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Different Estimation Methods for A New Extension Three Parameter Log-Logistic Distribution with Applications

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Abstract: This article presents a new extension of the three-parameter log-logistic distribution, the so-called heavy-tailed log-logistic (HTLL) distribution. Some important mathematical properties of the new HTLL distribution are calculated. In addition, some numerical results of moments for the HTLL distribution are calculated. Extensive simulations were performed to investigate the estimation of the model parameters using many established approaches, including maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramer-von Mises (CVM), Anderson-Darling (AD), and right-tail Anderson-Darling (RTAD). The simulation results show that the AD approach has the highest efficiency among these approaches. The usefulness of the newly proposed model is demonstrated by analyzing two real data sets.

Keywords: Statistical model; Log-logistic distribution; Heavy tailed; Simulation; Anderson-Darling; Maximum Likelihood; Cramer von Mises.

1 Introduction

The Fisk distribution is another name for the log-logistic (LL) distribution, which is a distribution that is used in the literature on income distribution [1]. Arnold [2] referred to the Fisk distribution as the Pareto Type III distribution and added an extra location parameter. Other authors, such as [3,4], have referred to the Fisk distribution as the LL distribution. They have also included an additional location parameter. There is further information on the LL model that may be found in [5]. Statistical inference of entropy for the LL distribution utilizing progressive type II censoring are studied by [6]. For the purpose of enhancing the capacity and adaptability of the LL distribution, a number of authors have investigated several generalized variants of the distribution. The following are a few instances that are particularly noteworthy: the Kumaraswamy-LL [7], the transmuted LL [8], the transmuted generalized LL [10,11], the beta-LL [9], the Marshall-Olkin LL [12], the alpha power LL [13,14], the McDonald LL [15], the Zografos-Balakrishnan LL [16], the exponentiated LL [17], the odd Lomax LL [19], the truncated Cauchy power [20], the extended LL [21], the Kavya-Manoharan LL [22], and alpha power exponentiated LL [23] distributions .

The probability density function (pdf) and the cumulative distribution function (cdf) of the LL distribution are

$$g(z;\eta,\ \vartheta) = \frac{\vartheta}{\eta^{\vartheta}}\, z^{\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-2}, \quad z>0, \quad \eta,\ \vartheta>0, \tag{1}$$

and

$$G(z;\eta,\ \vartheta) = 1-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}, \quad z>0, \quad \eta,\ \vartheta>0, \tag{2}$$

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where η and ϑ are two shape parameters.
Various generalizations of probability distributions have been developed in the literature in order to provide distributions with additional shape characteristics. These generalizations include the variable transformation, the exponentiation approach, the quantile method, the combination of two or more distributions or models, and many others. As a result, a large number of unique distributions have been extended and improved over the last decade. Just one example: the compounded Bell–G by [24], truncated Cauchy power Weibull–G by [25], alpha power transformed Weibull–G by [26], generalized truncated Fréchet–G by [27], generalized inverted Kumaraswamy–G by [28], Type II exponentiated half logistic–G by [29], Marshall-Olkin odd Burr III–G by [30], new extended cosine–G by [31], odd generalized N-H–G by [32], sine-exponentiated Weibull–G by [33], Type II half-logistic odd Fréchet–G by [34], sine Burr–G by [35], exponentiated M–G by [36], generalized odd Burr III–G by [37], truncated burr X–G by [38], Topp-Leone odd Fréchet–G by [39], new truncated Muth–G by [40], ratio exponentiated general–G by [41] and odd inverse power generalized Weibull–G by [42]. Recently, [43] suggested the heavy-tailed (HT) family of distributions as an alternative model for fitting HT data. The cdf, pdf survival function (sf) and hazard rate function (hrf) of the HT distributions are

$$F(z;\tau,\kappa)=\frac{\tau\ G(z;\kappa)}{\tau-1+G(z;\kappa)},\tau>1,z\in R, \tag{3}$$

$$f(z;\tau,\kappa)=\frac{\tau\ (\tau-1)g(z;\kappa)}{[\tau-1+G(z;\kappa)]^2}, \tag{4}$$

$$\overline{F}(z;\tau,\kappa)=\frac{(\tau-1)\ \overline{G}(z;\kappa)}{\tau-1+G(z;\kappa)},$$

and

$$h(z;\tau,\kappa)=\frac{\tau(\tau-1)g(z;\kappa)}{\overline{F}(z;\tau,\kappa)[\tau-1+G(z;\kappa)]}.$$

where, $G(z;\kappa)$ is the cdf of the baseline distribution with the vector of parameters κ.
In this paper, we introduce a novel three-parameter lifetime distribution by combining the HT family of distributions with the LL distribution. It can be regarded as a new addition to the HT family that utilizes the properties of the LL distribution. This new addition is called the heavy tailed log-logistic (HTLL) distribution.

There are seven different sections that make up the rest of the article. In addition, the HTLL distribution is shown in section 2. To determine the basic mathematical and statistical properties of the HTLL distribution, section 3 is particularly important. Sections 4 and 5 focus on the estimation of the model parameters. The structure of the applications can be seen in section 6. Concluding considerations are contained in section 7.

2 Heavy Tailed log-logistic Distribution

In this section, we discuss the construction of the HTLL distribution by inserting (1) and (1) in (3) and (4), we get the cdf and pdf of the HTLL distribution as follows

$$F(z;\tau,\eta,\ \vartheta)=\frac{\tau\left[1-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}},z>0,\tau>1,\eta,\ \vartheta>0, \tag{5}$$

and

$$f(z;\tau,\eta,\ \vartheta)=\frac{\tau\ (\tau-1)\vartheta\ z^{\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-2}}{\eta^{\vartheta}\left[\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]^{2}}. \tag{6}$$

The sf, hrf, reversed hrf and cumulative hrf of the HTLL distribution are

$$\overline{F}(z;\tau,\eta,\ \vartheta)=\frac{(\tau-1)\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}}{\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}},$$

$$h(z;\tau,\eta,\ \vartheta)=\frac{\tau\ (\tau-1)\vartheta\ z^{\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}}{\eta^{\vartheta}\left[\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]},$$

$$\tau(z;\tau,\eta,\ \vartheta)=\frac{(\tau-1)\vartheta\ z^{\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-2}}{\eta^{\vartheta}\left[1-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]\left[\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]},$$

and

$$H(z;\tau,\eta,\ \vartheta)=-\log\left[\frac{(\tau-1)\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}}{\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}}\right].$$

Figure 2 shows the pdf and hrf plots for the HTLL distribution. From Figure 2 we can note that the pdf can be decreasing, right skewed and unimodal, but the hrf can be decreasing, increasing, up-side-down or J-shaped.

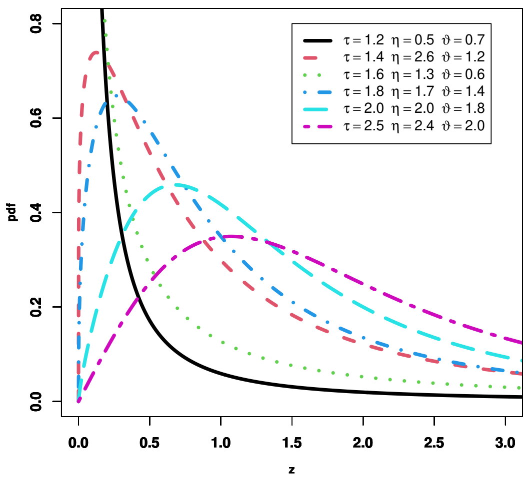

Fig. 1 The hrf plots for the HTLL distribution.

3 Statistical Properties

In this section, we discuss the statistical properties of the HTLL distribution. In particular, the quantiles, the moments and the moment generating function. In this section, we discuss the statistical properties of the HTLL distribution. In particular, the quantiles, the moments and the moment generating function.

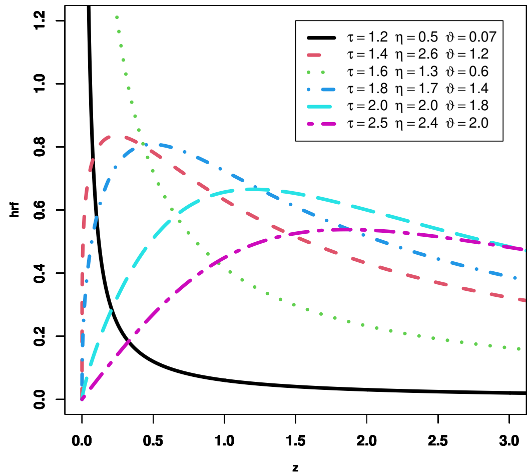

Fig. 2 The hrf plots for the HTLL distribution.

3.1 Quantile Function

Quantiles are fundamentally important for estimates (e.g. quantile estimators) and simulations. The u^{th}-quantile x_u of the HTLL distribution is provided via

$$Q(u) = \eta\left(\left[1 - \frac{u\,(\tau-1)}{(\tau-u)}\right]^{-1} - 1\right)^{\frac{1}{\vartheta}}. \tag{7}$$

The effects of the shape parameters on the skewness and kurtosis can be recognized using quantile measures.

3.2 Moments

Some of the most important characteristics and properties of a distribution can be examined using moments. The r^{th}-moment of the HTLL distribution is calculated by

$$\mu_r^{'} = E(Z^r) = \int_0^\infty z^r f(z;\tau,\eta,\vartheta)dz = \int_0^\infty \frac{\tau(\tau-1)\vartheta z^{r+\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^\vartheta\right)^{-2}}{\eta^\vartheta\left[\tau-\left(1+\left(\frac{z}{\eta}\right)^\vartheta\right)^{-1}\right]^2}dz, \tag{8}$$

by employing the binomial expansion to the above equation

$$\left[\tau-\left(1+\left(\frac{z}{\eta}\right)^\vartheta\right)^{-1}\right]^{-2} = \sum_{i=0}^{\infty}\tau^{-2-i}(1+i)\left(1+\left(\frac{z}{\eta}\right)^\vartheta\right)^{-i},$$

then, we can rewrite the equation (8) as

$$\mu_r^{'} = (\tau-1)\vartheta\eta^{-\vartheta}\sum_{i=0}^{\infty}\tau^{-1-i}(1+i)\int_0^\infty z^{r+\vartheta-1}\left(1+\left(\frac{z}{\eta}\right)^\vartheta\right)^{-i-2}dz. \tag{9}$$

Let $v = \left(\frac{z}{\eta}\right)^{\vartheta}$, then

$$\mu_r' = (\tau - 1)\sum_{i=0}^{\infty} \tau^{-1-i}(1+i)\int_0^{\infty} v^{\frac{r}{\vartheta}}(1+v)^{-i-2}dv. \tag{10}$$

By using the beta prime function, the r^{th}-moment of the HTLL distribution is given by

$$\mu_r' = (\tau - 1)\sum_{i=0}^{\infty} \tau^{-1-i}(1+i)B\left(1+\frac{r}{\vartheta}, i+1-\frac{r}{\vartheta}\right), \quad r < \vartheta(i+1). \tag{11}$$

The moment generating function of the HTLL is given by

$$M_Z(t) = \int_0^{\infty} e^{-tz} f(z;\tau,\eta,\vartheta)dz = \int_0^{\infty}\sum_{k=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

$$= (\tau - 1)\sum_{r,i=0}^{\infty}\frac{(1+i)t^r}{\tau^{1+i}r!}B\left(1+\frac{r}{\vartheta}, i+1-\frac{r}{\vartheta}\right), r < \vartheta(i+1). \tag{12}$$

Tables 1 and 2 show the numerical values of the first moment μ_1' also the numerical values of variance (σ^2), coefficient of skewness (CS), coefficient of kurtosis (CK) and coefficient of variation (CV) associated with the HTLL distribution.

4 Statistical Inference

In this section, we will find different estimators for the parameters of our distribution τ, η, and ϑ, these estimators are maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramer von Mises Minimum Distance Estimation (CME), Anderson-Darling estimation (ADE) and the right - tail Anderson-Darling estimation (RTADE) their equations will be solved numerically using the R Development Core Team [44].

4.1 Maximum Likelihood Estimation Method

MLE is most frequently used in estimating the unknown parameters, it depends on maximizing the logarithm of the likelihood of the distribution, let $z_1, ..., z_n$ is a random sample has size n of the HTLL distribution the log-likelihood function is,

$$l_n = \sum_{i=1}^{n}\log(f(z_i;\tau,\eta,\vartheta)) = n\log(\tau) + n\log(\tau-1) + n\log(\vartheta) + (\vartheta-1)\sum_{i=1}^{n} z_i$$
$$-2\sum_{i=1}^{n}\log\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right) - n\vartheta\log(\eta) - 2\sum_{i=1}^{n}\log\left[\tau - \left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)^{-1}\right].$$

The method depends on finding partial derivative and and solve them simultaneously using R software , to get the values of τ , η and ϑ that maximize them, the partial derivatives of l_n are:

$$\frac{\partial l_n}{\partial \tau} = \frac{n}{\tau} + \frac{n}{\tau-1} - \frac{2}{\tau - (1+(\frac{z_i}{\eta})^{\vartheta})^{-1}},$$

$$\frac{\partial l_n}{\partial \vartheta} = \frac{n}{\vartheta} + \sum_{i=1}^{n}\log(z_i) - 2\sum_{i=1}^{n}\frac{(\frac{z_i}{\eta})^{\vartheta}\log(\frac{z_i}{\eta})}{1+(\frac{z_i}{\eta})^{\vartheta}} - n\log(\eta) - 2\sum_{i=1}^{n}\frac{(\frac{z_i}{\eta})^{\vartheta}\log(\frac{z_i}{\eta})(1+(\frac{z_i}{\eta})^{\vartheta})^{-2}}{\tau-(1+(\frac{z_i}{\eta})^{\vartheta})^{-1}},$$

and

$$\frac{\partial l_n}{\partial \eta} = \frac{-n\vartheta}{\eta} + 2\eta^{-\vartheta-1}\sum_{i=1}^{n}\frac{z_i^{\vartheta}}{1+(\frac{z_i}{\eta})^{\vartheta}} + 2\vartheta\eta^{-\vartheta-1}\sum_{i=1}^{n}\frac{z_i^{\vartheta}(1+(\frac{z_i}{\eta})^{\vartheta})^{-2}}{\tau-(1+(\frac{z_i}{\eta})^{\vartheta})^{-1}}.$$

Table 1 Results of μ_1', σ^2, σ, CS, CK, and CV associated with the HTLL distribution whera ϑ=1.5 .

η	τ	μ_1'	σ^2	σ	CS	CK	CV
0.1	0.1	1.3168	61.8203	7.8626	8.1068	76.0802	5.9709
	0.3	1.4175	66.5214	8.1561	7.7991	70.5222	5.7539
	0.5	1.4652	68.7497	8.2915	7.6643	68.1541	5.6590
	0.7	1.4969	70.2295	8.3803	7.5783	66.6649	5.5986
	0.9	1.5206	71.3401	8.4463	7.5155	65.5880	5.5545
	1.1	1.5397	72.2298	8.4988	7.4662	64.7493	5.5199
	1.3	1.5556	72.9721	8.5424	7.4258	64.0652	5.4915
	1.5	1.5692	73.6092	8.5796	7.3916	63.4892	5.4676
	1.7	1.5811	74.1671	8.6120	7.3620	62.9929	5.4468
	1.9	1.5917	74.6635	8.6408	7.3359	62.5576	5.4286
0.5	0.1	0.9637	33.1296	5.7558	10.2634	126.8441	5.9726
	0.3	1.5904	55.1612	7.4271	7.8682	75.1666	4.6700
	0.5	1.9929	69.3896	8.3300	6.9652	59.2310	4.1799
	0.7	2.3034	80.4480	8.9693	6.4323	50.7387	3.8939
	0.9	2.5622	89.6633	9.4691	6.0637	45.2620	3.6957
	1.1	2.7860	97.6443	9.8815	5.7865	41.3543	3.5469
	1.3	2.9843	104.7267	10.2336	5.5665	38.3855	3.4292
	1.5	3.1631	111.1188	10.5413	5.3857	36.0311	3.3326
	1.7	3.3264	116.9606	10.8148	5.2331	34.1046	3.2512
	1.9	3.4770	122.3509	11.0612	5.1018	32.4902	3.1813
0.7	0.1	0.5244	13.8236	3.7180	15.1344	283.2896	7.0895
	0.3	1.0583	28.9223	5.3779	10.3997	134.3748	5.0818
	0.5	1.4502	40.4355	6.3589	8.7507	95.5192	4.3849
	0.7	1.7787	50.2175	7.0864	7.8170	76.4953	3.9841
	0.9	2.0667	58.8918	7.6741	7.1887	64.9066	3.7133
	1.1	2.3258	66.7665	8.1711	6.7256	56.9902	3.5132
	1.3	2.5631	74.0239	8.6037	6.3644	51.1837	3.3567
	1.5	2.7830	80.7835	8.9880	6.0715	46.7124	3.2296
	1.7	2.9884	87.1293	9.3343	5.8272	43.1450	3.1235
	1.9	3.1817	93.1230	9.6500	5.6190	40.2212	3.0330
1.2	0.1	0.1594	1.2213	1.1051	40.9024	2381.4400	6.9344
	0.3	0.4411	4.4178	2.1019	21.7861	667.8110	4.7653
	0.5	0.7015	7.9500	2.8196	16.3605	374.2333	4.0194
	0.7	0.9484	11.6451	3.4125	13.5845	256.9633	3.5983
	0.9	1.1850	15.4343	3.9286	11.8407	194.6921	3.3153
	1.1	1.4133	19.2802	4.3909	10.6204	156.3411	3.1068
	1.3	1.6346	23.1601	4.8125	9.7074	130.4530	2.9442
	1.5	1.8496	27.0585	5.2018	8.9920	111.8503	2.8123
	1.7	2.0592	30.9646	5.5646	8.4126	97.8616	2.7023
	1.9	2.2638	34.8705	5.9051	7.9312	86.9736	2.6085

4.2 Least Squares Method

LSE method was introduced by [45], it depends on minimizing its functions to get the estimates of the parameters, and the least squares function is:

$$LSE(\tau,\eta,\vartheta)=\sum_{i=1}^{n}\left[F(z_{(i)},\tau,\eta,\vartheta)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[\frac{\tau\left[1-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}}-\frac{i}{n+1}\right]^2.$$

From now and on we will use the following abbreviations

$$F_i=F(z_{(i)},\tau,\eta,\vartheta), \tag{13}$$

Table 2 Results of μ_1', σ^2, σ, CS, CK, and CV associated with the HTLL distribution whera ϑ=2.1 .

η	τ	μ_1'	σ^2	σ	CS	CK	CV
0.1	0.1	1.7424	81.7153	9.0397	6.9914	56.9489	5.1882
	0.3	1.8464	86.5979	9.3058	6.7778	53.6052	5.0400
	0.5	1.8940	88.8380	9.4254	6.6858	52.1955	4.9765
	0.7	1.9250	90.2985	9.5026	6.6276	51.3147	4.9364
	0.9	1.9479	91.3801	9.5593	6.5855	50.6808	4.9074
	1.1	1.9661	92.2375	9.6040	6.5526	50.1891	4.8849
	1.3	1.9811	92.9467	9.6409	6.5257	49.7894	4.8664
	1.5	1.9939	93.5507	9.6722	6.5031	49.4539	4.8510
	1.7	2.0050	94.0761	9.6993	6.4836	49.1655	4.8376
	1.9	2.0148	94.5408	9.7232	6.4665	48.9133	4.8259
0.5	0.1	1.4578	50.4701	7.1042	8.2450	82.3898	4.8733
	0.3	2.3598	82.4524	9.0803	6.3470	49.4433	3.8480
	0.5	2.9219	102.4983	10.1241	5.6334	39.2753	3.4649
	0.7	3.3487	117.7563	10.8516	5.2132	33.8569	3.2406
	0.9	3.6980	130.2692	11.4136	4.9231	30.3613	3.0864
	1.1	3.9961	140.9588	11.8726	4.7052	27.8671	2.9710
	1.3	4.2572	150.3324	12.2610	4.5327	25.9724	2.8800
	1.5	4.4902	158.7031	12.5977	4.3910	24.4700	2.8056
	1.7	4.7009	166.2796	12.8949	4.2716	23.2408	2.7431
	1.9	4.8934	173.2093	13.1609	4.1690	22.2111	2.6895
0.7	0.1	0.7941	21.3870	4.6246	12.1315	182.4239	5.8235
	0.3	1.5745	44.1227	6.6425	8.3631	87.3592	4.2187
	0.5	2.1387	61.0738	7.8150	7.0517	62.5090	3.6541
	0.7	2.6029	75.2441	8.6743	6.3087	50.3173	3.3326
	0.9	3.0049	87.6387	9.3616	5.8087	42.8810	3.1155
	1.1	3.3629	98.7576	9.9377	5.4402	37.7957	2.9551
	1.3	3.6875	108.8964	10.4353	5.1527	34.0623	2.8299
	1.5	3.9857	118.2487	10.8742	4.9197	31.1851	2.7283
	1.7	4.2622	126.9505	11.2672	4.7252	28.8881	2.6435
	1.9	4.5204	135.1015	11.6233	4.5596	27.0045	2.5713
1.2	0.1	0.2265	1.9028	1.3794	32.8968	1535.0000	6.0892
	0.3	0.6214	6.8189	2.6113	17.6305	435.3168	4.2021
	0.5	0.9829	12.1842	3.4906	13.2882	245.7623	3.5513
	0.7	1.3231	17.7402	4.2119	11.0619	169.7207	3.1833
	0.9	1.6473	23.3874	4.8360	9.6607	129.1978	2.9358
	1.1	1.9584	29.0724	5.3919	8.6791	104.1739	2.7533
	1.3	2.2584	34.7655	5.8962	7.9431	87.2347	2.6108
	1.5	2.5488	40.4462	6.3597	7.3657	75.0339	2.4952
	1.7	2.8306	46.1009	6.7898	6.8973	65.8398	2.3987
	1.9	3.1046	51.7203	7.1917	6.5076	58.6696	2.3165

$$F_{\tau_i} = \frac{-\left[1-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}\right]\left[1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right]}{\left[\tau-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}\right]^{2}}, \tag{14}$$

$$F_{\eta_i} = \frac{-\tau^{2}(\tau-1)\eta^{-\vartheta-1}\vartheta z_i^{\vartheta}\left[1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right]^{-2}}{\left[\tau-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}\right]^{2}}, \tag{15}$$

and

$$F_{\vartheta_i}=\frac{\tau(\tau-1)(\frac{z_i}{\eta})^{\vartheta}\left[1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right]^{-2}\log(\frac{z_i}{\eta})}{\left[\tau-\left(1+\left(\frac{z_{(i)}}{\eta}\right)^{\vartheta}\right)^{-1}\right]^{2}}. \tag{16}$$

The needed steps are finding partial derivative and equating these partial derivative with zero then solve them simultaneously but as this system non-expressible in closed form, we use numerical methods and depend on R software. Gaining benefits from (13) to (16) the following partial derivative arises

$$\frac{\partial LSE(\tau,\eta,\vartheta)}{\partial\tau}=2\sum_{i=1}^{n}F_{\tau_i}\left[F_i-\frac{i}{n+1}\right],$$

$$\frac{\partial LSE(\tau,\eta,\vartheta)}{\partial\eta}=2\sum_{i=1}^{n}F_{\eta_i}\left[F_i-\frac{i}{n+1}\right],$$

and

$$\frac{\partial LSE(\tau,\eta,\vartheta)}{\partial\vartheta}=2\sum_{i=1}^{n}F_{\vartheta_i}\left[F_i-\frac{i}{n+1}\right].$$

4.3 Weighted Least Squares Method

WLSE was introduced by [45], minimizes the WLSE function to get the estimates of the parameters, the weighted least square function is defined by,

$$WLSE(\tau,\eta,\vartheta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(z_{(i)},\tau,\eta,\vartheta)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{\tau\left[1-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z}{\eta}\right)^{\vartheta}\right)}-\frac{i}{n+1}\right]^2.$$

Doing same steps done before in LSE, the following partial derivatives are

$$\frac{\partial WLSE(\tau,\eta,\vartheta)}{\partial\tau}=2\sum_{i=1}^{n}F_{\tau_i}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F_i-\frac{i}{n+1}\right],$$

$$\frac{\partial WLSE(\tau,\eta,\vartheta)}{\partial\eta}=2\sum_{i=1}^{n}F_{\eta_i}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F_i-\frac{i}{n+1}\right],$$

and

$$\frac{\partial WLSE(\tau,\eta,\vartheta)}{\partial\vartheta}=2\sum_{i=1}^{n}F_{\vartheta_i}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F_i-\frac{i}{n+1}\right].$$

4.4 Cramer von Mises Minimum Distance Estimation Method

CME finds the estimates of τ, η and ϑ by minimizing Cramer von Mises minimum distance estimation function , and this function is

$$CME(\tau,\eta,\vartheta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(z_{(i)},\tau,\eta,\vartheta)-\frac{2i-1}{2n}\right]^2=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{\tau\left[1-\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_{n+i-1}}{\eta}\right)^{\vartheta}\right)}-\frac{2i-1}{2n}\right]^2. \tag{17}$$

Using the abbreviations from (13) to (16) the following partial derivatives arises,

$$\frac{\partial CME(\tau,\eta,\vartheta)}{\partial\tau} = 2\sum_{i=1}^{n} F_{\tau_i}\left[F_i - \frac{2i-1}{2n}\right]$$

$$\frac{\partial CME(\tau,\eta,\vartheta)}{\partial\eta} = 2\sum_{i=1}^{n} F_{\eta_i}\left[F_i - \frac{2i-1}{2n}\right],$$

and

$$\frac{\partial CME(\tau,\eta,\vartheta)}{\partial\vartheta} = 2\sum_{i=1}^{n} F_{\vartheta_i}\left[F_i - \frac{2i-1}{2n}\right].$$

These equations requiring numerical solutions, so we use R software to solve them simultaneously to get the estimates of τ, η and ϑ

4.5 Anderson-Darling Method

Anderson-Darling Estimation (ADE) Method introduced by ade , to get the estimates of τ, η and ϑ we minimize the Anderson-Darling function with respect to τ,ηandϑ, this function is:

$$A(\tau,\eta,\vartheta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log(F(z_{(i)},\tau,\eta,\vartheta)) + \log(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))\right]$$

$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(\frac{\tau\left[1-\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)}\right) + \log\left(1-\frac{\tau\left[1-\left(1+\left(\frac{z_{n+i-1}}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_{n+i-1}}{\eta}\right)^{\vartheta}\right)}\right)\right].$$

To get the estimates of τ,ηandϑ we first find the partial derivatives of $A(\tau,\eta,\vartheta)$ using the abbreviations from (13) to (16) to get the following:

$$\frac{\partial A(\tau,\eta,\vartheta)}{\partial\tau} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\tau_i}}{(F(z_{(i)},\tau,\eta,\vartheta))} - \frac{F_{\tau_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right],$$

$$\frac{\partial A(\tau,\eta,\vartheta)}{\partial\eta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\eta_i}}{(F(z_{(i)},\tau,\eta,\vartheta))} - \frac{F_{\eta_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right],$$

and

$$\frac{\partial A(\tau,\eta,\vartheta)}{\partial\vartheta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\vartheta_i}}{(F(z_{(i)},\tau,\eta,\vartheta))} - \frac{F_{\vartheta_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right].$$

4.6 The Right - Tail Anderson-Darling Estimation Method

The Right - tail Anderson-Darling Method (RADE) gets the estimates of τ, η and ϑ by minimizing the following function

$$RADE(\tau,\eta,\vartheta) = \frac{n}{2} - 2\sum_{i=1}^{n}F(z_{(i)},\tau,\eta,\vartheta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))\right]$$

$$= \frac{n}{2} - 2\sum_{i=1}^{n}\frac{\tau\left[1-\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_i}{\eta}\right)^{\vartheta}\right)} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1-\frac{\tau\left[1-\left(1+\left(\frac{z_{n+i-1}}{\eta}\right)^{\vartheta}\right)^{-1}\right]}{\tau-\left(1+\left(\frac{z_{n+i-1}}{\eta}\right)^{\vartheta}\right)}\right)\right].$$

Doing the usual steps of finding the partial derivatives of $RADE$ with the help of the abbreviations from (13) to (16)

$$\frac{\partial RADE(\tau,\eta,\vartheta)}{\partial\tau} = -2\sum_{i=1}^{n} F_{\tau_i} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\tau_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right],$$

$$\frac{\partial RADE(\tau,\eta,\vartheta)}{\partial\eta} = -2\sum_{i=1}^{n} F_{\eta_i} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\eta_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right],$$

and

$$\frac{\partial RADE(\tau,\eta,\vartheta)}{\partial\vartheta} = -2\sum_{i=1}^{n} F_{\vartheta_i} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{F_{\vartheta_{n+i-1}}}{(1-F(z_{(n+i-1)},\tau,\eta,\vartheta))}\right].$$

5 Numerical Outcomes

In this section we find the numerical study using R software to get the best method of estimation using the previous methods, to estimate (τ,η,ϑ) of the HTLL, we generate random samples of different sizes of $n = (25,50,75,100,125,150,175,200,225,250)$ from HTLL with replication 1000 times with different initial values of (τ,η,ϑ).

The estimates of the mean and mean square error (MSE) exist in tables . These tables show:

- the estimated parameters (τ,η,ϑ) goes closer to their initial values as n increases.
- the values of MSE decreases as n increases.
- To compare between the different methods of estimation use the ranks for all tables which are summarized in tables .

Table 3 simulation data $\tau = 1.3$, $\eta = 0.5$ and $\vartheta = 1.2$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	1.3548	1.3755	1.3699	1.35860	1.36321	1.35996	0.4617	0.4836	0.4787	0.46457	0.47363	0.47854	1.2695	1.1903	1.2058	1.27479	1.22181	1.24402
	Rbias	0.0421	0.0581	0.0538	0.04508	0.04862	0.04613	0.0766	0.0328	0.0427	0.07087	0.05274	0.04293	0.0579	0.0081	0.0049	0.06233	0.01817	0.03668
	MSE	0.0105	0.0153	0.0139	0.01465	0.01194	0.02021	0.0203	0.0219	0.0209	0.02216	0.02020	0.02150	0.0586	0.0639	0.0577	0.07656	0.04806	0.07151
50	Mean	1.3541	1.3639	1.3556	1.34890	1.35330	1.34976	0.4597	0.4743	0.4732	0.46570	0.46914	0.47116	1.2312	1.1923	1.2022	1.23242	1.20919	1.22169
	Rbias	0.0417	0.0492	0.0428	0.03761	0.04100	0.03828	0.0807	0.0514	0.0536	0.06861	0.06171	0.05768	0.026	0.0065	0.0018	0.02702	0.00766	0.01807
	MSE	0.0071	0.01	0.0077	0.00778	0.00749	0.00655	0.0109	0.0122	0.0115	0.01164	0.01107	0.01222	0.0232	0.028	0.0243	0.03123	0.02294	0.02953
75	Mean	1.3519	1.3576	1.3522	1.34747	1.35046	1.34539	0.4589	0.4691	0.4681	0.45887	0.45951	0.46464	1.2209	1.1939	1.2027	1.22878	1.21334	1.21981
	Rbias	0.0399	0.0443	0.0401	0.03651	0.03881	0.03492	0.0822	0.0618	0.0638	0.08227	0.08098	0.07072	0.0174	0.0051	0.0022	0.02398	0.01112	0.01651
	MSE	0.006	0.0073	0.0061	0.00643	0.00569	0.00566	0.0085	0.0092	0.0092	0.00921	0.00830	0.00855	0.0159	0.0186	0.0164	0.02150	0.01611	0.01940
100	Mean	1.3485	1.3557	1.3507	1.34923	1.35274	1.34612	0.4566	0.4612	0.4604	0.45913	0.45712	0.46346	1.217	1.1993	1.2065	1.21576	1.20732	1.21447
	Rbias	0.0373	0.0428	0.039	0.03787	0.04057	0.03548	0.0867	0.0776	0.0792	0.08175	0.08577	0.07308	0.0142	0.0006	0.0054	0.01313	0.00610	0.01206
	MSE	0.0048	0.0066	0.0053	0.00598	0.00536	0.00531	0.0065	0.007	0.0067	0.00704	0.00643	0.00682	0.0108	0.0128	0.0115	0.01434	0.01169	0.01512
125	Mean	1.3513	1.354	1.3513	1.34977	1.35276	1.34709	0.4565	0.4634	0.4614	0.46330	0.46254	0.46658	1.2136	1.1968	1.2038	1.21161	1.20148	1.21081
	Rbias	0.0394	0.0416	0.0395	0.03829	0.04058	0.03622	0.0871	0.0731	0.0773	0.07340	0.07492	0.06684	0.0113	0.0027	0.0032	0.00968	0.00123	0.00901
	MSE	0.0047	0.0059	0.005	0.00544	0.00500	0.00483	0.0057	0.0059	0.0056	0.00605	0.00563	0.00604	0.0088	0.0105	0.0092	0.01121	0.00899	0.01140
150	Mean	1.3501	1.3514	1.3488	1.34746	1.34808	1.34396	0.4542	0.462	0.4602	0.46038	0.46094	0.46403	1.2138	1.1992	1.2051	1.21358	1.20717	1.21115
	Rbias	0.0386	0.0395	0.0375	0.03651	0.03699	0.03382	0.0915	0.076	0.0796	0.07924	0.07811	0.07193	0.0115	0.0006	0.0043	0.01132	0.00597	0.00929
	MSE	0.0043	0.0053	0.0045	0.00481	0.00434	0.00414	0.0052	0.0055	0.0052	0.00587	0.00530	0.00513	0.0073	0.0089	0.0076	0.00869	0.00686	0.00826
175	Mean	1.3507	1.3553	1.3516	1.34786	1.34994	1.34275	0.4571	0.4609	0.4603	0.45678	0.45599	0.46235	1.2087	1.1958	1.2016	1.21399	1.20804	1.21240
	Rbias	0.039	0.0426	0.0397	0.03682	0.03841	0.03289	0.0858	0.0782	0.0794	0.08643	0.08801	0.07530	0.0072	0.0035	0.0013	0.01166	0.00670	0.01033
	MSE	0.0041	0.0053	0.0045	0.00470	0.00429	0.00412	0.0046	0.0051	0.0047	0.00496	0.00474	0.00459	0.0055	0.0071	0.0059	0.00862	0.00691	0.00877
200	Mean	1.3486	1.3533	1.3506	1.35055	1.35169	1.34246	0.4579	0.4606	0.4595	0.45580	0.45557	0.46312	1.2099	1.1995	1.2041	1.20665	1.20164	1.20537
	Rbias	0.0374	0.041	0.0389	0.03889	0.03976	0.03266	0.0842	0.0788	0.0811	0.08840	0.08885	0.07377	0.0083	0.0004	0.0034	0.00555	0.00136	0.00448
	MSE	0.0038	0.0051	0.0043	0.00478	0.00439	0.00368	0.0044	0.0048	0.0045	0.00503	0.00476	0.00428	0.005	0.0065	0.0054	0.00725	0.00592	0.00734
225	Mean	1.349	1.353	1.3513	1.35015	1.35256	1.34416	0.4568	0.4592	0.458	0.45627	0.45546	0.46238	1.2064	1.1964	1.2005	1.20679	1.20219	1.20478
	Rbias	0.0377	0.0407	0.0395	0.03858	0.04043	0.03397	0.0864	0.0815	0.0841	0.08746	0.08907	0.07524	0.0054	0.003	0.0004	0.00566	0.00183	0.00398
	MSE	0.0037	0.0046	0.004	0.00443	0.00434	0.00370	0.004	0.0041	0.0041	0.00470	0.00453	0.00406	0.0046	0.0059	0.005	0.00649	0.00535	0.00668
250	Mean	1.3483	1.3518	1.3501	1.34878	1.35028	1.34296	0.4546	0.4561	0.4555	0.45798	0.45740	0.46341	1.2084	1.201	1.2041	1.20862	1.20470	1.20773
	Rbias	0.0372	0.0398	0.0386	0.03752	0.03868	0.03305	0.0908	0.0878	0.0891	0.08404	0.08520	0.07319	0.007	0.0008	0.0034	0.00718	0.00392	0.00644
	MSE	0.0037	0.0044	0.0039	0.00406	0.00384	0.00334	0.0041	0.0043	0.0043	0.00411	0.00390	0.00366	0.0041	0.0053	0.0045	0.00553	0.00440	0.00538

Table 4 Simulation results at $\tau = 1.3$, $\eta = 0.9$ and $\vartheta = 1.2$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	1.3080	1.3280	1.3265	1.3189	1.3276	1.3277	0.9104	0.9518	0.9411	0.9263	0.9352	0.9248	1.2728	1.2034	1.2153	1.2763	1.2285	1.2651
	Rbias	0.0062	0.0215	0.0204	0.0145	0.0213	0.0213	0.0116	0.0576	0.0457	0.0292	0.0391	0.0275	0.0607	0.0029	0.0127	0.0635	0.0237	0.0543
	MSE	0.0192	0.0255	0.0253	0.0250	0.0257	0.0239	0.0162	0.0198	0.0185	0.0197	0.0172	0.0207	0.0561	0.0644	0.0575	0.0733	0.0491	0.0783
50	Mean	1.3016	1.3083	1.3065	1.3046	1.3103	1.3138	0.9196	0.9469	0.9363	0.9375	0.9401	0.9309	1.2296	1.1935	1.2041	1.2215	1.1996	1.2136
	Rbias	0.0012	0.0064	0.0050	0.0035	0.0079	0.0106	0.0217	0.0521	0.0403	0.0416	0.0445	0.0344	0.0246	0.0054	0.0034	0.0179	0.0003	0.0113
	MSE	0.0104	0.0113	0.0104	0.0115	0.0113	0.0123	0.0072	0.0107	0.0096	0.0111	0.0100	0.0111	0.0215	0.0273	0.0235	0.0302	0.0231	0.0309
75	Mean	1.3013	1.3057	1.3037	1.3018	1.3055	1.3071	0.9220	0.9424	0.9339	0.9339	0.9370	0.9308	1.2225	1.1977	1.2065	1.2187	1.2038	1.2125
	Rbias	0.0010	0.0044	0.0028	0.0014	0.0042	0.0054	0.0245	0.0471	0.0376	0.0376	0.0411	0.0342	0.0188	0.0019	0.0054	0.0156	0.0031	0.0104
	MSE	0.0076	0.0079	0.0077	0.0083	0.0084	0.0083	0.0055	0.0084	0.0069	0.0080	0.0067	0.0078	0.0142	0.0175	0.0150	0.0198	0.0152	0.0194
100	Mean	1.2948	1.2971	1.2966	1.2936	1.2961	1.3004	0.9253	0.9421	0.9341	0.9354	0.9372	0.9267	1.2154	1.1984	1.2047	1.2107	1.2003	1.2079
	Rbias	0.0040	0.0022	0.0026	0.0049	0.0030	0.0003	0.0281	0.0468	0.0379	0.0394	0.0413	0.0297	0.0128	0.0013	0.0039	0.0089	0.0002	0.0065
	MSE	0.0051	0.0058	0.0054	0.0057	0.0055	0.0059	0.0039	0.0070	0.0057	0.0071	0.0059	0.0064	0.0102	0.0129	0.0109	0.0151	0.0120	0.0148
125	Mean	1.2918	1.2919	1.2922	1.2938	1.2954	1.2990	0.9202	0.9375	0.9292	0.9397	0.9408	0.9309	1.2240	1.2076	1.2144	1.2033	1.1966	1.2018
	Rbias	0.0063	0.0062	0.0060	0.0048	0.0035	0.0008	0.0224	0.0416	0.0325	0.0441	0.0453	0.0343	0.0200	0.0063	0.0120	0.0027	0.0028	0.0015
	MSE	0.0047	0.0047	0.0046	0.0050	0.0052	0.0052	0.0037	0.0063	0.0053	0.0062	0.0050	0.0047	0.0086	0.0102	0.0089	0.0098	0.0080	0.0101
150	Mean	1.2920	1.2899	1.2890	1.2895	1.2924	1.2949	0.9239	0.9414	0.9372	0.9355	0.9350	0.9299	1.2129	1.1999	1.2054	1.2140	1.2070	1.2098
	Rbias	0.0062	0.0078	0.0085	0.0081	0.0058	0.0039	0.0266	0.0460	0.0413	0.0394	0.0389	0.0332	0.0107	0.0001	0.0045	0.0116	0.0058	0.0082
	MSE	0.0037	0.0037	0.0037	0.0042	0.0041	0.0045	0.0039	0.0055	0.0049	0.0055	0.0047	0.0048	0.0071	0.0089	0.0077	0.0092	0.0074	0.0093
175	Mean	1.2923	1.2881	1.2884	1.2846	1.2862	1.2898	0.9233	0.9457	0.9383	0.9391	0.9392	0.9295	1.2129	1.2000	1.2054	1.2118	1.2061	1.2118
	Rbias	0.0059	0.0092	0.0089	0.0119	0.0106	0.0079	0.0259	0.0507	0.0426	0.0435	0.0435	0.0327	0.0107	0.0000	0.0045	0.0098	0.0051	0.0099
	MSE	0.0032	0.0032	0.0031	0.0032	0.0032	0.0033	0.0033	0.0053	0.0042	0.0051	0.0040	0.0041	0.0055	0.0070	0.0059	0.0079	0.0062	0.0079
200	Mean	1.2888	1.2861	1.2883	1.2850	1.2853	1.2885	0.9305	0.9469	0.9376	0.9346	0.9371	0.9289	1.2059	1.1965	1.2004	1.2109	1.2054	1.2093
	Rbias	0.0086	0.0107	0.0090	0.0115	0.0113	0.0088	0.0339	0.0521	0.0418	0.0385	0.0413	0.0321	0.0050	0.0029	0.0003	0.0090	0.0045	0.0078
	MSE	0.0029	0.0030	0.0028	0.0031	0.0030	0.0031	0.0031	0.0052	0.0044	0.0054	0.0042	0.0040	0.0048	0.0064	0.0053	0.0065	0.0053	0.0067
225	Mean	1.2895	1.2851	1.2869	1.2846	1.2871	1.2892	0.9278	0.9467	0.9373	0.9422	0.9403	0.9355	1.2072	1.1985	1.2027	1.2023	1.1987	1.1997
	Rbias	0.0081	0.0115	0.0101	0.0118	0.0099	0.0083	0.0308	0.0518	0.0414	0.0469	0.0448	0.0394	0.0060	0.0013	0.0023	0.0019	0.0011	0.0002
	MSE	0.0026	0.0026	0.0025	0.0027	0.0029	0.0027	0.0032	0.0052	0.0041	0.0051	0.0040	0.0040	0.0050	0.0064	0.0054	0.0061	0.0049	0.0062
250	Mean	1.2886	1.2852	1.2860	1.2821	1.2826	1.2860	0.9275	0.9439	0.9374	0.9410	0.9426	0.9333	1.2105	1.2019	1.2058	1.2047	1.2009	1.2041
	Rbias	0.0088	0.0114	0.0108	0.0138	0.0134	0.0107	0.0305	0.0488	0.0416	0.0455	0.0473	0.0370	0.0088	0.0016	0.0049	0.0040	0.0008	0.0034
	MSE	0.0023	0.0024	0.0022	0.0024	0.0024	0.0023	0.0030	0.0046	0.0038	0.0044	0.0038	0.0036	0.0045	0.0055	0.0047	0.0055	0.0045	0.0057

Table 5 Simulation results at $\tau = 1.9$, $\eta = 0.9$ and $\vartheta = 1.8$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	1.9444	2.0051	1.9870	1.3232	1.3153	1.3278	0.8919	0.9036	0.8996	0.9123	0.9344	0.9094	1.8948	1.7891	1.8065	1.9159	1.8450	1.8720
	Rbias	0.0234	0.0553	0.0458	0.0179	0.0117	0.0214	0.0089	0.0040	0.0004	0.0137	0.0382	0.0104	0.0527	0.0060	0.0036	0.0644	0.0250	0.0400
	MSE	0.0343	0.0525	0.0402	0.0772	0.0217	0.0248	0.0294	0.0336	0.0317	0.0236	0.0279	0.0179	0.1168	0.1419	0.1233	0.1901	0.1224	0.1593
50	Mean	1.9588	1.9943	1.9785	1.3067	1.3126	1.3182	0.8966	0.9012	0.9006	0.9121	0.9155	0.9100	1.8454	1.7873	1.8059	1.8600	1.8243	1.8366
	Rbias	0.0310	0.0496	0.0413	0.0051	0.0097	0.0140	0.0037	0.0014	0.0007	0.0134	0.0172	0.0111	0.0252	0.0070	0.0033	0.0334	0.0135	0.0203
	MSE	0.0181	0.0293	0.0256	0.0100	0.0100	0.0109	0.0150	0.0152	0.0150	0.0100	0.0088	0.0101	0.0485	0.0581	0.0511	0.0756	0.0541	0.0701
75	Mean	1.9628	1.9880	1.9786	1.3051	1.3106	1.3152	0.8857	0.8885	0.8874	0.9122	0.9116	0.9073	1.8282	1.7898	1.8033	1.8428	1.8175	1.8249
	Rbias	0.0330	0.0463	0.0413	0.0039	0.0082	0.0117	0.0159	0.0127	0.0140	0.0135	0.0128	0.0081	0.0157	0.0056	0.0018	0.0238	0.0097	0.0138
	MSE	0.0134	0.0223	0.0200	0.0081	0.0074	0.0113	0.0106	0.0105	0.0104	0.0076	0.0069	0.0068	0.0303	0.0384	0.0327	0.0479	0.0353	0.0432
100	Mean	1.9659	1.9868	1.9764	1.3035	1.3073	1.3070	0.8828	0.8838	0.8839	0.9073	0.9057	0.9063	1.8218	1.7931	1.8041	1.8277	1.8112	1.8182
	Rbias	0.0347	0.0457	0.0402	0.0027	0.0056	0.0054	0.0191	0.0180	0.0179	0.0081	0.0064	0.0070	0.0121	0.0038	0.0023	0.0154	0.0062	0.0101
	MSE	0.0116	0.0200	0.0162	0.0058	0.0048	0.0056	0.0084	0.0080	0.0080	0.0064	0.0055	0.0059	0.0242	0.0312	0.0261	0.0340	0.0267	0.0344
125	Mean	1.9628	1.9862	1.9807	1.3034	1.3067	1.3043	0.8793	0.8791	0.8776	0.9025	0.9008	0.9037	1.8207	1.7970	1.8070	1.8238	1.8130	1.8203
	Rbias	0.0331	0.0454	0.0425	0.0026	0.0051	0.0033	0.0230	0.0232	0.0249	0.0028	0.0009	0.0041	0.0115	0.0017	0.0039	0.0132	0.0072	0.0113
	MSE	0.0107	0.0176	0.0148	0.0043	0.0043	0.0047	0.0059	0.0058	0.0059	0.0051	0.0045	0.0052	0.0181	0.0232	0.0194	0.0273	0.0216	0.0257
150	Mean	1.9639	1.9841	1.9762	1.3005	1.3054	1.3064	0.8837	0.8828	0.8828	0.9068	0.9032	0.9019	1.8212	1.8027	1.8100	1.8224	1.8118	1.8147
	Rbias	0.0336	0.0443	0.0401	0.0004	0.0041	0.0049	0.0181	0.0192	0.0191	0.0076	0.0035	0.0021	0.0118	0.0015	0.0056	0.0124	0.0066	0.0082
	MSE	0.0095	0.0155	0.0125	0.0032	0.0033	0.0036	0.0048	0.0044	0.0046	0.0055	0.0039	0.0040	0.0161	0.0199	0.0170	0.0210	0.0167	0.0211
175	Mean	1.9660	1.9819	1.9770	1.2991	1.3008	1.3045	0.8837	0.8833	0.8829	0.9067	0.9063	0.9005	1.8163	1.7991	1.8066	1.8186	1.8098	1.8184
	Rbias	0.0348	0.0431	0.0405	0.0007	0.0006	0.0034	0.0181	0.0185	0.0190	0.0074	0.0070	0.0005	0.0091	0.0005	0.0037	0.0103	0.0054	0.0102
	MSE	0.0090	0.0147	0.0122	0.0037	0.0032	0.0036	0.0049	0.0046	0.0047	0.0042	0.0043	0.0046	0.0131	0.0165	0.0140	0.0186	0.0148	0.0194
200	Mean	1.9694	1.9847	1.9807	1.3024	1.3059	1.3059	0.8806	0.8796	0.8793	0.9059	0.9028	0.9017	1.8100	1.7971	1.8024	1.8148	1.8082	1.8126
	Rbias	0.0365	0.0446	0.0425	0.0019	0.0046	0.0046	0.0216	0.0226	0.0230	0.0065	0.0031	0.0019	0.0056	0.0016	0.0013	0.0082	0.0046	0.0070
	MSE	0.0095	0.0152	0.0133	0.0027	0.0028	0.0026	0.0040	0.0038	0.0037	0.0039	0.0032	0.0037	0.0124	0.0160	0.0136	0.0165	0.0132	0.0153
225	Mean	1.9683	1.9810	1.9793	1.3018	1.3043	1.3033	0.8800	0.8795	0.8784	0.9035	0.9017	0.9023	1.8131	1.8019	1.8073	1.8100	1.8041	1.8081
	Rbias	0.0360	0.0427	0.0417	0.0014	0.0033	0.0026	0.0222	0.0228	0.0240	0.0039	0.0019	0.0025	0.0073	0.0011	0.0040	0.0056	0.0023	0.0045
	MSE	0.0090	0.0138	0.0116	0.0026	0.0026	0.0026	0.0036	0.0034	0.0035	0.0030	0.0027	0.0029	0.0107	0.0149	0.0119	0.0141	0.0115	0.0144
250	Mean	1.9699	1.9866	1.9815	1.3020	1.3057	1.3062	0.8824	0.8813	0.8810	0.9054	0.9013	0.8999	1.8111	1.7969	1.8033	1.8079	1.8024	1.8065
	Rbias	0.0368	0.0456	0.0429	0.0015	0.0044	0.0048	0.0195	0.0207	0.0211	0.0060	0.0014	0.0002	0.0062	0.0017	0.0018	0.0044	0.0014	0.0036
	MSE	0.0087	0.0133	0.0113	0.0027	0.0024	0.0025	0.0032	0.0030	0.0030	0.0028	0.0023	0.0025	0.0085	0.0113	0.0092	0.0120	0.0100	0.0123

Table 6 Simulation results at $\tau = 1.9$, $\eta = 1.5$ and $\vartheta = 1.8$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	1.9814	2.0374	2.0284	1.3245	1.3271	1.3334	1.4825	1.4991	1.4946	1.5158	1.5348	1.5275	1.8944	1.7836	1.8015	1.9069	1.8371	1.8619
	Rbias	0.0428	0.0723	0.0676	0.0189	0.0208	0.0257	0.0117	0.0006	0.0036	0.0105	0.0232	0.0184	0.0525	0.0091	0.0008	0.0594	0.0206	0.0344
	MSE	0.0439	0.0559	0.0544	0.0261	0.0235	0.0260	0.0865	0.0939	0.0894	0.0480	0.0277	0.0333	0.1228	0.1289	0.1156	0.1699	0.1178	0.1487
50	Mean	1.9720	2.0062	1.9920	1.3067	1.3074	1.3128	1.4701	1.4775	1.4767	1.5235	1.5378	1.5304	1.8629	1.8051	1.8186	1.8502	1.8162	1.8271
	Rbias	0.0379	0.0559	0.0484	0.0051	0.0057	0.0098	0.0199	0.0150	0.0155	0.0156	0.0252	0.0202	0.0349	0.0028	0.0103	0.0279	0.0090	0.0150
	MSE	0.0216	0.0328	0.0260	0.0134	0.0126	0.0145	0.0432	0.0442	0.0441	0.0241	0.0163	0.0202	0.0542	0.0601	0.0527	0.0692	0.0516	0.0655
75	Mean	1.9748	2.0014	1.9901	1.3002	1.3033	1.3097	1.4644	1.4662	1.4663	1.5335	1.5374	1.5246	1.8307	1.7968	1.8081	1.8392	1.8173	1.8259
	Rbias	0.0394	0.0534	0.0474	0.0001	0.0026	0.0074	0.0237	0.0225	0.0224	0.0224	0.0249	0.0164	0.0170	0.0018	0.0045	0.0218	0.0096	0.0144
	MSE	0.0164	0.0255	0.0206	0.0098	0.0098	0.0106	0.0319	0.0332	0.0328	0.0174	0.0135	0.0140	0.0339	0.0415	0.0363	0.0439	0.0347	0.0446
100	Mean	1.9665	1.9834	1.9743	1.2940	1.2945	1.3015	1.4700	1.4729	1.4722	1.5396	1.5418	1.5306	1.8385	1.8156	1.8242	1.8246	1.8119	1.8196
	Rbias	0.0350	0.0439	0.0391	0.0046	0.0042	0.0011	0.0200	0.0181	0.0186	0.0264	0.0278	0.0204	0.0214	0.0086	0.0134	0.0137	0.0066	0.0109
	MSE	0.0124	0.0193	0.0160	0.0059	0.0056	0.0074	0.0226	0.0235	0.0229	0.0152	0.0106	0.0123	0.0251	0.0340	0.0282	0.0303	0.0242	0.0298
125	Mean	1.9697	1.9899	1.9799	1.2855	1.2868	1.2911	1.4728	1.4739	1.4739	1.5373	1.5396	1.5321	1.8271	1.8008	1.8119	1.8294	1.8165	1.8222
	Rbias	0.0367	0.0473	0.0421	0.0111	0.0102	0.0069	0.0181	0.0174	0.0174	0.0248	0.0264	0.0214	0.0151	0.0005	0.0066	0.0164	0.0091	0.0123
	MSE	0.0113	0.0174	0.0149	0.0046	0.0041	0.0056	0.0190	0.0195	0.0194	0.0121	0.0101	0.0106	0.0203	0.0260	0.0222	0.0251	0.0207	0.0258
150	Mean	1.9684	1.9845	1.9751	1.2816	1.2840	1.2885	1.4680	1.4691	1.4698	1.5464	1.5448	1.5359	1.8222	1.7995	1.8094	1.8175	1.8081	1.8111
	Rbias	0.0360	0.0445	0.0395	0.0142	0.0123	0.0088	0.0213	0.0206	0.0201	0.0309	0.0298	0.0240	0.0124	0.0003	0.0052	0.0097	0.0045	0.0062
	MSE	0.0106	0.0159	0.0131	0.0037	0.0034	0.0040	0.0153	0.0160	0.0159	0.0118	0.0085	0.0083	0.0172	0.0215	0.0183	0.0195	0.0162	0.0198
175	Mean	1.9721	1.9828	1.9797	1.2842	1.2843	1.2879	1.4640	1.4659	1.4641	1.5492	1.5518	1.5437	1.8160	1.8003	1.8070	1.8114	1.8022	1.8084
	Rbias	0.0380	0.0436	0.0420	0.0122	0.0121	0.0093	0.0240	0.0228	0.0240	0.0328	0.0345	0.0291	0.0089	0.0002	0.0039	0.0063	0.0012	0.0047
	MSE	0.0100	0.0141	0.0120	0.0036	0.0033	0.0037	0.0140	0.0144	0.0142	0.0109	0.0091	0.0095	0.0132	0.0178	0.0146	0.0177	0.0143	0.0172
200	Mean	1.9742	1.9860	1.9810	1.2833	1.2846	1.2872	1.4703	1.4712	1.4704	1.5456	1.5473	1.5442	1.8103	1.7959	1.8023	1.8159	1.8064	1.8076
	Rbias	0.0391	0.0452	0.0426	0.0129	0.0118	0.0098	0.0198	0.0192	0.0198	0.0304	0.0315	0.0295	0.0057	0.0023	0.0013	0.0088	0.0036	0.0042
	MSE	0.0100	0.0130	0.0112	0.0027	0.0026	0.0032	0.0113	0.0125	0.0123	0.0103	0.0078	0.0085	0.0117	0.0151	0.0128	0.0165	0.0128	0.0158
225	Mean	1.9680	1.9821	1.9747	1.2833	1.2823	1.2869	1.4743	1.4738	1.4743	1.5523	1.5560	1.5456	1.8146	1.7997	1.8063	1.8061	1.7992	1.8030
	Rbias	0.0358	0.0432	0.0393	0.0129	0.0136	0.0101	0.0171	0.0175	0.0172	0.0349	0.0373	0.0304	0.0081	0.0002	0.0035	0.0034	0.0005	0.0016
	MSE	0.0087	0.0126	0.0099	0.0025	0.0023	0.0026	0.0105	0.0110	0.0107	0.0108	0.0080	0.0074	0.0104	0.0131	0.0111	0.0137	0.0110	0.0139
250	Mean	1.9681	1.9812	1.9764	1.2837	1.2845	1.2877	1.4673	1.4672	1.4663	1.5493	1.5505	1.5421	1.8112	1.7967	1.8035	1.8089	1.8021	1.8073
	Rbias	0.0358	0.0427	0.0402	0.0125	0.0119	0.0095	0.0218	0.0219	0.0224	0.0329	0.0337	0.0281	0.0062	0.0019	0.0020	0.0049	0.0012	0.0041
	MSE	0.0086	0.0118	0.0103	0.0023	0.0023	0.0023	0.0098	0.0099	0.0099	0.0093	0.0074	0.0073	0.0093	0.0123	0.0101	0.0124	0.0102	0.0131

Table 7 Simulation results at $\tau = 1.9$, $\eta = 1.5$ and $\vartheta = 2.4$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	1.9572	2.0378	2.0179	1.3585	1.3542	1.3767	1.4997	1.5056	1.5058	1.4708	1.4845	1.4481	2.5593	2.5579	2.3911	2.4216	2.4585	2.5213
	Rbias	0.0301	0.0725	0.0621	0.0450	0.0417	0.0590	0.0002	0.0038	0.0039	0.0195	0.0103	0.0346	0.0664	0.0658	0.0037	0.0090	0.0244	0.0505
	MSE	0.0516	0.0706	0.0656	0.0414	0.0256	0.0313	0.0481	0.0474	0.0474	0.0520	0.0374	0.0421	0.3390	0.2185	0.2114	0.1958	0.2107	0.3142
50	Mean	1.9753	2.0221	2.0089	1.3402	1.3441	1.3619	1.4788	1.4816	1.4795	1.4627	1.4675	1.4434	2.4780	2.4546	2.3801	2.4040	2.4347	2.4559
	Rbias	0.0396	0.0643	0.0573	0.0309	0.0339	0.0476	0.0141	0.0123	0.0137	0.0249	0.0217	0.0377	0.0325	0.0227	0.0083	0.0017	0.0144	0.0233
	MSE	0.0324	0.0518	0.0432	0.0129	0.0131	0.0204	0.0243	0.0248	0.0237	0.0275	0.0233	0.0262	0.1370	0.0909	0.1176	0.1024	0.1000	0.1292
75	Mean	1.9740	2.0114	1.9947	1.3336	1.3380	1.3471	1.4785	1.4795	1.4787	1.4529	1.4536	1.4407	2.4513	2.4498	2.4004	2.4168	2.4228	2.4380
	Rbias	0.0389	0.0586	0.0498	0.0259	0.0293	0.0362	0.0143	0.0137	0.0142	0.0314	0.0310	0.0395	0.0214	0.0207	0.0002	0.0070	0.0095	0.0159
	MSE	0.0239	0.0384	0.0319	0.0057	0.0059	0.0094	0.0162	0.0160	0.0155	0.0221	0.0187	0.0183	0.0718	0.0589	0.0733	0.0623	0.0554	0.0692
100	Mean	1.9754	2.0070	1.9981	1.3350	1.3392	1.3457	1.4816	1.4797	1.4789	1.4604	1.4586	1.4481	2.4326	2.4290	2.3908	2.4053	2.4091	2.4215
	Rbias	0.0397	0.0563	0.0516	0.0269	0.0301	0.0352	0.0122	0.0136	0.0141	0.0264	0.0276	0.0346	0.0136	0.0121	0.0038	0.0022	0.0038	0.0090
	MSE	0.0191	0.0299	0.0254	0.0057	0.0060	0.0071	0.0119	0.0112	0.0109	0.0191	0.0161	0.0168	0.0609	0.0420	0.0503	0.0440	0.0469	0.0594
125	Mean	1.9882	2.0079	1.9989	1.3362	1.3361	1.3411	1.4669	1.4687	1.4680	1.4503	1.4533	1.4480	2.4320	2.4279	2.3892	2.4056	2.4161	2.4211
	Rbias	0.0464	0.0568	0.0521	0.0278	0.0278	0.0316	0.0221	0.0209	0.0213	0.0332	0.0311	0.0347	0.0133	0.0116	0.0045	0.0023	0.0067	0.0088
	MSE	0.0201	0.0283	0.0237	0.0043	0.0037	0.0056	0.0108	0.0103	0.0103	0.0156	0.0136	0.0140	0.0435	0.0352	0.0421	0.0362	0.0339	0.0425
150	Mean	1.9896	2.0028	2.0025	1.3335	1.3372	1.3361	1.4701	1.4730	1.4696	1.4557	1.4524	1.4521	2.4259	2.4235	2.3932	2.4058	2.4102	2.4259
	Rbias	0.0471	0.0541	0.0540	0.0258	0.0286	0.0278	0.0199	0.0180	0.0203	0.0295	0.0317	0.0319	0.0108	0.0098	0.0028	0.0024	0.0043	0.0108
	MSE	0.0188	0.0254	0.0226	0.0036	0.0035	0.0040	0.0084	0.0076	0.0079	0.0139	0.0117	0.0136	0.0393	0.0266	0.0329	0.0278	0.0307	0.0421
175	Mean	1.9876	2.0030	1.9982	1.3364	1.3356	1.3372	1.4721	1.4720	1.4711	1.4540	1.4568	1.4546	2.4214	2.4181	2.3964	2.4063	2.4095	2.4134
	Rbias	0.0461	0.0542	0.0517	0.0280	0.0274	0.0286	0.0186	0.0187	0.0193	0.0307	0.0288	0.0303	0.0089	0.0075	0.0015	0.0026	0.0039	0.0056
	MSE	0.0165	0.0233	0.0199	0.0037	0.0030	0.0034	0.0074	0.0067	0.0068	0.0125	0.0103	0.0113	0.0311	0.0231	0.0295	0.0249	0.0256	0.0315
200	Mean	1.9864	2.0049	1.9990	1.3357	1.3388	1.3364	1.4733	1.4719	1.4716	1.4549	1.4513	1.4540	2.4118	2.4196	2.3973	2.4079	2.4022	2.4092
	Rbias	0.0455	0.0552	0.0521	0.0274	0.0298	0.0280	0.0178	0.0187	0.0190	0.0301	0.0324	0.0306	0.0049	0.0082	0.0011	0.0033	0.0009	0.0038
	MSE	0.0150	0.0218	0.0185	0.0033	0.0031	0.0032	0.0065	0.0062	0.0063	0.0112	0.0094	0.0102	0.0293	0.0205	0.0251	0.0218	0.0234	0.0292
225	Mean	1.9904	2.0014	1.9971	1.3338	1.3370	1.3351	1.4670	1.4678	1.4672	1.4507	1.4478	1.4509	2.4163	2.4109	2.3905	2.3988	2.4093	2.4114
	Rbias	0.0476	0.0534	0.0511	0.0260	0.0285	0.0270	0.0220	0.0214	0.0219	0.0329	0.0348	0.0327	0.0068	0.0045	0.0040	0.0005	0.0039	0.0047
	MSE	0.0153	0.0210	0.0184	0.0028	0.0028	0.0026	0.0065	0.0058	0.0061	0.0103	0.0094	0.0106	0.0235	0.0193	0.0235	0.0203	0.0192	0.0237
250	Mean	1.9912	2.0013	1.9995	1.3351	1.3373	1.3357	1.4662	1.4668	1.4655	1.4495	1.4488	1.4507	2.4143	2.4125	2.3941	2.4018	2.4063	2.4073
	Rbias	0.0480	0.0533	0.0524	0.0270	0.0287	0.0274	0.0225	0.0222	0.0230	0.0336	0.0341	0.0328	0.0060	0.0052	0.0025	0.0007	0.0026	0.0030
	MSE	0.0148	0.0195	0.0170	0.0026	0.0027	0.0024	0.0058	0.0052	0.0054	0.0094	0.0088	0.0090	0.0219	0.0164	0.0202	0.0172	0.0180	0.0212

Table 8 Simulation results at $\tau = 1.9$, $\eta = 2.5$ and $\vartheta = 2.4$

n		τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
25	Mean	2.0160	2.0903	2.0677	1.3357	1.3426	1.3344	2.4531	2.4646	2.4660	2.4890	2.5134	2.5061	2.5196	2.3745	2.4011	2.5794	2.4750	2.5326
	Rbias	0.0610	0.1002	0.0883	0.0274	0.0327	0.0264	0.0188	0.0142	0.0136	0.0044	0.0054	0.0024	0.0498	0.0106	0.0005	0.0748	0.0313	0.0553
	MSE	0.0642	0.0981	0.0906	0.0396	0.0565	0.0296	0.1359	0.1425	0.1383	0.0849	0.0577	0.0608	0.2171	0.2529	0.2221	0.3206	0.1999	0.2863
50	Mean	2.0057	2.0522	2.0350	1.3121	1.3139	1.3213	2.4463	2.4500	2.4514	2.5300	2.5374	2.5195	2.4635	2.3799	2.4060	2.4855	2.4373	2.4631
	Rbias	0.0556	0.0801	0.0711	0.0093	0.0107	0.0163	0.0215	0.0200	0.0194	0.0120	0.0150	0.0078	0.0265	0.0084	0.0025	0.0356	0.0155	0.0263
	MSE	0.0383	0.0558	0.0508	0.0178	0.0154	0.0212	0.0719	0.0738	0.0754	0.0524	0.0372	0.0383	0.0918	0.1119	0.0984	0.1462	0.1056	0.1361
75	Mean	1.9982	2.0284	2.0177	1.2951	1.2954	1.3042	2.4511	2.4567	2.4534	2.5565	2.5653	2.5373	2.4478	2.3941	2.4127	2.4300	2.4037	2.4253
	Rbias	0.0517	0.0676	0.0619	0.0038	0.0036	0.0032	0.0195	0.0173	0.0186	0.0226	0.0261	0.0149	0.0199	0.0025	0.0053	0.0125	0.0015	0.0105
	MSE	0.0290	0.0427	0.0368	0.0090	0.0085	0.0096	0.0492	0.0530	0.0497	0.0340	0.0246	0.0255	0.0630	0.0813	0.0694	0.0671	0.0537	0.0730
100	Mean	1.9996	2.0285	2.0192	1.2904	1.2908	1.2999	2.4604	2.4603	2.4588	2.5499	2.5583	2.5337	2.4212	2.3783	2.3960	2.4482	2.4247	2.4359
	Rbias	0.0524	0.0676	0.0627	0.0074	0.0071	0.0000	0.0158	0.0159	0.0165	0.0200	0.0233	0.0135	0.0088	0.0090	0.0017	0.0201	0.0103	0.0150
	MSE	0.0230	0.0361	0.0309	0.0061	0.0057	0.0068	0.0352	0.0352	0.0363	0.0260	0.0211	0.0229	0.0396	0.0535	0.0445	0.0653	0.0497	0.0642
125	Mean	1.9910	2.0231	2.0103	1.2884	1.2907	1.2971	2.4596	2.4553	2.4569	2.5555	2.5595	2.5433	2.4266	2.3922	2.4053	2.4293	2.4123	2.4198
	Rbias	0.0479	0.0648	0.0580	0.0089	0.0071	0.0022	0.0161	0.0179	0.0172	0.0222	0.0238	0.0173	0.0111	0.0033	0.0022	0.0122	0.0051	0.0083
	MSE	0.0197	0.0298	0.0255	0.0050	0.0053	0.0062	0.0302	0.0305	0.0309	0.0216	0.0199	0.0210	0.0368	0.0467	0.0401	0.0467	0.0365	0.0457
150	Mean	1.9922	2.0133	2.0058	1.2840	1.2831	1.2885	2.4527	2.4517	2.4505	2.5575	2.5642	2.5490	2.4259	2.4011	2.4124	2.4201	2.4082	2.4164
	Rbias	0.0485	0.0596	0.0557	0.0123	0.0130	0.0089	0.0189	0.0193	0.0198	0.0230	0.0257	0.0196	0.0108	0.0004	0.0052	0.0084	0.0034	0.0068
	MSE	0.0183	0.0256	0.0229	0.0044	0.0039	0.0044	0.0247	0.0258	0.0258	0.0199	0.0164	0.0174	0.0285	0.0374	0.0314	0.0377	0.0306	0.0371
175	Mean	1.9943	2.0211	2.0118	1.2854	1.2833	1.2903	2.4437	2.4388	2.4391	2.5575	2.5678	2.5484	2.4174	2.3907	2.4029	2.4047	2.3934	2.3978
	Rbias	0.0496	0.0637	0.0588	0.0112	0.0128	0.0075	0.0225	0.0245	0.0243	0.0230	0.0271	0.0194	0.0072	0.0039	0.0012	0.0020	0.0028	0.0009
	MSE	0.0174	0.0251	0.0215	0.0036	0.0030	0.0032	0.0223	0.0221	0.0230	0.0169	0.0151	0.0154	0.0233	0.0293	0.0247	0.0277	0.0227	0.0286
200	Mean	1.9957	2.0142	2.0072	1.2856	1.2850	1.2908	2.4548	2.4512	2.4525	2.5565	2.5624	2.5464	2.4186	2.3991	2.4076	2.4078	2.3994	2.4066
	Rbias	0.0504	0.0601	0.0564	0.0111	0.0116	0.0071	0.0181	0.0195	0.0190	0.0226	0.0249	0.0185	0.0078	0.0004	0.0032	0.0032	0.0002	0.0027
	MSE	0.0170	0.0235	0.0198	0.0026	0.0026	0.0030	0.0184	0.0192	0.0196	0.0166	0.0148	0.0159	0.0202	0.0255	0.0218	0.0273	0.0222	0.0278
225	Mean	1.9924	2.0113	2.0092	1.2831	1.2835	1.2899	2.4494	2.4455	2.4444	2.5576	2.5604	2.5432	2.4181	2.3993	2.4072	2.4043	2.3978	2.3996
	Rbias	0.0486	0.0586	0.0575	0.0130	0.0127	0.0078	0.0202	0.0218	0.0222	0.0230	0.0242	0.0173	0.0075	0.0003	0.0030	0.0018	0.0009	0.0002
	MSE	0.0160	0.0218	0.0204	0.0023	0.0025	0.0024	0.0173	0.0179	0.0185	0.0164	0.0144	0.0168	0.0181	0.0233	0.0197	0.0253	0.0207	0.0246
250	Mean	1.9935	2.0103	2.0055	1.2837	1.2835	1.2887	2.4478	2.4444	2.4438	2.5600	2.5658	2.5525	2.4152	2.3995	2.4062	2.4114	2.4027	2.4066
	Rbias	0.0492	0.0580	0.0556	0.0126	0.0127	0.0087	0.0209	0.0222	0.0225	0.0240	0.0263	0.0210	0.0063	0.0002	0.0026	0.0048	0.0011	0.0028
	MSE	0.0161	0.0210	0.0179	0.0020	0.0021	0.0023	0.0167	0.0177	0.0169	0.0142	0.0145	0.0164	0.0158	0.0213	0.0174	0.0213	0.0173	0.0214

Table 9 Ranks of all tables

initial parameters	n	τ						η						ϑ					
		MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE	MLE	LSE	WLSE	CME	ADE	RTADE
$\tau = 1.3$, $\eta = 0.5$ and $\vartheta = 1.2$	25	1	6	4	2.5	2.5	5	5	3	1.5	6	1.5	4	4	3	1	6	2	5
	50	3	6	5	3	3	1	4	2.5	1	5.5	2.5	5.5	4	3	1.5	6	1.5	5
	75	3.5	6	5	3.5	2	1	5	1.5	3.5	6	1.5	3.5	3.5	3.5	1	6	2	5
	100	1	6	3	4	5	2	5	4	2.5	6	2.5	1	4	2	1	5.5	3	5.5
	125	2	6	4.5	3	4.5	1	5.5	2.5	2.5	5.5	2.5	2.5	4	2	2	5.5	1	5.5
	150	3.5	6	5	3.5	2	1	5	3	3	6	3	1	4.5	3	2	6	1	4.5
	175	2	6	5	4	3	1	2.5	4	2.5	5.5	5.5	1	2	3.5	1	5.5	3.5	5.5
	200	2	6	3	4	5	1	2.5	4	2.5	6	5	1	4	2	2	5.5	2	5.5
	225	1.5	6	3	4	5	1.5	2.5	2.5	4	5.5	5.5	1	3	4	1	6	2	5
	250	2	6	4	4	4	1	4.5	4.5	6	3	2	1	4	2	2	6	2	5
$\tau = 1.3$, $\eta = 0.9$ and $\vartheta = 1.2$	25	1	6	3.5	2	5	3.5	1	6	4.5	3	2	4.5	4	2.5	2.5	6	1	5
	50	1	4.5	2	3	4.5	6	1	5.5	2	5.5	4	3	3.5	3.5	2	5.5	1	5.5
	75	1	4	2	3	5	6	1	6	3.5	5	3.5	2	4	2	2	6	2	5
	100	2	4	1	6	4	4	1	6	2	5	4	3	4	3	2	6	1	5
	125	5	5	1	3	5	2	1	5.5	3	5.5	4	2	4.5	6	4.5	2.5	1	2.5
	150	2	2	4.5	6	2	4.5	1	6	4	5	2.5	2.5	4	2	2	6	2	5
	175	2	3	1	4.5	6	4.5	1	6	3.5	5	3.5	2	4	2	1	5.5	3	5.5
	200	1	4	2	6	4	4	1.5	6	4.5	4.5	3	1.5	2.5	4	1	5.5	2.5	5.5
	225	1	4	2	6	5	3	1	6	3.5	5	3.5	2	4	6	4	4	1	2
	250	1	4	2	5.5	5.5	3	1	6	3	4.5	4.5	2	3.5	2	3.5	5.5	1	5.5
$\tau = 1.3$, $\eta = 0.9$ and $\vartheta = 1.8$	25	3	6	5	4	1	2	3.5	5	2	3.5	6	1	3.5	3.5	1	6	2	5
	50	4	6	5	1	2	3	3.5	6	1	3.5	3.5	3.5	3	3	1	6	3	5
	75	4	6	5	1.5	1.5	3	6	3.5	5	3.5	2	1	3	3	1	6	3	5
	100	4	6	5	2	2	2	6	4.5	4.5	3	1	2	3	3	1	6	3	5
	125	4	6	5	1	2	3	5	4	6	2	1	3	3.5	2	1	6	3.5	5
	150	4	6	5	1	2	3	4.5	4.5	4.5	4.5	1.5	1.5	4	2	2	6	2	5
	175	4	6	5	2.5	1	2.5	5	4	6	1.5	1.5	3	2.5	2.5	1	5.5	4	5.5
	200	4	6	5	1	3	2	6	4.5	4.5	3	1.5	1.5	2.5	4	1	6	2.5	5
	225	4	6	5	2	2	2	5	4	6	3	1	2	3.5	3.5	2	5.5	1	5.5
	250	4	6	5	2	1	3	5	4	6	3	1.5	1.5	4	3	2	5.5	1	5.5
$\tau = 1.3$, $\eta = 1.5$ and $\vartheta = 1.8$	25	4	6	5	2	1	3	6	3.5	3.5	1	3.5	3.5	4	3	1	6	2	5
	50	4	6	5	1.5	1.5	3	6	3.5	3.5	1	3.5	3.5	4.5	2.5	2.5	6	1	4.5
	75	4	6	5	1	2	3	5	6	4	2	3	1	4	2	2	6	2	5
	100	4	6	5	3	1	2	3.5	3.5	3.5	6	3.5	1	4.5	4.5	2.5	6	1	2.5
	125	4	6	5	3	1	2	3.5	3.5	3.5	6	3.5	1	3	4	1.5	6	1.5	5
	150	4	6	5	3	1	2	3.5	5	2	6	3.5	1	4	3	2	5.5	1	5.5
	175	4	6	5	3	1	2	3.5	3.5	3.5	6	3.5	1	3.5	3.5	2	6	1	5
	200	4	6	5	3	1	2	3.5	3.5	3.5	6	3.5	1	3	3	1	6	3	5
	225	4	6	5	2	2	2	1	5	3	6	4	2	3	2	4	5.5	1	5.5
	250	4	6	5	2	2	2	1.5	4.5	4.5	4.5	4.5	1.5	4	3	2	5.5	1	5.5
$\tau = 1.3$, $\eta = 1.5$ and $\vartheta = 2.4$	25	2	6	5	3.5	1	3.5	3	1.5	4	6	1.5	5	6	4.5	2	1	3	4.5
	50	3.5	6	5	1	2	3.5	4	2.5	1	5.5	2.5	5.5	6	2.5	4	1	2.5	5
	75	4	6	5	1	2	3	3	1.5	1.5	6	4	5	6	3.5	3.5	2	1	5
	100	4	6	5	1	2	3	2	2	2	5	4	6	6	3	4	1	2	5
	125	4	6	5	2	1	3	3	1	2	5.5	4	5.5	6	4	3	1.5	1.5	5
	150	4	6	5	1	2	3	2.5	1	2.5	5	4	6	5.5	2	3.5	1	3.5	5.5
	175	4	6	5	2.5	1	2.5	2	1	3	6	4	5	6	3.5	2	1	3.5	5
	200	4	6	5	2	2	2	2	1	3	5	5	5	6	4	3	2	1	5
	225	4	6	5	1.5	3	1.5	3	1	2	5	5	5	5.5	3	4	2	1	5.5
	250	4	6	5	1.5	3	1.5	2.5	1	2.5	6	5	4	6	3	3	1	3	5
$\tau = 1.3$, $\eta = 2.5$ and $\vartheta = 2.4$	25	4	6	5	2	3	1	5	6	4	3	2	1	3.5	3.5	1.5	6	1.5	5
	50	4	6	5	1.5	1.5	3	5	5	5	3	2	1	3	3	1	6	3	5
	75	4	6	5	3	1	2	4.5	4.5	4.5	4.5	2	1	4	4	2	4	1	6
	100	4	6	5	3	1	2	2	4.5	6	4.5	3	1	1.5	3.5	1.5	6	3.5	5
	125	4	6	5	2	2	2	1.5	6	4.5	4.5	3	1.5	3.5	3.5	1.5	6	1.5	5
	150	4	6	5	2	2	2	1.5	3.5	6	5	3.5	1.5	4	2.5	2.5	6	1	5
	175	4	6	5	3	2	1	3.5	5	6	2	3.5	1	5	6	1.5	4	1.5	3
	200	4	6	5	1	3	2	2	5.5	5.5	4	3	1	4	2.5	2.5	6	1	5
	225	4	6	5	2	3	1	2	5	6	3.5	3.5	1	4.5	2	4.5	6	2	2
	250	4	6	5	1	3	2	1.5	5.5	5.5	3	4	1.5	4	2	3	5.5	1	5.5
$\sum Ranks$		195	340.5	262.5	159.5	153.5	149	194	240	219	270	191	146	241	186.5	126	296.5	114	295

Noticing sum of ranks for all estimation methods we find that the sum of ranks of ADE is the smallest sum among all methods, so it is superior than all other methods to estimate τ, η and ϑ.

6 Applications

In this section, we introduce two real data sets to illustrate the importance and potentiality of the HTLL distribution.

data set1: The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported in [46]. The data are as follows: 1.08 1.95 1.05 0.96 1.21 1.2 1.3 1.68 0.77 2.78 1.59 2.15 1.16 5.55 1.71 1 1.08 1.24 1.22 0.74 1.07 1.39 4.02 2.16 2.53 2.54 2.4 2.45 0.1 1.15 1.53 1.09 1.72 2.51 1.46 4.32 2.31 1.76 1.36 1.96 1.83 1.07 1.34 2.54 1.97 0.33 2.02 1.12 0.59 1.6 3.27 0.59 3.61 1.63 1.22 3.47 1.44 0.92 0.56 4.58 1.13 0.72 2.93 2.3 3.42 0.93 2.13 2.22 1 1.02 1.08 0.44.

data set2: we look at statistics on the number of months it takes for renal dialysis patients to get infected, as stated in [47]. The times of infection data set is: (6.5 27.5 12.5 22.5 14.5 21.5 2.5 22.5 3.5 2.5 3.5 4.5 11.5 25.5 3.5 14.5 7.5 12.5 7.5 10.5 7.5 21.5 8.5 9.5 5.5 6.5 13.5 7.5). We now perform a normalization procedure by dividing this data by thirty, resulting in values between 0 and 1. The data set has been changed to: (0.216667 0.75 0.38333 0.716667 0.416667 0.483333 0.08333 0.483333 0.85 0.08333 0.116667 0.116667 0.316667 0.716667 0.116667 0.75 0.25 0.35 0.216667 0.28333 0.25 0.45 0.25 0.25 0.15 0.18333 0.416667 0.916667).

The data sets are utilized to assess the goodness of fit of the HTLL distribution. The suggested model is compared with Burr_X (BX) [48], exponentiated Kavya-Manoharan Burr_X (EKM_BX) [49], Kumaraswamy Burr-III (Kum_BIII) [50], Kavya-Manoharan Burr_X (KM_BX) [51], Kumaraswamy Burr-II (Kum_BII) [50], Kumaraswamy Rayleigh (Kum_R) [50], exponentiated Burr III (E_BIII) [50], Burr III (B_III) [52], and Rayleigh (R) models.

The maximum likelihood estimators (MLEs) and standard errors (SEs) of the model parameters are computed. In order to assess the competitive models, various criteria are taken into account, including the Kolmogorov-Smirnov (KS) test, p-value (P_value) test, the Cramer-Von-Mises test (W), and the Anderson-Darling test (A).

Tables 10 and 11 show the MLEs with SEs also shows the numerical values for the KS, P_value, W and A statistics for the datasets. Figures 3 and 4 shows the estimated pdfs, cdfs, complementary cdf (ccdfs) and PP plots of competitive models for the datasets. The graphs in Figures 3 and 4 demonstrate that the HTLL model fits the both datasets well.

Table 10 Measures of fitting for data set1.

Model	HTLL	KM_Bx	EKM_Bx	Kum_R	Bx	Kum_BIII	R	E_BIII	BIII	Kum_BII
KS	0.0684	0.0868	0.0869	0.0966	0.0966	0.1082	0.1090	0.1121	0.1122	0.1125
P_value	0.8896	0.6499	0.6481	0.5126	0.5125	0.3678	0.3588	0.3259	0.3255	0.3220
W	0.0544	0.1335	0.1338	0.1850	0.1850	0.1035	0.1834	0.1002	0.1002	0.2013
A	0.3830	0.7918	0.7931	1.0871	1.0871	0.6813	1.0779	0.7633	0.7633	1.1718
$\hat{\vartheta}$	3.0121	0.4432	0.4428	0.3435	0.4783	0.0531	0.4890	1.4688	2.3399	2.8863
SE($\hat{\vartheta}$)	0.2971	0.0383	0.0385	4.1823	0.0384	0.0024	0.0288	NaN	0.2152	21.3309
$\hat{\tau}$	1.0021	1.0806	1.1321	0.9362	0.9361	0.0831		2.3398	1.9163	1.7309
SE($\hat{\tau}$)	0.0002	0.1531	0.9066	0.1459	0.1459	0.0098		0.2152	0.2260	0.2869
$\hat{\eta}$	11.8065		0.9466	0.6660		19.6719		1.3048		2.2708
SE($\hat{\eta}$)	2.5528		0.8768	8.1106		NaN		NaN		16.7823
$\hat{\beta}$						1.4081				
SE($\hat{\beta}$)						NaN				

Table 11 Measures of fitting for data set2.

Model	HTLL	KM_Bx	EKM_Bx	Kum_BII	Bx	Kum_R	Kum_BIII	R	E_BIII	BIII
KS	0.1010	0.1226	0.1236	0.1363	0.1366	0.1366	0.1567	0.1974	0.2408	0.2408
P_value	0.9375	0.7939	0.7860	0.6753	0.6728	0.6727	0.4975	0.2253	0.0777	0.0777
W	0.0451	0.0586	0.0582	0.1136	0.0757	0.0757	0.2428	0.0747	0.1041	0.1041
A	0.3834	0.4405	0.4348	0.7050	0.5363	0.5363	1.4534	0.5316	0.6521	0.6521
$\hat{\vartheta}$	2.4285	1.8647	1.9141	0.0021	2.0201	1.4691	0.1644	2.2288	2.4317	168.5345
SE($\hat{\vartheta}$)	0.3762	0.2767	0.2932	0.0002	0.2723	36.2809	0.5083	0.2106	35.5214	109.9480
$\hat{\tau}$	25.2077	0.8620	0.4080	2.0812	0.7483	0.7483	138.5359		143.8583	0.0050
SE($\hat{\tau}$)	472.2860	0.1852	0.6061	0.5789	0.1773	0.1773	91.4053		210.4000	0.0031
$\hat{\eta}$	0.3123		2.4529	58.7150		2.7778	62.5949		0.0024	
SE($\hat{\eta}$)	0.9958		4.2921	92.6985		68.5982	193.6156		0.0005	
$\hat{\beta}$				10.1234						
SE($\hat{\beta}$)				16.0371						

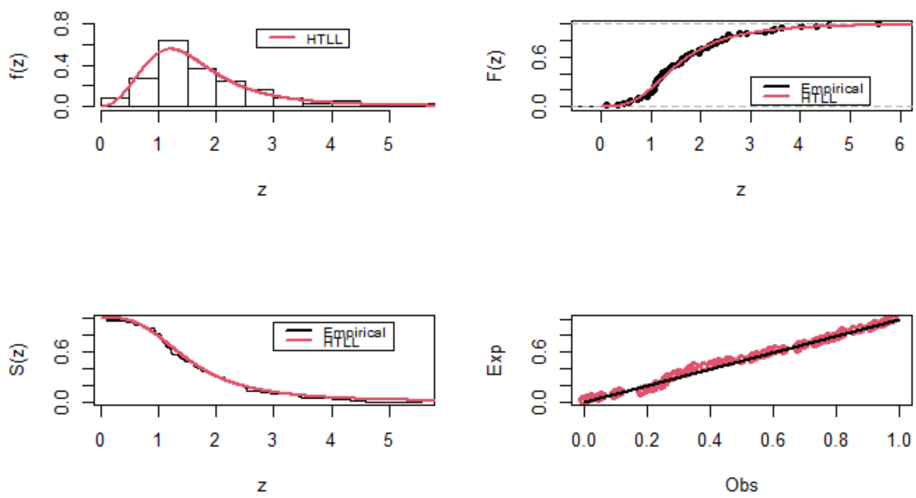

Fig. 3 Estimated pdf, cdf, ccdf and pp plots of data1

7 Concluding Remarks

This article introduces a novel extension of the three-parameter log-logistic distribution, which is referred to as the heavy-tailed log-logistic (HTLL) distribution. We establish a number of significant mathematical features that are associated with the new HTLL distribution. In addition to this, the equations for the HTLL distribution are used to derive various numerical results of moments. Six different estimation methods, such as maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramer-von Mises (CVM), Anderson-Darling (AD), and right-tail Anderson-Darling (RTAD) are computed to estimate the parameter of the HTLL distribution. When compared to various other ways, the findings of the simulation indicate that the AD strategy is the most efficient alternative. The analysis of two actual data sets demonstrates that the recently presented model is applicable in a number of different situations.

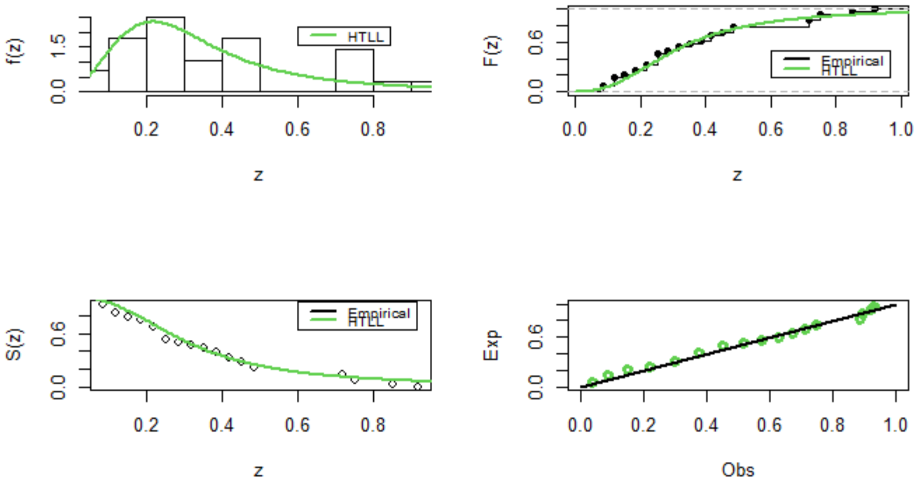

Fig. 4 Estimated pdf, cdf, ccdf and pp plots of data2

Availability of Data and Materials

Any data that supports the fndings of this study is included in the article.

Confict of Interest

The authors declare no competing interests.

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