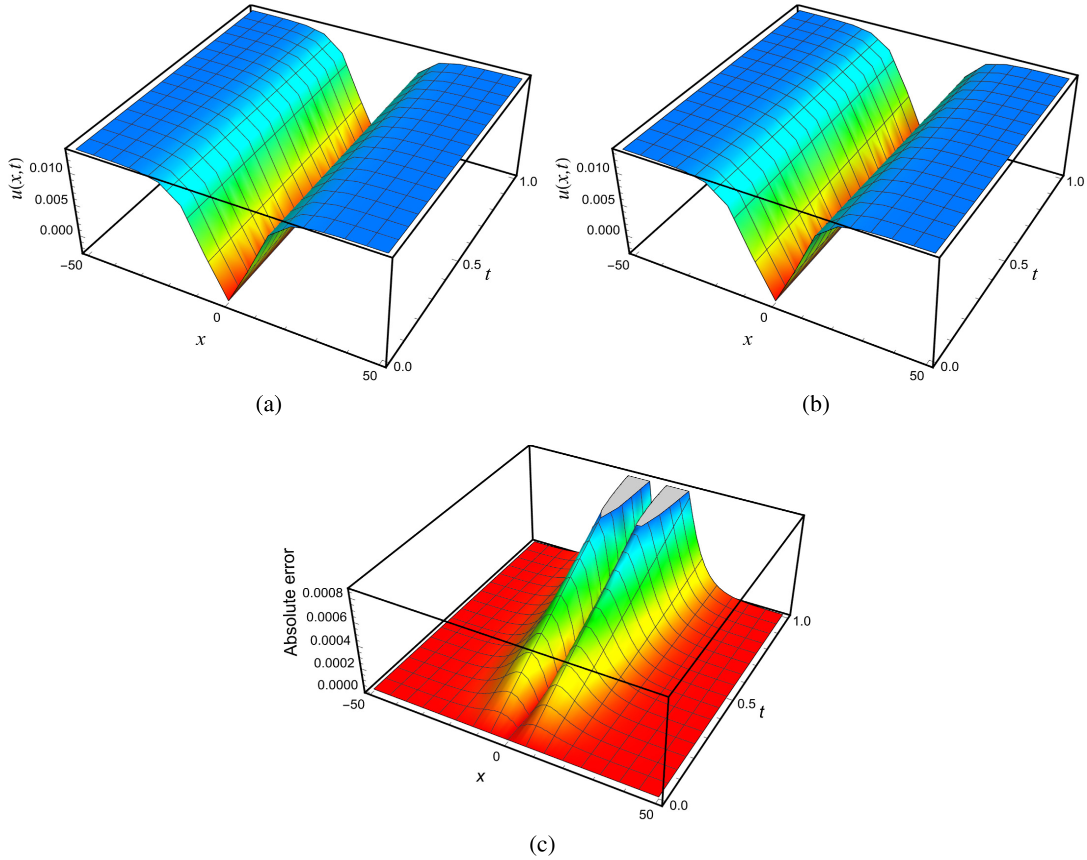

Fig. 5: 3D-Plot of u(x,t)for considered Example 4.2 when $c = 0.01$, $p = 2.5$ with $\alpha = 1$. (a) Third approximate solution (b) Exact solution (c) Absolute error $|u_exat - u_app|$.

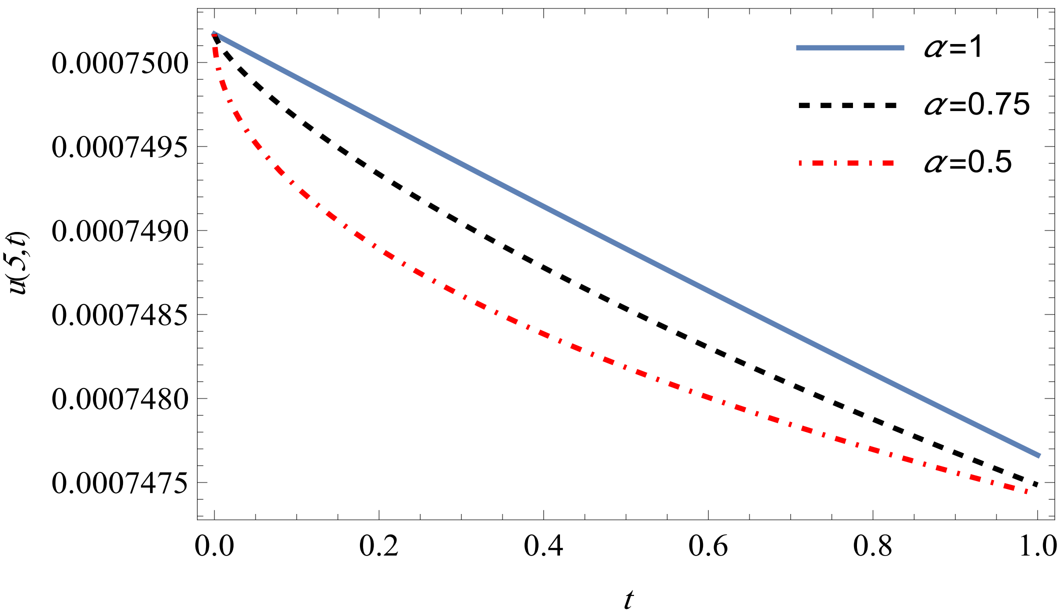

Fig. 6: Plot of the third approximate solution of $u(x,t)$ with respect to t at $x = 5$ for considered Example 4.2 when $c = 0.01$ and $p = 2.5$, with distinct α.

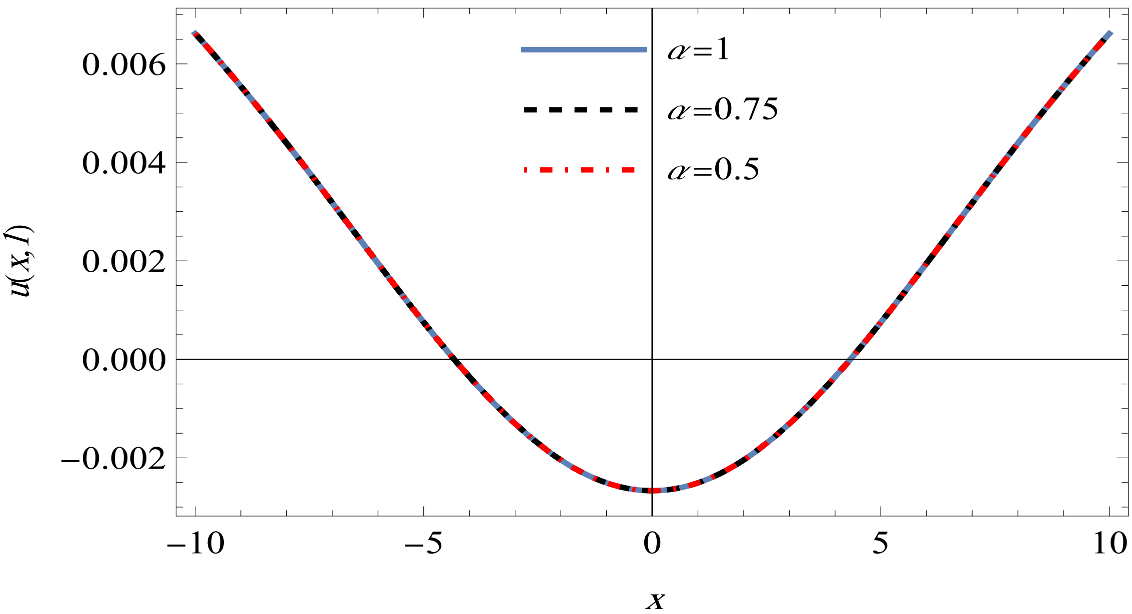

Fig. 7: Plot of the third approximate solution of $u(x,t)$ with respect to x at $t=1$ for considered Example 4.2 when $c=0.01$, $p=2.5$, with distinct α.

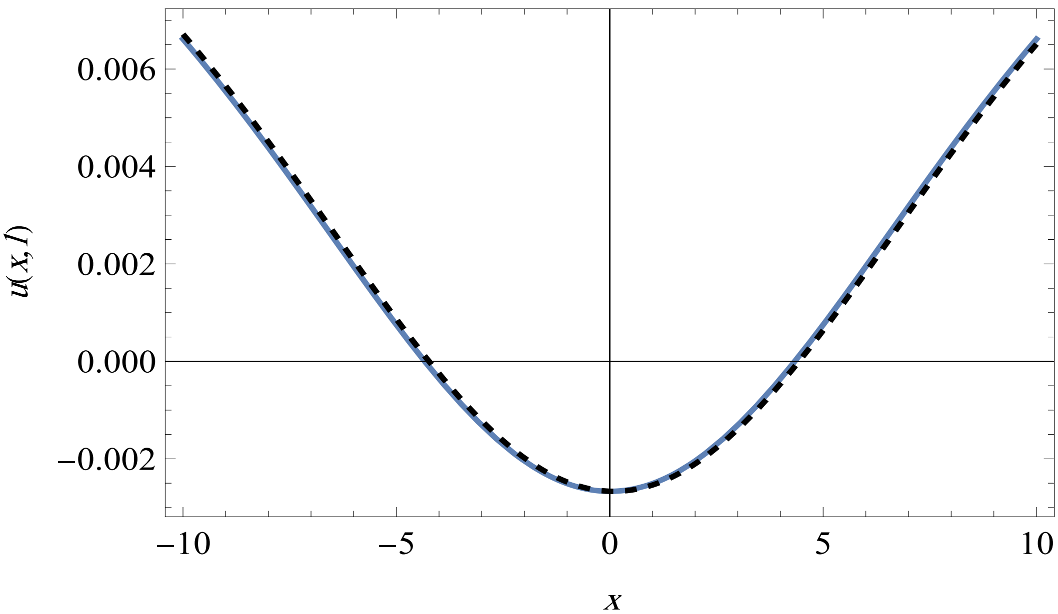

Fig. 8: Plot of the third approximate solution of $u(x,t)$ (blue) and the exact solution (black-dashed line) with respect to x for considered Example 4.1 when $\omega = 1$, $\gamma = 0.1$, $t = 0.1$ and α=1.

5 Conclusion

In this paper, ARA-RPSM was successfully implemented to get accurate solution of the time-fractional Kaup-Kupershmidt equation. Results obtained show great agreement with known techniques. The performance of ARA-RPSM shows its effectiveness, precision, and ability to yield analytical and numerical solutions to an extensive range of fractional physical phenomena that arise in physics and engineering. In future work, we intend to use ARA-RPSM to solve fractional integral equations.

Table 4: Error analysis for Example 4.2 when $c = 0.01$, $p = 2.5$, and $k = 3$ with $\alpha = 1$.

t	x	Error	t	x	Error
0.25	1	6.92109×10^{-5}	0.75	1	1.48982×10^{-4}
	2	1.42901×10^{-4}		2	3.7541×10^{-4}
	3	2.05759×10^{-4}		3	5.73018×10^{-4}
	4	2.54136×10^{-4}		4	7.2948×10^{-4}
	5	2.86425×10^{-4}		5	8.38458×10^{-4}
0.5	1	1.18954×10^{-4}	1	1	1.59143×10^{-4}
	2	2.68238×10^{-4}		2	6.92109×10^{-5}
	3	3.97051×10^{-4}		3	1.42901×10^{-4}
	4	4.97626×10^{-4}		4	2.05759×10^{-4}
	5	5.66245×10^{-4}		5	2.54136×10^{-4}

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