Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100115

Solving Fredholm and Fractional Integral Equations through Orthogonal Pentagonal Metric Spaces

Poh Soon JosephNg[1,*], *Gunaseelan Mani*[2], *Shoba Sree Ramulu*[3], *Hasanen A. Hammad*[4,5] *and M. Abdel-Aty*[5,6,7]

[1] Faculty of Data Science and Information Technology, INTI International University, Malaysia
[2] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai-602 105,Tamil Nadu, India
[3] Department of Mathematics, S.A. Engineering College, Chennai -600077, Tamil Nadu, India
[4] Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia
[5] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[6] Department of Computer Science and Information Engineering, Chung Hua University, Hsinchu, Taiwan 30012, R.O.C.
[7] Deanship of Graduate Studies and Research, Ahlia University, P.O. Box 10878 Manama, Bahrain

Received: 2 Sep. 2023, Revised: 18 Nov. 2023, Accepted: 20 Dec. 2023
Published online: 1 Jan. 2024

Abstract: In this paper, we introduce a novel mathematical concept called the orthogonal neutrosophic pentagonal metric space. The topological properties of this space are thoroughly examined, and illustrative examples are provided to support its applicability. Moreover, fixed point results under contraction conditions are established using this space, thereby extending and unifying previous findings in a similar vein. Our discoveries significantly contribute to the progress of mathematical theory and its diverse applications across various fields. Ultimately, as an application, the existence and uniqueness of the Fredholm and fractional integral equations are discussed for inclusive innovation.

Keywords: Fixed point technique; orthogonal neutrosophic pentagonal metric space; integral equation, existence solution, inclusive innovation

2010 Mathematics Subject Classification. Primary 26A33, 35A20; Secondary 33B15.

1 Introduction

Fractional differential equations find applications in various scientific disciplines and engineering fields due to their ability to describe non-local and memory-dependent phenomena [1,2,3,4,5,6,7,8,9,10]. One significant application is in the modeling of complex physical systems, such as in physics and mechanics, where fractional differential equations provide a more accurate representation of the dynamics of viscoelastic materials, anomalous diffusion, and fractal phenomena [11,12,13,14,15]. In finance, fractional differential equations are employed to model the behavior of stock prices, option pricing, and risk management, considering the presence of long-range dependence and memory effects. Fractional differential equations also play a crucial role in the study of biological systems, including population dynamics, tumor growth, and the dynamics of infectious diseases, accounting for the non-local interactions and memory effects. Additionally, fractional differential equations are utilized in image processing, signal processing, and data analysis for denoising, image enhancement, and time series analysis, enabling improved performance in handling non-local and non-Markovian data [16,17,18,19,20,21,22].

The notion of MS (metric spaces) and the Banach contraction principle are the backbone of the field of fixed point theory. The axiomatic interpretation of metric space attracts thousands of researchers towards spaciousness. So far, there have been many generalizations on MS. This tells us about the beauty, attraction, and expansion of the notion of MS. The notion of fuzzy sets was presented by Zadeh [23]. The adjective ”fuzzy” seems to be a very popular, and frequent, one in contemporary studies concerning the logical and set-theoretical foundations of mathematics. The main reason for

* Corresponding author e-mail: joseph.ng@newinti.edu.my

this quick improvement So, every formal description of the real world, or some of its aspects, is, in every case, only an approximation and an idealization of the actual state. Notions like fuzzy sets, fuzzy orderings, fuzzy languages, etc., enable us to handle, and to study, the degree of uncertainty mentioned above in a purely mathematical and formal way.

The fuzzy set notion has succeeded in moving many mathematical structures within its concept [24], Schweizer and Sklar [25] defined the concept of continuous $t-$norms. The fuzzy MS concpet was initiated by Kramosil and Michalek [26]. They extended the notion of fuzziness to traditional conceptions of metric and MS via continuous $t-$norms and contrasted the results with those derived from other, particularly probabilistic, statistical generalizations of MS. Garbiec [27] provided the fuzzy interpretation of Banach contraction principle in fuzzy MS. Reham et al. [28] proved some $\alpha-\phi-$fuzzy cone contraction results with integral type application.

Fuzzy MS only deal with membership functions. An intuitionistic fuzzy MS was established by Park [29] that is used to deal with both membership and non-membership functions. Konwar [30] presented the concept of an intuitionistic fuzzy $b-$MS and proved several fixed-point theorems. Kirişci and Simsek [31] introduced the notion of neutrosophic MS that is used to deal with membership, non-membership and naturalness. Simsek and Kirişci [32] proved some amazing fixed-point results in the context of neutrosophic MS. Sowndrarajan et al. [33] proved some fixed point results in the setting of neutrosophic MS. Itoh [34] proved an application regarding random differential equations in Banach spaces. Mlaiki [35] coined the concept of controlled MS and proved several fixed-point results for contraction mappings. Sezen [36] presented the notion of controlled fuzzy MS and proved various contraction mapping results. For related articles, see [37]-[44]. Gordji et al. [45] introduced orthogonal MS and proved fixed point theorem. Gunaseelan, Arul Joseph, Choonkil and Sungsik [46] proved fixed theorems under orthogonal O-contractions on b-complete MS. Senthil Kumar, Arul Joseph, Gunaseelan, and Santosh Kumar [47] proposed orthogonal L^* contraction and proved fixed point theorems on orthogonal branciariMS. Gunaseelan, Arul Joseph, Vidhya and Fahd [48] proposed orthogonal neutrosophic rectangular MS and proved fixed point theorems. Janardhanan, Mani, Ege, Vidhya and George [49] proposed orthogonal neutrosophic 2-MS and proved fixed point theorems. Gunaseelan, Subbarayan, Mitrovic, Aloqaily and Mlaiki [50] introduced neutrosophic pendagonal metric space and prove fixed point theorems.

In this paper, we introduce the notion of orthogonal neutrosophic pentagonal MS ($\mathscr{O}\mathscr{N}\mathscr{P}\mathscr{M}\mathscr{S}$) and we prove fixed point theorems. The main objectives of this paper are as follows:

– To introduce the notion of $\mathscr{O}\mathscr{N}\mathscr{P}\mathscr{M}\mathscr{S}$
– To prove several fixed-point theorems for contraction mappings
– To find the existence and uniqueness solution to Fredholm and fractional integral equations.

2 Preliminaries work

In this part, the researchers provide some definitions that will be useful for readers to understand the main section.

Definition 1.([29]) *A binary operation* $*: [0,1]\times[0,1]\to[0,1]$ *is called a continuous triangle norm if for all* $\rho,\tau,\tilde{n},\beta\in[0,1]$ *s.t.*

(1) $\rho*\tau=\tau*\rho$,
(2) $*$ is continuous,
(3) $\rho*1=\rho$,
(4) $(\rho*\tau)*\tilde{n}=\rho*(\tau*\tilde{n})$,
(5) If $\rho\leq\tilde{n}$ and $\tau\leq\beta$, then $\rho*\tau\leq\tilde{n}*\beta$.

Definition 2.([29]) *A binary operation* $\circ: [0,1]\times[0,1]\to[0,1]$ *is called a continuous triangle co-norm if for all* $\rho,\tau,\tilde{n},\beta\in[0,1]$:

(1) $\rho\circ\tau=\tau\circ\rho$,
(2) $\circ$ is continuous,
(3) $\rho\circ 0=0$,
(4) $(\rho\circ\tau)\circ\tilde{n}=\rho\circ(\tau\circ\tilde{n})$,
(5) If $\rho\leq\tilde{n}$ and $\tilde{n}\leq\beta$, then $\rho\circ\tau\leq\tilde{n}\circ\beta$.

Definition 3.([34]) *Given* $\xi,\Gamma: \Lambda\times\Lambda\to[1,\mathbb{R}^+)$ *are non-comparable functions, if* $\partial: \Lambda\times\Lambda\to[0,\mathbb{R}^+)$ *satisfies the following conditions:*

(a) $\partial(\rho,\Omega)=0$ iff $\rho=\Omega$;
(b) $\partial(\rho,\Omega)=\partial(\Omega,\rho)$;
(c) $\partial(\rho,\Omega)\leq\xi(\rho,\sigma)\partial(\rho,\sigma)+\Gamma(\sigma,\Omega)\partial(\sigma,\Omega)$;

for all $\rho,\Omega,\sigma\in\Lambda$, then, (Λ,∂) is called a double controlled MS.

Definition 4.([37]) *Suppose $\Lambda\neq\emptyset$ and $\xi,\Gamma:\Lambda\times\Lambda\to[1,\mathbb{R}^+)$ are given non-comparable functions, $*$ is a continuous t-norm and Γ is a fuzzy set on $\Lambda\times\Lambda\times(0,+\infty)$ is said to be a fuzzy double controlled metric on Λ, for all $\rho,\Omega,\sigma\in\Lambda$ if:*

(i)$\Gamma(\rho,\Omega,0)=0$;
(ii)$\Gamma(\rho,\Omega,\wp)=1$ for all $\wp>0$, $\iff$ $\rho=\Omega$;
(iii)$\Gamma(\rho,\Omega,\wp)=\Gamma(\Omega,\rho,\wp)$;
(iv)$\Gamma(\rho,\sigma,\wp+\wp)\geq\Gamma\left(\rho,\Omega,\frac{\wp}{\xi(\rho,\Omega)}\right)*\Gamma\left(\Omega,\sigma,\frac{\wp}{\Gamma(\Omega,\sigma)}\right)$;
(v)$\Gamma(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ is left continuous.

Then, $(\Lambda,\Gamma,\Xi,*)$ is said to be a fuzzy double controlled MS.

Definition 5.([30]) *Take $\Lambda\neq\emptyset$. Let $*$ and $\circ$ be a continuous t-norm and continuous t-co-norm, $\mathfrak{b}\geq1$ and Γ,Ξ be fuzzy sets on $\Lambda\times\Lambda\times(0,+\infty)$. If $(\Lambda,\Gamma,\Xi,*,\circ)$ fullfils $\forall\ \rho,\Omega\in\Lambda$ and $\kappa,\wp>0$:*

(I)$\Gamma(\rho,\Omega,\wp)+\Xi(\rho,\Omega,\wp)\leq1$;
(II)$\Gamma(\rho,\Omega,\wp)>0$;
(III)$\Gamma(\rho,\Omega,\wp)=1\Leftrightarrow\rho=\Omega$;
(IV)$\Gamma(\rho,\Omega,\wp)=\Gamma(\Omega,\rho,\wp)$;
(V)$\Gamma(\rho,\sigma,\mathfrak{b}(\wp+\kappa))\geq\Gamma(\rho,\Omega,\wp)*\Gamma(\Omega,\sigma,\kappa)$;
(VI)$\Gamma(\rho,\Omega,\cdot)$ is a increasing mapping of $\mathbb{R}^+$ and $\lim_{\wp\to\mathbb{R}^+}\Gamma(\rho,\Omega,\wp)=1$;
(VII)$\Xi(\rho,\Omega,\wp)>0$;
(VIII)$\Xi(\rho,\Omega,\wp)=0\Leftrightarrow\rho=\Omega$;
(IX)$\Xi(\rho,\Omega,\wp)=\Xi(\Omega,\rho,\wp)$;
(X)$\Xi(\rho,\sigma,\mathfrak{b}(\wp+\kappa))\leq\Xi(\rho,\Omega,\wp)\circ\Xi(\Omega,\sigma,\kappa)$;
(XI)$\Xi(\rho,\Omega,\cdot)$ is a decreasing mapping of $\mathbb{R}^+$ and $\lim_{\wp\to\mathbb{R}^+}\Xi(\rho,\Omega,\wp)=0$,

Then, $(\Lambda,\Gamma,\Xi,*,\circ)$ is an intuitionistic fuzzy $\mathfrak{b}$-MS.

Definition 6.([31]) *Let $\Lambda\neq\emptyset,*$ is a continuous t-norm, $\circ$ be a continuous t-co-norm, and $\Gamma,\Xi,\mathscr{S}$ are neutrosophic sets on $\Lambda\times\Lambda\times(0,+\infty)$ is said to be a neutosophic metric on Λ, if $\forall\ \rho,\Omega,\sigma\in\Lambda$, the following axioms are satisfied:*

(1)$\Gamma(\rho,\Omega,\wp)+\Xi(\rho,\Omega,\wp)+\mathscr{S}(\rho,\Omega,\wp)\leq3$;
(2)$\Gamma(\rho,\Omega,\wp)>0$;
(3)$\Gamma(\rho,\Omega,\wp)=1,\ \forall\ \wp>0$, iff $\rho=\Omega$;
(4)$\Gamma(\rho,\Omega,\wp)=\Gamma(\Omega,\rho,\wp)$;
(5)$\Gamma(\rho,\sigma,\wp+\kappa)\geq\Gamma(\rho,\Omega,\wp)*\Gamma(\Omega,\sigma,\kappa)$;
(6)a continuous map $\Gamma(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ and $\lim_{\wp\to\mathbb{R}^+}\Gamma(\rho,\Omega,\wp)=1$;
(7)$\Xi(\rho,\Omega,\wp)<1$;
(8)$\Xi(\rho,\Omega,\wp)=0$ for all $\wp>0$, iff $\rho=\Omega$;
(9)$\Xi(\rho,\Omega,\wp)=\Xi(\Omega,\rho,\wp)$;
(10)$\Xi(\rho,\sigma,\wp+\kappa)\leq\Xi(\rho,\Omega,\wp)\circ\Xi(\Omega,\sigma,\kappa)$;
(11)a continuous map $\Xi(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ and $\lim_{\wp\to\mathbb{R}^+}\Xi(\rho,\Omega,\wp)=0$;
(12)$\mathscr{S}(\rho,\Omega,\wp)<1$;
(13)$\mathscr{S}(\rho,\Omega,\wp)=0$ for all $\wp>0$, iff $\rho=\Omega$;
(14)$\mathscr{S}(\rho,\Omega,\wp)=\mathscr{S}(\Omega,\rho,\wp)$;
(15)$\mathscr{S}(\rho,\sigma,\wp+\kappa)\leq\mathscr{S}(\rho,\Omega,\wp)\circ\mathscr{S}(\Omega,\sigma,\kappa)$;
(16)$\mathscr{S}(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ is continuous and $\lim_{\wp\to\mathbb{R}^+}\mathscr{S}(\rho,\Omega,\wp)=0$;
(17)If $\wp\leq0$, then $\Gamma(\rho,\Omega,\wp)=0,\Xi(\rho,\Omega,\wp)=0$;

Then, $(\Lambda,\Gamma,\Xi,\mathscr{S},*,\circ)$ is called a neutrosophic MS.

Definition 7.[45] *Let $(\Lambda,\perp)$ be an $\mathscr{O}$-set. A sequence $\{\eta_\pi\}$ is said to be an orthogonal sequence (briefly, $\mathscr{O}$-sequence) if*

$$(\forall\pi\in\mathbb{N},\upsilon_\pi\perp\upsilon_{\pi+1})\quad or\quad(\forall\pi\in\mathbb{N},\upsilon_{\pi+1}\perp\upsilon_\pi).$$

Definition 8.[45] *Let $(\Lambda,\perp)$ be an $\mathscr{O}$-set. A mapping $\omega:\Lambda\to\Lambda$ is said to be $\perp$-preserving if $\omega\upsilon\perp\omega\vartheta$ whenever $\upsilon\perp\vartheta$.*

Definition 9.[50] *Let $\Lambda\neq\emptyset$, $*$ and $\circ$ be a continuous t-norm and t-co-norm, Γ,Ξ,Φ be neutrosophic sets on $\Lambda\times\Lambda\times(0,+\infty)$ is called a neutrosophic pentagonal metric on Λ, if for every $\rho,\sigma\in\Lambda$ and all distinct $\upsilon,\Omega,\eta,\sigma\in\Lambda$ and for all $\wp,\hat{s},\hat{w},\rho>0$, the following conditions are satisfied:*

(i)$\Gamma(\rho,\Omega,\wp)+\Xi(\rho,\Omega,\wp)+\Phi(\rho,\Omega,\wp)\leq 3$;
(ii)$\Gamma(\rho,\Omega,\wp)>0$;
(iii)$\Gamma(\rho,\Omega,\wp)=1$ *for all* $\wp>0$, *iff* $\rho=\Omega$;
(iv)$\Gamma(\rho,\Omega,\wp)=\Gamma(\Omega,\rho,\wp)$;
(v)$\Gamma(\rho,\sigma,\wp+\kappa+\mu+\check{e})\geq\Gamma(\rho,\Omega,\wp)*\Gamma(\Omega,\upsilon,\kappa)*\Gamma(\upsilon,\vartheta,\mu)*\Gamma(\vartheta,\sigma,\check{e})$;
(vi)a continuous map $\Gamma(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *and* $\lim_{\wp\to\mathbb{R}^+}\Gamma(\rho,\Omega,\wp)=1$;
(vii)$\Xi(\rho,\Omega,\wp)<1$;
(viii)$\Xi(\rho,\Omega,\wp)=0$ *for all* $\wp>0$, *iff* $\rho=\Omega$;
(ix)$\Xi(\rho,\Omega,\wp)=\Xi(\Omega,\rho,\wp)$;
(x)$\Xi(\rho,\sigma,\wp+\kappa+\mu+\check{e})\leq\Xi(\rho,\Omega,\wp)\circ\Xi(\Omega,\upsilon,\kappa)\circ\Xi(\upsilon,\vartheta,\mu)\circ\Xi(\vartheta,\sigma,\check{e})$;
(xi)a continuous map $\Xi(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *and* $\lim_{\wp\to\mathbb{R}^+}\Xi(\rho,\Omega,\wp)=0$;
(xii)$\Phi(\rho,\Omega,\wp)<1$;
(xiii)$\Phi(\rho,\Omega,\wp)=0$ *for all* $\wp>0$, *iff* $\rho=\Omega$;
(xiv)$\Phi(\rho,\Omega,\wp)=\Phi(\Omega,\rho,\wp)$;
(xv)$\Phi(\rho,\sigma,\wp+\kappa+\mu+\check{e})\leq\Phi(\rho,\Omega,\wp)\circ\Phi(\Omega,\upsilon,\kappa)\circ\Phi(\upsilon,\vartheta,\mu)\circ\Phi(\vartheta,\sigma,\check{e})$;
(xvi)a continuous map $\Phi(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *and* $\lim_{\wp\to\mathbb{R}^+}\Phi(\rho,\Omega,\wp)=0$;
(xvii)If $\wp\leq 0$, *then* $\Gamma(\rho,\Omega,\wp)=0,\Xi(\rho,\Omega,\wp)=1$ *and* $\mathscr{S}(\rho,\Omega,\wp)=1$.

Then, $(\Lambda,\Gamma,\Xi,\Phi,*,\circ)$ *is said to be neutrosophic pentagonal MS.*

3 Main results

We start this part with the following definition:

Definition 10.*Let* $\Lambda\neq\emptyset$, $*$ *and* $\circ$ *be a continuous t-norm, t-co-norm, and* Γ,Ξ,Φ *be neutrosophic sets on* $\Lambda\times\Lambda\times(0,+\infty)$ *is called a orthogonal neutrosophic pentagonal metric on* Λ, *if for each* $\rho,\sigma\in\Lambda$ *and all distinct* $\upsilon,\Omega,\eta,\sigma\in\Lambda$, *the following conditions are satisfied:*

(i)$\Gamma(\rho,\Omega,\wp)+\Xi(\rho,\Omega,\wp)+\Phi(\rho,\Omega,\wp)\leq 3$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(ii)$\Gamma(\rho,\Omega,\wp)>0$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(iii)$\Gamma(\rho,\Omega,\wp)=1$ *iff* $\rho=\Omega$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(iv)$\Gamma(\rho,\Omega,\wp)=\Gamma(\Omega,\rho,\wp)$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(v)$\Gamma(\rho,\sigma,\wp+\kappa+\mu+\check{e})\geq\Gamma(\rho,\Omega,\wp)*\Gamma(\Omega,\upsilon,\kappa)*\Gamma(\upsilon,\vartheta,\mu)*\Gamma(\vartheta,\sigma,\check{e})$ *for all* $\rho,\Omega\in\Lambda$, $\wp,\hat{s},\hat{w},\rho>0$ *s.t.* $\rho\perp\sigma$, $\rho\perp\Omega$, $\Omega\perp\upsilon$, $\upsilon\perp\eta$ *and* $\eta\perp\sigma$;
(vi)$\Gamma(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\wp\to\mathbb{R}^+}\Gamma(\rho,\Omega,\wp)=1$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(vii)$\Xi(\rho,\Omega,\wp)<1$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(viii)$\Xi(\rho,\Omega,\wp)=0$ *iff* $\rho=\Omega$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(ix)$\Xi(\rho,\Omega,\wp)=\Xi(\Omega,\rho,\wp)$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(x)$\Xi(\rho,\sigma,\wp+\kappa+\mu+\check{e})\leq\Xi(\rho,\Omega,\wp)\circ\Xi(\Omega,\upsilon,\kappa)\circ\Xi(\upsilon,\vartheta,\mu)\circ\Xi(\vartheta,\sigma,\check{e})$ *for all* $\rho,\Omega\in\Lambda$, $\wp,\hat{s},\hat{w},\rho>0$ *s.t.* $\rho\perp\sigma$, $\rho\perp\Omega$, $\Omega\perp\upsilon$, $\upsilon\perp\eta$ *and* $\eta\perp\sigma$;
(xi)$\Xi(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\wp\to\mathbb{R}^+}\Xi(\rho,\Omega,\wp)=0$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(xii)$\Phi(\rho,\Omega,\wp)<1$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(xiii)$\Phi(\rho,\Omega,\wp)=0$ *iff* $\rho=\Omega$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(xiv)$\Phi(\rho,\Omega,\wp)=\Phi(\Omega,\rho,\wp)$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(xv)$\Phi(\rho,\sigma,\wp+\kappa+\mu+\check{e})\leq\Phi(\rho,\Omega,\wp)\circ\Phi(\Omega,\upsilon,\kappa)\circ\Phi(\upsilon,\vartheta,\mu)\circ\Phi(\vartheta,\sigma,\check{e})$ *for all* $\rho,\Omega\in\Lambda$, $\wp,\hat{s},\hat{w},\rho>0$ *s.t.* $\rho\perp\sigma$, $\rho\perp\Omega$, $\Omega\perp\upsilon$, $\upsilon\perp\eta$ *and* $\eta\perp\sigma$;
(xvi)$\Phi(\rho,\Omega,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\wp\to\mathbb{R}^+}\Phi(\rho,\Omega,\wp)=0$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$;
(xvii)If $\wp\leq 0$, *then* $\Gamma(\rho,\Omega,\wp)=0,\Xi(\rho,\Omega,\wp)=1$ *and* $\mathscr{S}(\rho,\Omega,\wp)=1$ *for all* $\rho,\Omega\in\Lambda$, $\wp>0$ *s.t.* $\rho\perp\Omega$ *and* $\Omega\perp\rho$.

Then, $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ *is said to be an* $\mathscr{ONPMS}$.

*Example 1.*Let $\Lambda = \{1,2,3,4,5\}$. Define $\Gamma, \Xi, \Phi \colon \Lambda \times \Lambda \times (0,+\infty) \to [0,1]$ as

$$\Gamma(\rho,\Omega,\wp) = \begin{cases} 1, & \text{if} \quad \rho = \Omega \\ \frac{\wp}{\wp+\max\{\rho,\Omega\}}, & \text{if} \quad \text{otherwise}, \end{cases}$$

$$\Xi(\rho,\Omega,\wp) = \begin{cases} 0, & \text{if} \quad \rho = \Omega \\ \frac{\max\{\rho,\Omega\}}{\wp+\max\{\rho,\Omega\}}, & \text{if} \quad \text{otherwise}, \end{cases}$$

and

$$\Phi(\rho,\Omega,\wp) = \begin{cases} 0, & \text{if} \quad \rho = \Omega \\ \frac{\max\{\rho,\Omega\}}{\wp}, & \text{if} \quad \text{otherwise}, \end{cases}$$

and a binary relation $\perp$ by $\upsilon \perp \Omega$ iff $\upsilon, \Omega \geq 0$. Then, $(\Lambda, \Gamma, \Xi, \Phi, *, \circ, \perp)$ is an $\mathscr{ONPMS}$ with continuous t-norm $\rho * \tau = \rho\tau$ and continuous t-co-norm, $\rho \circ \bar{a} = \max\{\rho, \bar{a}\}$.

*Proof.*Now, we show (v), (x) and (xv) others are obvious.
Let $\rho = 1, \Omega = 2$, $\upsilon = 3$, $\sigma = 4$ and $\eta = 5$. Then

$$\Gamma(1,5,\wp+\kappa+\mu+\zeta) = \frac{\wp+\kappa+\mu+\zeta}{\wp+\kappa+\mu+\zeta+\max\{1,5\}} = \frac{\wp+\kappa+\mu+\zeta}{\wp+\kappa+\mu+\zeta+5}.$$

On the other hand,

$$\Gamma(1,2,\wp) = \frac{\wp}{\wp+\max\{1,2\}} = \frac{\wp}{\wp+2},$$

$$\Gamma(2,3,\kappa) = \frac{\kappa}{\kappa+\max\{2,3\}} = \frac{\kappa}{\kappa+3},$$

$$\Gamma(3,4,\mu) = \frac{\mu}{\mu+\max\{3,4\}} = \frac{\mu}{\mu+4}$$

and

$$\Gamma(4,5,\zeta) = \frac{\zeta}{\zeta+\max\{4,5\}} = \frac{\zeta}{\zeta+5}.$$

That is,

$$\frac{\wp+\kappa+\mu+\zeta}{\wp+\kappa+\mu+\zeta+5} \geq \frac{\wp}{\wp+2}\frac{\kappa}{\kappa+3}\frac{\mu}{\mu+4}\frac{\zeta}{\zeta+5}.$$

Then, it satisfies all $\wp, \kappa, \mu, \zeta > 0$. Hence,

$$\Gamma(\rho,\eta,\wp+\kappa+\mu+\zeta) \geq \Gamma(\rho,\Omega,\wp) * \Gamma(\Omega,\upsilon,\kappa) * \Gamma(\upsilon,\sigma,\mu) * \Gamma(\sigma,\eta,\zeta).$$

Now,

$$\Xi(1,5,\wp+\kappa+\mu+\zeta) = \frac{\max\{1,5\}}{\wp+\kappa+\mu+\zeta+\max\{1,5\}} = \frac{5}{\wp+\kappa+\mu+\zeta+5}.$$

On the other hand,

$$\Xi(1,2,\wp) = \frac{\max\{1,2\}}{\wp+\max\{1,2\}} = \frac{2}{\wp+2},$$

$$\Xi(2,3,\kappa) = \frac{\max\{2,3\}}{\kappa+\max\{2,3\}} = \frac{3}{\kappa+3},$$

$$\Xi(3,4,\mu) = \frac{\max\{3,4\}}{\mu+\max\{3,4\}} = \frac{4}{\mu+4}$$

and

$$\Xi(4,5,\zeta) = \frac{\max\{4,5\}}{\zeta+\max\{4,5\}} = \frac{5}{\zeta+5},$$

that is,

$$\frac{5}{\wp+\kappa+\mu+\zeta+5} \leq \max\left\{\frac{2}{\wp+2}, \frac{3}{\kappa+3}, \frac{4}{\mu+4}, \frac{5}{\zeta+5}\right\}.$$

Then, it satisfies all $\wp, \kappa, \mu, \zeta > 0$. Hence,

$$\Xi(\rho,\eta,\wp+\kappa+\mu+\zeta) \leq \Xi(\rho,\Omega,\wp) \circ \Xi(\upsilon,\sigma,\hat{s}) \circ \Xi(\upsilon,\sigma,\hat{w}) \circ \Xi(\sigma,\eta,\hat{y}).$$

Now,

$$\Phi(1,5,\wp+\kappa+\mu+\zeta) = \frac{\max\{1,5\}}{\wp+\kappa+\mu+\zeta} = \frac{5}{\wp+\kappa+\mu+\zeta}.$$

On the other hand,

$$\Phi(1,2,\wp) = \frac{\max\{1,2\}}{\wp} = \frac{2}{\wp},$$

$$\Phi(2,3,\kappa) = \frac{\max\{2,3\}}{\kappa} = \frac{3}{\kappa},$$

$$\Phi(3,4,\mu) = \frac{\max\{3,4\}}{\mu} = \frac{4}{\mu}$$

and

$$\Phi(4,5,\zeta) = \frac{\max\{4,5\}}{\zeta} = \frac{5}{\zeta},$$

that is,

$$\frac{5}{\wp+\kappa+\mu+\zeta} \leq \max\left\{\frac{2}{\wp}, \frac{3}{\kappa}, \frac{4}{\mu}, \frac{5}{\zeta}\right\}.$$

Then, it satisfies all $\wp, \kappa, \mu, \zeta > 0$. Hence,

$$\Phi(\rho,\eta,\wp+\kappa+\mu+\zeta) \leq \Phi(\rho,\Omega,\wp) \circ \Phi(\Omega,\upsilon,\kappa) \circ \Phi(\upsilon,\sigma,\mu) \circ \Phi(\sigma,\eta,\mu).$$

Therefore, $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is an $\mathscr{ONPMS}$.

Definition 11. *Let $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is an $\mathscr{ONPMS}$, an open ball is then defined $\Phi(\rho,r,\wp)$ with center ρ, radius $r, 0<r<1$ and $\wp>0$ as follows:*

$$\Phi(\rho,r,\wp) = \{\Omega \in \Lambda : \Gamma(\rho,\Omega,\wp) > 1-r, \Xi(\rho,\Omega,\wp) < r, \Phi(\rho,\Omega,\wp) < r\}.$$

Definition 12. *Let $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is an $\mathscr{ONPMS}$ and $\{\rho_\pi\}$ be an $\mathscr{O}$-sequence in Λ. Then $\{\rho_\pi\}$ is called:*

(a)an $\mathscr{O}$-convergent exists if there exists $\rho \in \Lambda$ s.t.

$$\lim_{\pi\to\mathbb{R}^+} \Gamma(\rho_\pi,\rho,\wp) = 1, \lim_{\pi\to\mathbb{R}^+} \Xi(\rho_\pi,\rho,\wp) = 0, \lim_{\pi\to\mathbb{R}^+} \Phi(\rho_\pi,\rho,\wp) = 0 \quad \textit{for all} \quad \wp > 0,$$

(b)an $\mathscr{O}$-Cauchy sequence $\iff$ for each $\bar{a} > 0, \wp > 0$, there exists $\pi_0 \in \mathbb{N}$ s.t.

$$\Gamma(\rho_\pi, \rho_{\pi+\mathfrak{q}}, \wp) \geq 1 - \bar{a}, \Xi(\rho_\pi, \rho_{\pi+\mathfrak{q}}, \wp) \leq \bar{a}, \Xi(\rho_\pi, \rho_{\pi+\mathfrak{q}}, \wp) \leq \bar{a} \quad \textit{for all} \quad \pi, \mathfrak{m} \geq \pi_0,$$

If every $\mathscr{O}$-Cauchy sequence $\mathscr{O}$-convergent in Λ, then $(\Lambda, \Gamma, \Xi, \Phi, *, \circ, \perp)$ is said to be a complete $\mathscr{ONPMS}$.

Lemma 1.*Let $\{\rho_\pi\}$ be an $\mathscr{O}$-Cauchy sequence in $\mathscr{ONPMS}$ $(\Lambda, \Gamma, \Xi, \Phi, *, \circ, \perp)$ s.t. $\rho_\pi \neq \rho_\mathfrak{m}$ whenever $\mathfrak{m}, \pi \in \mathbb{N}$ with $\pi \neq \mathfrak{m}$. Then, an $\mathscr{O}$-sequence $\{\rho_\pi\}$ can an $\mathscr{O}$-converge to, at most, one limit point.*

*Proof.*Contrarily, assume that $\rho_\pi \to \rho$ and $\rho_\pi \to \Omega$, for $\rho \neq \Omega$. Then, $\lim_{\pi\to\mathbb{R}^+} \Gamma(\rho_\pi, \rho, \wp) = 1, \lim_{\pi\to\mathbb{R}^+} \Xi(\rho_\pi, \rho, \wp) = 0, \lim_{\pi\to\mathbb{R}^+} \Phi(\rho_\pi, \rho, \wp) = 0$, and $\lim_{\pi\to\mathbb{R}^+} \Gamma(\rho_\pi, \Omega, \wp) = 1, \lim_{\pi\to\mathbb{R}^+} \Xi(\rho_\pi, \Omega, \wp) = 0, \lim_{\pi\to\mathbb{R}^+} \Phi(\rho_\pi, \Omega, \wp) = 0$, for all $\wp > 0$. Suppose

$$\begin{aligned}
\Gamma(\rho, \Omega, \wp) &\geq \Gamma\left(\rho, \rho_\pi, \frac{\wp}{4}\right) * \Gamma\left(\rho_\pi, \rho_{\pi+1}, \frac{\wp}{4}\right) * \Gamma\left(\rho_{\pi+1}, \rho_{\pi+2}, \frac{\wp}{4}\right) * \Gamma\left(\rho_{\pi+2}, \Omega, \frac{\wp}{4}\right) \\
&\to 1 * 1 * 1 * 1, \quad \text{as} \quad \pi \to \mathbb{R}^+, \\
\Xi(\rho, \Omega, \wp) &\leq \Xi\left(\rho, \rho_\pi, \frac{\wp}{4}\right) \circ \Xi\left(\rho_\pi, \rho_{\pi+1}, \frac{\wp}{4}\right) \circ \Xi\left(\rho_{\pi+1}, \rho_{\pi+2}, \frac{\wp}{4}\right) \circ \Xi\left(\rho_{\pi+2}, \Omega, \frac{\wp}{4}\right) \\
&\to 0 \circ 0 \circ 0 \circ 0, \quad \text{as} \quad \pi \to \mathbb{R}^+, \\
\Phi(\rho, \Omega, \wp) &\leq \Phi\left(\rho, \rho_\pi, \frac{\wp}{4}\right) \circ \Phi\left(\rho_\pi, \rho_{\pi+1}, \frac{\wp}{4}\right) \circ \Phi\left(\rho_{\pi+1}, \rho_{\pi+2}, \frac{\wp}{4}\right) \circ \Phi\left(\rho_{\pi+2}, \Omega, \frac{\wp}{4}\right) \\
&\to 0 \circ 0 \circ 0 \circ 0, \quad \text{as} \quad \pi \to \mathbb{R}^+.
\end{aligned}$$

That is $\Gamma(\rho, \Omega, \wp) \geq 1 * 1 * 1 = 1, \Xi(\rho, \Omega, \wp) \leq 0 \circ 0 \circ 0 = 0$ and $\Phi(\rho, \Omega, \wp) \leq 0 \circ 0 \circ 0 = 0$. Therefore $\rho = \Omega$, that is, an $\mathscr{O}$-sequence, an $\mathscr{O}$-converges to, at most, one limit point.

Lemma 2.*Let $(\Lambda, \Gamma, \Xi, \Phi, *, \circ, \perp)$ is an $\mathscr{ONPMS}$. If for some $0 < \theta < 1$ and for any $\rho, \Omega \in \Lambda, \wp > 0$,*

$$\Gamma(\rho, \Omega, \wp) \geq \Gamma\left(\rho, \Omega, \frac{\wp}{\theta}\right), \Xi(\rho, \Omega, \wp) \leq \Xi\left(\rho, \Omega, \frac{\wp}{\theta}\right), \Phi(\rho, \Omega, \wp) \leq \Phi\left(\rho, \Omega, \frac{\wp}{\theta}\right) \tag{1}$$

then $\rho = \Omega$.

Proof.(1) implies that

$$\begin{aligned}
\Gamma(\rho, \Omega, \wp) \geq \Gamma\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right), \Xi(\rho, \Omega, \wp) \leq \Xi\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right), \\
\Phi(\rho, \Omega, \wp) \leq \Phi\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right), \pi \in \mathbb{N}, \wp > 0.
\end{aligned}$$

Now,

$$\begin{aligned}
\Gamma(\rho, \Omega, \wp) &\geq \lim_{\pi\to\mathbb{R}^+} \Gamma\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right) = 1, \\
\Xi(\rho, \Omega, \wp) &\leq \lim_{\pi\to\mathbb{R}^+} \Xi\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right) = 0, \\
\Phi(\rho, \Omega, \wp) &\leq \lim_{\pi\to\mathbb{R}^+} \Phi\left(\rho, \Omega, \frac{\wp}{\theta^\pi}\right) = 0, \wp > 0.
\end{aligned}$$

Also, by the definition of (iii), (viii), (xiii), we have, $\rho = \Omega$.

Theorem 1.*Suppose $(\Lambda, \Gamma, \Xi, \Phi, *, \circ, \perp)$ is a complete $\mathscr{ONPMS}$ with $0 < \theta < 1$ and suppose that*

$$\lim_{\wp\to\mathbb{R}^+} \Gamma(\rho, \Omega, \wp) = 1, \lim_{\wp\to\mathbb{R}^+} \Xi(\rho, \Omega, \wp) = 0 \quad \textit{and} \quad \lim_{\wp\to\mathbb{R}^+} \Phi(\rho, \Omega, \wp) = 0 \tag{2}$$

for all $\rho, \Omega \in \Lambda$ and $\wp > 0$. Let $\omega \colon \Lambda \to \Lambda$ be a mapping satisfying

(i)ω is an $\perp$-preserving mapping;
(ii)

$$\Gamma(\omega\rho,\omega\Omega,\theta\wp)\geq\Gamma(\rho,\Omega,\wp),$$
$$\Xi(\omega\rho,\omega\Omega,\theta\wp)\leq\Xi(\rho,\Omega,\wp)\quad \textit{and}\quad \Phi(\omega\rho,\omega\Omega,\theta\wp)\leq\Phi(\rho,\Omega,\wp) \tag{3}$$

for all $\rho,\Omega\in\Lambda$ and $\wp>0$.
Then, ω has a unique fixed point.

*Proof.*Since $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$, there exists $\rho_0\in\Lambda$ s.t.

$$\rho_0\perp\mathfrak{u},\ \text{ for all } \mathfrak{u}\in\Lambda.$$

Since, ω is an $\perp$-preserving mapping, we have

$$\rho_0\perp\omega\rho_0.$$

Now, we define an $\mathscr{O}$-sequence ρ_π by $\rho_\pi=\omega^\pi\rho_0=\omega\rho_{\pi-1},$ for all $\pi\in\mathbb{N}$. By utilizing (3) for all $\wp>0$, we obtain

$$\Gamma(\rho_\pi,\rho_{\pi+1},\theta\wp)=\Gamma(\omega\rho_{\pi-1},\omega\rho_\pi,\theta\wp)\geq\Gamma(\rho_{\pi-1},\rho_\pi,\wp)\geq\Gamma\left(\rho_{\pi-2},\rho_{\pi-1},\frac{\wp}{\theta}\right)$$
$$\geq\Gamma\left(\rho_{\pi-3},\rho_{\pi-2},\frac{\wp}{\theta^2}\right)\geq\cdots\geq\Gamma\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+1},\theta\wp)=\Xi(\omega\rho_{\pi-1},\omega\rho_\pi,\theta\wp)\leq\Xi(\rho_{\pi-1},\rho_\pi,\wp)\leq\Xi\left(\rho_{\pi-2},\rho_{\pi-1},\frac{\wp}{\theta}\right)$$
$$\leq\Xi\left(\rho_{\pi-3},\rho_{\pi-2},\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Xi\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right).$$

and

$$\Phi(\rho_\pi,\rho_{\pi+1},\theta\wp)=\Phi(\omega\rho_{\pi-1},\omega\rho_\pi,\wp)\leq\Phi(\rho_{\pi-1},\rho_\pi,\wp)\leq\Phi\left(\rho_{\pi-2},\rho_{\pi-1},\frac{\wp}{\theta}\right)$$
$$\leq\Phi\left(\rho_{\pi-3},\rho_{\pi-2},\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Phi\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right).$$

Hence,

$$\Gamma(\rho_\pi,\rho_{\pi+1},\theta\wp)\geq\Gamma\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+1},\theta\wp)\leq\Xi\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+1},\theta\wp)\leq\Phi\left(\rho_0,\rho_1,\frac{\wp}{\theta^{\pi-1}}\right). \tag{4}$$

Consequently,

$$\Gamma(\rho_\pi,\rho_{\pi+2},\theta\wp)=\Gamma(\omega\rho_{\pi-1},\omega\rho_{\pi+1},\theta\wp)\geq\Gamma(\rho_{\pi-1},\rho_{\pi+1},\wp)\geq\Gamma\left(\rho_{\pi-2},\rho_\pi,\frac{\wp}{\theta}\right)$$
$$\geq\Gamma\left(\rho_{\pi-3},\rho_{\pi-1},\frac{\wp}{\theta^2}\right)\geq\cdots\geq\Gamma\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+2},\theta\wp)=\Xi(\omega\rho_{\pi-1},\omega\rho_{\pi+1},\theta\wp)\leq\Xi(\rho_{\pi-1},\rho_{\pi+1},\wp)\leq\Xi\left(\rho_{\pi-2},\rho_\pi,\frac{\wp}{\theta}\right)$$
$$\leq\Xi\left(\rho_{\pi-3},\rho_{\pi-1},\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Xi\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right)$$

and

$$\Phi(\rho_\pi,\rho_{\pi+2},\theta\wp)=\Phi(\omega\rho_{\pi-1},\omega\rho_{\pi+1},\wp)\leq\Phi(\rho_{\pi-1},\rho_{\pi+1},\wp)\leq\Phi\left(\rho_{\pi-2},\rho_\pi,\frac{\wp}{\theta}\right)$$
$$\leq\Phi\left(\rho_{\pi-3},\rho_{\pi-1},\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Phi\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right),$$

which implies that

$$\Gamma(\rho_\pi,\rho_{\pi+2},\theta\wp)\geq\Gamma\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+2},\theta\wp)\leq\Xi\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+2},\theta\wp)\leq\Phi\left(\rho_0,\rho_2,\frac{\wp}{\theta^{\pi-1}}\right).\tag{5}$$

It follows that

$$\Gamma(\rho_\pi,\rho_{\pi+3},\theta\wp)=\Gamma(\omega\rho_{\pi-1},\omega\rho_{\pi+2},\theta\wp)\geq\Gamma(\rho_{\pi-1},\rho_{\pi+2},\wp)\geq\Gamma\left(\rho_{\pi-2},\rho_{\pi+1},\frac{\wp}{\theta}\right)$$
$$\geq\Gamma\left(\rho_{\pi-3},\rho_\pi,\frac{\wp}{\theta^2}\right)\geq\cdots\geq\Gamma\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+3},\theta\wp)=\Xi(\omega\rho_{\pi-1},\omega\rho_{\pi+2},\theta\wp)\leq\Xi(\rho_{\pi-1},\rho_{\pi+2},\wp)\leq\Xi\left(\rho_{\pi-2},\rho_{\pi+1},\frac{\wp}{\theta}\right)$$
$$\leq\Xi\left(\rho_{\pi-3},\rho_\pi,\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Xi\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right)$$

and

$$\Phi(\rho_\pi,\rho_{\pi+3},\theta\wp)=\Phi(\omega\rho_{\pi-1},\omega\rho_{\pi+2},\wp)\leq\Phi(\rho_{\pi-1},\rho_{\pi+2},\wp)\leq\Phi\left(\rho_{\pi-2},\rho_{\pi+1},\frac{\wp}{\theta}\right)$$
$$\leq\Phi\left(\rho_{\pi-3},\rho_\pi,\frac{\wp}{\theta^2}\right)\leq\cdots\leq\Phi\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right).$$

Hence,

$$\Gamma(\rho_\pi,\rho_{\pi+3},\theta\wp)\geq\Gamma\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+3},\theta\wp)\leq\Xi\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+3},\theta\wp)\leq\Phi\left(\rho_0,\rho_3,\frac{\wp}{\theta^{\pi-1}}\right).\tag{6}$$

Similarly, for $j=1,2,3,.....$, we obtain

$$\Gamma(\rho_\pi,\rho_{\pi+3j+1},\theta\wp)\geq\Gamma\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp)\leq\Xi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp)\leq\Phi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right),\tag{7}$$

$$\Gamma(\rho_\pi,\rho_{\pi+3j+2},\theta\wp)\geq\Gamma\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+3j+2},\theta\wp)\leq\Xi\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+3j+2},\theta\wp)\leq\Phi\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right),\tag{8}$$

$$\Gamma(\rho_\pi,\rho_{\pi+3j+3},\theta\wp)\geq\Gamma\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right),$$
$$\Xi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp)\leq\Xi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right)\quad\text{and}\quad\Phi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp)\leq\Phi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right).\tag{9}$$

By using 4, we have

$$\Gamma(\rho_0,\rho_4,\wp)\geq\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)$$
$$\geq\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right),$$

$$\begin{aligned}\Xi(\rho_0,\rho_4,\wp) &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_4,\wp) &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma(\rho_0,\rho_7,\wp) &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Gamma\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Gamma\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Gamma\left(\rho_6,\rho_7,\frac{\wp}{4}\right)\\ &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_4,\wp) &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Xi\left(\rho_6,\rho_7,\frac{\wp}{4}\right)\\ &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_4,\wp) &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Phi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Phi\left(\rho_6,\rho_7,\frac{\wp}{4}\right)\\ &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right).\end{aligned}$$

For each $j=1,2,3,....$, we have

$$\begin{aligned}\Gamma(\rho_0,\rho_{3j+1},\wp) &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right)\\ &*\cdots*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j-1}}\right),\end{aligned}$$

$$\Xi(\rho_0,\rho_{3j+1},\wp) \leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)$$
$$\circ\,\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right)$$
$$\circ\cdots\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j-1}}\right)$$

and

$$\Phi(\rho_0,\rho_{3j+1},\wp) \leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)$$
$$\circ\,\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^5}\right)$$
$$\circ\cdots\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j-1}}\right).$$

Now, from 7, we get

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\geq \Gamma\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+4}}\right)*\cdots*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j+\pi-2}}\right),\end{aligned} \tag{10}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\leq \Xi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\,\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\,\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j+\pi-2}}\right)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\leq \Phi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\,\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\,\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{3j+\pi-2}}\right).\end{aligned} \tag{12}$$

By using 4 and 5, we get

$$\begin{aligned}\Gamma(\rho_0,\rho_5,\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^2}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_5,\wp) &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^2}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_5,\wp) &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma(\rho_0,\rho_8,\wp) &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Gamma\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Gamma\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Gamma\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_8,\wp) &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Xi\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\le \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_8,\wp) &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Phi\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\le \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right).\end{aligned}$$

Hence, for each $j = 1,2,3,....$, we can write

$$\begin{aligned}\Gamma(\rho_0,\rho_{3j+2},\wp) &\ge \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right)\\ &*\cdots*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j-1}}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_{3j+2},\wp) \le\ & \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right)\\ &\circ\cdots\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_{3j+2},\wp) \le\ & \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^5}\right)\\ &\circ\cdots\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j-1}}\right).\end{aligned}$$

Now, from 7, we get

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+2},\theta\wp) \ge\ & \Gamma\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right)\\ \ge\ & \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{\pi+4}}\right)*\cdots*\Gamma\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j+\pi-2}}\right),\end{aligned}\tag{13}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3j+2},\theta\wp) \le\ & \Xi\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right)\\ \le\ & \Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Xi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j+\pi-2}}\right)\end{aligned}\tag{14}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+3j+2},\theta\wp) \le\ & \Phi\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right)\\ \le\ & \Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j+\pi-2}}\right).\end{aligned}\tag{15}$$

By using 4 and 5, we have

$$\begin{aligned}\Gamma(\rho_0,\rho_6,\wp) \ge\ & \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_6,\frac{\wp}{4}\right)\\ \ge\ & \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^2}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_6,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_6,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^2}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_6,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_6,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma(\rho_0,\rho_9,\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Gamma\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Gamma\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Gamma\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\ &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_9,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Xi\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_9,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Phi\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right).\end{aligned}$$

For each $j = 1,2,3,....$, we obtain that

$$\begin{aligned}\Gamma(\rho_0,\rho_{3j+3},\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right)\\ &*\cdots*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{3j-1}}\right),\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_{3j+3},\wp) \leq\, &\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right)\\ &\circ\cdots\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{3j-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_{3j+3},\wp) \leq\, &\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^2}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^3}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^4}\right)\circ\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^5}\right)\\ &\circ\cdots\circ\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{3j-1}}\right).\end{aligned}$$

Now, from 7, we get

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\geq \Gamma\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)*\Gamma\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi+4}}\right)*\cdots*\Gamma\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{3j+\pi-2}}\right),\end{aligned} \tag{16}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\leq \Xi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{3j+\pi-2}}\right)\end{aligned} \tag{17}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\leq \Phi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi}}\right)\\ &\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+1}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+2}}\right)\circ\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi+3}}\right)\\ &\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{\pi+4}}\right)\circ\cdots\circ\Phi\left(\rho_0,\rho_2,\frac{\wp}{4\theta^{3j+\pi-2}}\right).\end{aligned} \tag{18}$$

Using (10)-(18), for each case $\pi \to \mathbb{R}^+$, we deduce that

$$\begin{aligned}\lim_{\pi\to\mathbb{R}^+}\Gamma(\rho_\pi,\rho_{\pi+\mathrm{i}},\wp) &= 1*1*\cdots*1 = 1,\\ \lim_{\pi\to\mathbb{R}^+}\Xi(\rho_\pi,\rho_{\pi+\mathrm{i}},\wp) &= 0\circ0\circ\cdots\circ0 = 0\end{aligned}$$

and

$$\lim_{\pi\to\mathbb{R}^+}\Phi(\rho_\pi,\rho_{\pi+\mathfrak{i}},\wp)=0\circ 0\circ\cdots\circ 0=0.$$

Therefore, $\{\rho_\pi\}$ is an $\mathcal{O}$-Cauchy sequence. Since $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathcal{ONPMS}$, there exists

$$\lim_{\pi\to\mathbb{R}^+}\rho_\pi=\rho.$$

Now, fact that ρ is a fixed point of ω, utilizing (v), (x), (xv) and 2, we obtain that

$$\begin{aligned}
\Gamma(\rho,\omega\rho,\wp)&\geq\Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4}\right)*\Gamma\left(\rho_{\pi+1},\rho_{\pi+2},\frac{\wp}{4}\right)*\Gamma\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\
&=\Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\omega\rho_{\pi-1},\omega\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\omega\rho_\pi,\omega\rho_{\pi+1},\frac{\wp}{4}\right)\\
&*\Gamma\left(\omega\rho_{\pi+1},\omega\rho,\frac{\wp}{4}\right)\\
&\geq\Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\rho_{\pi-1},\rho_\pi,\frac{\wp}{4\theta^{\pi-2}}\right)*\Gamma\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4\theta^{\pi-1}}\right)\\
&*\Gamma\left(\rho_{\pi+1},\rho,\frac{\wp}{4\theta}\right)\\
&\to 1*1*1*1=1\quad\text{as}\quad\pi\to\mathbb{R}^+,
\end{aligned}$$

$$\begin{aligned}
\Xi(\rho,\omega\rho,\wp)&\leq\Xi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Xi\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi+1},\rho_{\pi+2},\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\
&=\Xi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Xi\left(\omega\rho_{\pi-1},\omega\rho_\pi,\frac{\wp}{4}\right)\circ\Xi\left(\omega\rho_\pi,\omega\rho_{\pi+1},\frac{\wp}{4}\right)\\
&\circ\Xi\left(\omega\rho_{\pi+1},\omega\rho,\frac{\wp}{4}\right)\\
&\leq\Xi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi-1},\rho_\pi,\frac{\wp}{4\theta^{\pi-2}}\right)\circ\Xi\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4\theta^{\pi-1}}\right)\\
&\circ\Xi\left(\rho_{\pi+1},\rho,\frac{\wp}{4\theta}\right)\\
&\to 0\circ 0\circ 0\circ 0=0\quad\text{as}\quad\pi\to\mathbb{R}^+,
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\rho,\omega\rho,\wp)&\leq\Phi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Phi\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi+1},\rho_{\pi+2},\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\
&=\Phi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Phi\left(\omega\rho_{\pi-1},\omega\rho_\pi,\frac{\wp}{4}\right)\circ\Phi\left(\omega\rho_\pi,\omega\rho_{\pi+1},\frac{\wp}{4}\right)\\
&\circ\Phi\left(\omega\rho_{\pi+1},\omega\rho,\frac{\wp}{4}\right)\\
&\leq\Phi\left(\rho,\rho_\pi,\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi-1},\rho_\pi,\frac{\wp}{4\theta^{\pi-2}}\right)\circ\Phi\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4\theta^{\pi-1}}\right)\\
&\circ\Phi\left(\rho_{\pi+1},\rho,\frac{\wp}{4\theta}\right)\\
&\to 0\circ 0\circ 0\circ 0=0\quad\text{as}\quad\pi\to\mathbb{R}^+,
\end{aligned}$$

Therefore, $\omega\rho=\rho$. Consider $\tilde{n}$ be a another fixed point of ω s.t. $\omega\tilde{n}=\tilde{n}\neq\rho=\omega\rho$. Then

$$\rho\perp\tilde{n}.$$

Since, ω is an $\perp$-preserving, we have

$$\omega\rho\perp\omega\tilde{n}.$$

Then,

$$1 \geq \Gamma(\tilde{n},\rho,\wp) = \Gamma(\omega\tilde{n},\omega\rho,\wp) \geq \Gamma\left(\tilde{n},\rho,\frac{\wp}{\theta}\right) = \Gamma\left(\omega\tilde{n},\omega\rho,\frac{\wp}{\theta}\right)$$
$$\geq \left(\tilde{n},\rho,\frac{\wp}{\theta^2}\right) \geq \cdots \geq \Gamma\left(\tilde{n},\rho,\frac{\wp}{\theta^\pi}\right) \to 1 \quad \text{as} \quad \pi \to \mathbb{R}^+,$$
$$0 \leq \Xi(\tilde{n},\rho,\wp) = \Xi(\omega\tilde{n},\omega\rho,\wp) \leq \Xi\left(\tilde{n},\rho,\frac{\wp}{\theta}\right) = \Xi\left(\omega\tilde{n},\omega\rho,\frac{\wp}{\theta}\right)$$
$$\leq \Xi\left(\tilde{n},\rho,\frac{\wp}{\theta^2}\right) \leq \cdots \leq \Xi\left(\tilde{n},\rho,\frac{\wp}{\theta^\pi}\right) \to 0 \quad \text{as} \quad \pi \to \mathbb{R}^+,$$

and

$$0 \leq \Phi(\tilde{n},\rho,\wp) = \Phi(\omega\tilde{n},\omega\rho,\wp) \leq \Phi\left(\tilde{n},\rho,\frac{\wp}{\theta}\right) = \Phi\left(\omega\tilde{n},\omega\rho,\frac{\wp}{\theta}\right)$$
$$\leq \Phi\left(\tilde{n},\rho,\frac{\wp}{\theta^2}\right) \leq \cdots \leq \Phi\left(\tilde{n},\rho,\frac{\wp}{\theta^\pi}\right) \to 0 \quad \text{as} \quad \pi \to \mathbb{R}^+,$$

by using (iii), (viii) and (xiii), $\rho = \tilde{n}$.

Definition 13. *Let $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ be an $\mathscr{ONPMS}$. A map $\omega\colon \Lambda \to \Lambda$ is an $\mathscr{ONPC}$(orthogonal neutrosophic pentagonal contraction) if there exists $0 < \theta < 1$, such that*

$$\frac{1}{\Gamma(\mathscr{P}\rho,\mathscr{P}\Omega,\wp)} - 1 \leq \theta\left[\frac{1}{\Gamma(\rho,\Omega,\wp)} - 1\right] \tag{19}$$
$$\Xi(\mathscr{P}\rho,\mathscr{P}\Omega,\wp) \leq \theta\Xi(\rho,\Omega,\wp), \tag{20}$$

and

$$\Phi(\mathscr{P}\rho,\mathscr{P}\Omega,\wp) \leq \theta\Phi(\rho,\Omega,\wp), \tag{21}$$

for all $\rho,\Omega \in \Lambda$ and $\wp > 0$.
Here, we show the theorem for $\mathscr{ONPC}$(orthogonal neutrosophic pentagonal contraction).

Theorem 2. *Let $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ be a complete $\mathscr{ONPMS}$ and suppose that*

$$\lim_{\wp\to\mathbb{R}^+} \Gamma(\rho,\Omega,\wp) = 1, \lim_{\wp\to\mathbb{R}^+} \Xi(\rho,\Omega,\wp) = 0 \quad \textit{and} \quad \lim_{\wp\to\mathbb{R}^+} \Phi(\rho,\Omega,\wp) = 0 \tag{22}$$

for all $\rho,\Omega \in \Lambda$ and $\wp > 0$. Let $\omega\colon \Lambda \to \Lambda$ be a mapping satisfying

(i)ω is an $\perp$-preserving mapping;
(ii)ω is an $\mathscr{ONPC}$.

Then, ω has a unique fixed point.

Proof. Since $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$, there exists $\rho_0 \in \Lambda$ such that

$$\rho_0\perp\mathfrak{u}, \text{ for all } \mathfrak{u} \in \Lambda.$$

Since, ω is an $\perp$-preserving mapping, we have

$$\rho_0\perp\omega\rho_0.$$

Now, we define an $\mathscr{O}$-sequence ρ_π by $\rho_\pi = \omega^\pi \rho_0 = \omega \rho_{\pi-1},$ for all $\pi \in \mathbb{N}$. By using (19), (20) and (21) for all $\wp > 0$, we deduce

$$\begin{aligned}
\frac{1}{\Gamma(\rho_\pi, \rho_{\pi+1}, \wp)} - 1 &= \frac{1}{\Gamma(\omega\rho_{\pi-1}, \omega\rho_\pi, \wp)} - 1 \\
&\leq \theta \left[\frac{1}{\Gamma(\rho_{\pi-1}, \rho_\pi, \wp)} \right] = \frac{\theta}{\Gamma(\rho_{\pi-1}, \rho_\pi, \wp)} - \theta \\
\Rightarrow \frac{1}{\Gamma(\rho_\pi, \rho_{\pi+1}, \wp)} &\leq \frac{\theta}{\Gamma(\rho_{\pi-1}, \rho_\pi, \wp)} + (1-\theta) \\
&\leq \frac{\theta^2}{\Gamma(\rho_{\pi-2}, \rho_{\pi-1}, \wp)} + \theta(1-\theta) + (1-\theta).
\end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned}
\frac{1}{\Gamma(\rho_\pi, \rho_{\pi+1}, \wp)} &\leq \frac{\theta^\pi}{\Gamma(\rho_0, \rho_1, \wp)} + \theta^{\pi-1}(1-\theta) + \theta^{\pi-2}(1-\theta) \\
&+ \cdots + \theta(1-\theta) + (1-\theta) \\
&\leq \frac{\theta^\pi}{\Gamma(\rho_0, \rho_1, \wp)} + (\theta^{\pi-1} + \theta^{\pi-2} + \cdots + 1)(1-\theta) \\
&\leq \frac{\theta^\pi}{\Gamma(\rho_0, \rho_1, \wp)} + (1-\theta^\pi).
\end{aligned}$$

It follows that

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0, \rho_1, \wp)} + (1-\theta^\pi)} \leq \Gamma(\rho_\pi, \rho_{\pi+1}, \wp)$$

$$\begin{aligned}
\Xi(\rho_\pi, \rho_{\pi+1}, \wp) &= \Xi(\omega\rho_{\pi-1}, \omega\rho_\pi, \wp) \leq \theta \Xi(\rho_{\pi-1}, \rho_\pi, \wp) = \Xi(\omega\rho_{\pi-2}, \omega\rho_{\pi-1}, \wp) \\
&\leq \theta^2 \Xi(\rho_{\pi-2}, \rho_{\pi-1}, \wp) \leq \cdots \leq \theta^\pi \Xi(\rho_0, \rho_1, \wp)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\rho_\pi, \rho_{\pi+1}, \wp) &= \Phi(\omega\rho_{\pi-1}, \omega\rho_\pi, \wp) \leq \theta \Phi(\rho_{\pi-1}, \rho_\pi, \wp) = \Phi(\omega\rho_{\pi-2}, \omega\rho_{\pi-1}, \wp) \\
&\leq \theta^2 \Phi(\rho_{\pi-2}, \rho_{\pi-1}, \wp) \leq \cdots \leq \theta^\pi \Phi(\rho_0, \rho_1, \wp).
\end{aligned} \tag{23}$$

It again follows that

$$\begin{aligned}
\frac{1}{\Gamma(\rho_\pi, \rho_{\pi+2}, \wp)} - 1 &= \frac{1}{\Gamma(\omega\rho_{\pi-1}, \omega\rho_{\pi+1}, \wp)} - 1 \\
&\leq \theta \left[\frac{1}{\Gamma(\rho_{\pi-1}, \rho_{\pi+1}, \wp)} \right] \\
&= \frac{\theta}{\Gamma(\rho_{\pi-1}, \rho_{\pi+1}, \wp)} - \theta \\
\Rightarrow \frac{1}{\Gamma(\rho_\pi, \rho_{\pi+2}, \wp)} &\leq \frac{\theta}{\Gamma(\rho_{\pi-1}, \rho_{\pi+1}, \wp)} + (1-\theta) \\
&\leq \frac{\theta^2}{\Gamma(\rho_{\pi-2}, \rho_\pi, \wp)} + \theta(1-\theta) + (1-\theta)
\end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned}\frac{1}{\Gamma(\rho_\pi,\rho_{\pi+2},\wp)} &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_2,\wp)}+\theta^{\pi-1}(1-\theta)+\theta^{\pi-2}(1-\theta)\\ &+\cdots+\theta(1-\theta)+(1-\theta)\\ &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_2,\wp)}+(\theta^{\pi-1}+\theta^{\pi-2}+\cdots+1)(1-\theta)\\ &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_2,\wp)}+(1-\theta^\pi)\end{aligned}$$

Hence,

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0,\rho_2,\wp)}+(1-\theta^\pi)} \leq \Gamma(\rho_\pi,\rho_{\pi+2},\wp) \tag{24}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+2},\wp) &= \Xi(\omega\rho_{\pi-1},\omega\rho_{\pi+1},\wp) \leq \theta\Xi(\rho_{\pi-1},\rho_{\pi+1},\wp) = \Xi(\omega\rho_{\pi-2},\omega\rho_\pi,\wp)\\ &\leq \theta^2\Xi(\rho_{\pi-2},\rho_\pi,\wp) \leq \cdots \leq \theta^\pi\Xi(\rho_0,\rho_2,\wp)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+2},\wp) &= \Phi(\omega\rho_{\pi-1},\omega\rho_{\pi+1},\wp) \leq \theta\Phi(\rho_{\pi-1},\rho_{\pi+1},\wp) = \Phi(\omega\rho_{\pi-2},\omega\rho_\pi,\wp)\\ &\leq \theta^2\Phi(\rho_{\pi-2},\rho_\pi,\wp) \leq \cdots \leq \theta^\pi\Phi(\rho_0,\rho_2,\wp).\end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned}\frac{1}{\Gamma(\rho_\pi,\rho_{\pi+3},\wp)}-1 &= \frac{1}{\Gamma(\omega\rho_{\pi-1},\omega\rho_{\pi+2},\wp)}-1\\ &\leq \theta\left[\frac{1}{\Gamma(\rho_{\pi-1},\rho_{\pi+2},\wp)}\right]\\ &= \frac{\theta}{\Gamma(\rho_{\pi-1},\rho_{\pi+2},\wp)}-\theta\\ \Rightarrow \frac{1}{\Gamma(\rho_\pi,\rho_{\pi+3},\wp)} &\leq \frac{\theta}{\Gamma(\rho_{\pi-1},\rho_{\pi+2},\wp)}+(1-\theta)\\ &\leq \frac{\theta^2}{\Gamma(\rho_{\pi-2},\rho_{\pi+1},\wp)}+\theta(1-\theta)+(1-\theta).\end{aligned}$$

Proceeding in this way, we have

$$\begin{aligned}\frac{1}{\Gamma(\rho_\pi,\rho_{\pi+3},\wp)} &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_3,\wp)}+\theta^{\pi-1}(1-\theta)+\theta^{\pi-2}(1-\theta)\\ &+\cdots+\theta(1-\theta)+(1-\theta)\\ &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_3,\wp)}+(\theta^{\pi-1}+\theta^{\pi-2}+\cdots+1)(1-\theta)\\ &\leq \frac{\theta^\pi}{\Gamma(\rho_0,\rho_3,\wp)}+(1-\theta^\pi),\end{aligned}$$

yields,

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0,\rho_3,\wp)}+(1-\theta^\pi)} \leq \Gamma(\rho_\pi,\rho_{\pi+3},\wp) \tag{27}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3},\wp) &= \Xi(\omega\rho_{\pi-1},\omega\rho_{\pi+2},\wp) \leq \theta\Xi(\rho_{\pi-1},\rho_{\pi+2},\wp) = \Xi(\omega\rho_{\pi-2},\omega\rho_{\pi+1},\wp)\\ &\leq \theta^2\Xi(\rho_{\pi-2},\rho_{\pi+1},\wp) \leq \cdots \leq \theta^\pi\Xi(\rho_0,\rho_3,\wp)\end{aligned} \tag{28}$$

and

$$\Phi(\rho_\pi, \rho_{\pi+3}, \wp) = \Phi(\omega\rho_{\pi-1}, \omega\rho_{\pi+2}, \wp) \leq \theta\Phi(\rho_{\pi-1}, \rho_{\pi+2}, \wp) = \Phi(\omega\rho_{\pi-2}, \omega\rho_{\pi+1}, \wp) \leq \theta^2\Phi(\rho_{\pi-2}, \rho_{\pi+1}, \wp) \leq \cdots \leq \theta^\pi\Phi(\rho_0, \rho_3, \wp). \tag{29}$$

Similarly, for $j = 1,2,3,.....$, we have

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0,\rho_{3j+1},\wp)} + (1-\theta^\pi)} \leq \Gamma(\rho_\pi, \rho_{\pi+3j+1}, \wp)$$

$$\Xi(\rho_\pi, \rho_{\pi+3j+1}, \theta\wp) \leq \theta^\pi\Xi(\rho_0, \rho_{3j+1}, \wp) \quad \text{and} \quad \Phi(\rho_\pi, \rho_{\pi+3j+1}, \theta\wp) \leq \theta^\pi\Phi(\rho_0, \rho_{3j+1}, \wp), \tag{30}$$

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0,\rho_{3j+2},\wp)} + (1-\theta^\pi)} \leq \Gamma(\rho_\pi, \rho_{\pi+3j+2}, \wp)$$

$$\Xi(\rho_\pi, \rho_{\pi+3j+2}, \theta\wp) \leq \theta^\pi\Xi(\rho_0, \rho_{3j+2}, \wp) \quad \text{and} \quad \Phi(\rho_\pi, \rho_{\pi+3j+2}, \theta\wp) \leq \theta^\pi\Phi(\rho_0, \rho_{3j+2}, \wp), \tag{31}$$

$$\frac{1}{\frac{\theta^\pi}{\Gamma(\rho_0,\rho_{3j+3},\wp)} + (1-\theta^\pi)} \leq \Gamma(\rho_\pi, \rho_{\pi+3j+3}, \wp)$$

$$\Xi(\rho_\pi, \rho_{\pi+3j+3}, \theta\wp) \leq \theta^\pi\Xi(\rho_0, \rho_{3j+3}, \wp) \quad \text{and} \quad \Phi(\rho_\pi, \rho_{\pi+3j+3}, \theta\wp) \leq \theta^\pi\Phi(\rho_0, \rho_{3j+3}, \wp). \tag{32}$$

By using 23, we get

$$\begin{aligned}\Gamma(\rho_0, \rho_4, \wp) &\geq \Gamma\left(\rho_0, \rho_1, \frac{\wp}{4}\right) * \Gamma\left(\rho_1, \rho_2, \frac{\wp}{4}\right) * \Gamma\left(\rho_2, \rho_3, \frac{\wp}{4}\right) * \Gamma\left(\rho_3, \rho_4, \frac{\wp}{4}\right)\\
&\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^2)}\\
&* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^3)},\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0, \rho_4, \wp) &\leq \Xi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \Xi\left(\rho_1, \rho_2, \frac{\wp}{4}\right) \circ \Xi\left(\rho_2, \rho_3, \frac{\wp}{4}\right) \circ \Xi\left(\rho_3, \rho_4, \frac{\wp}{4}\right)\\
&\leq \Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^1\Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^2\Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^3\Xi\left(\rho_0, \rho_1, \frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0, \rho_4, \wp) &\leq \Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \Phi\left(\rho_1, \rho_2, \frac{\wp}{4}\right) \circ \Phi\left(\rho_2, \rho_3, \frac{\wp}{4}\right) \circ \Phi\left(\rho_3, \rho_4, \frac{\wp}{4}\right)\\
&\leq \Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta\Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta^2\Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta^3\Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma(\rho_0, \rho_7, \wp) &\geq \Gamma\left(\rho_0, \rho_1, \frac{\wp}{4}\right) * \Gamma\left(\rho_1, \rho_2, \frac{\wp}{4}\right) * \Gamma\left(\rho_2, \rho_3, \frac{\wp}{4}\right) * \Gamma\left(\rho_3, \rho_4, \frac{\wp}{4}\right)\\
&* \Gamma\left(\rho_4, \rho_5, \frac{\wp}{4}\right) * \Gamma\left(\rho_5, \rho_6, \frac{\wp}{4}\right) * \Gamma\left(\rho_6, \rho_7, \frac{\wp}{4}\right)\\
&\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^2)}\\
&* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^3)} * \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^4)}\\
&* \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^5)} * \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^6)},\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_7,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) * \Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right) * \Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right) * \Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &* \Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right) * \Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right) * \Xi\left(\rho_6,\rho_7,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^6\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_7,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right) \circ \Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right) \circ \Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ \Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right) \circ \Phi\left(\rho_5,\rho_6,\frac{\wp}{4}\right) \circ \Phi\left(\rho_6,\rho_7,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^6\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right).\end{aligned}$$

For each $j = 1,2,3,....$, we obtain that

$$\begin{aligned}\Gamma(\rho_0,\rho_{3j+1},\wp) \geq &\frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\ &* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^3)} * \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^4)}\\ &* \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^5)} * \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^6)}\\ &* \cdots * \frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^{3j})}\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_{3j+1},\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^6\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \cdots \circ \theta^{3j}\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_{3j+1},\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ \theta^6\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \cdots \circ \theta^{3j}\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right).\end{aligned}$$

Now, from 23, one has

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\geq \Gamma\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta)}\\ &* \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^2)}\\ &* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^3)} * \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^4)}\\ &* \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^5)} * \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^6)}\\ &* \cdots * \frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^{3j})},\end{aligned} \tag{33}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\leq \Xi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^6\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\cdots\circ\theta^{3j}\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\end{aligned} \tag{34}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+3j+1},\theta\wp) &\leq \Phi\left(\rho_0,\rho_{3j+1},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^6\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\cdots\circ\theta^{3j}\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right).\end{aligned} \tag{35}$$

By using 23, we can write

$$\begin{aligned}\Gamma(\rho_0,\rho_5,\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\ &* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_2,\frac{\wp}{4})}+(1-\theta^3)},\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_5,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Xi\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\leq \Xi(\rho_0,\rho_1,\frac{\wp}{4})\circ\theta\Xi(\rho_0,\rho_1,\frac{\wp}{4})\circ\theta^2\Xi(\rho_0,\rho_1,\frac{\wp}{4})\circ\theta^3\Xi\left(\rho_0,\rho_2,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_5,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_5,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^3\Phi\left(\rho_0,\rho_2,\frac{\wp}{4}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma(\rho_0,\rho_8,\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Gamma\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Gamma\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Gamma\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}}*\frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)}*\frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\ &*\frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^3)}*\frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^4)}\\ &*\frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^5)}*\frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_2,\frac{\wp}{4})}+(1-\theta^6)},\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_8,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &*\Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Xi\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^6\Xi\left(\rho_0,\rho_2,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_8,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\ &\circ\Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Phi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Phi\left(\rho_6,\rho_8,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^6\Phi\left(\rho_0,\rho_2,\frac{\wp}{4}\right).\end{aligned}$$

For each $j = 1,2,3,....,$ one can write

$$\begin{aligned}\Gamma(\rho_0,\rho_{3j+2},\wp) \geq &\frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\ &* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^3)} * \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^4)}\\ &* \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^5)} * \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_2,\frac{\wp}{4})}+(1-\theta^6)}\\ &*\cdots* \frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^{3j})}\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_{3j+2},\wp) \leq &\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^6\Xi\left(\rho_0,\rho_2,\frac{\wp}{4}\right)\circ\cdots\circ\theta^{3j}\Xi\left(\rho_0,\rho_2,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_{3j+2},\wp) \leq &\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\ &\circ\theta^6\Phi\left(\rho_0,\rho_2,\frac{\wp}{4}\right)\circ\cdots\circ\theta^{3j}\Phi\left(\rho_0,\rho_2,\frac{\wp}{4}\right).\end{aligned}$$

Now, from 23, we get

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+2},\theta\wp) \geq &\Gamma\left(\rho_0,\rho_{3j+2},\frac{\wp}{\theta^{\pi-1}}\right)\\ \geq &\frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta)}\\ &* \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^2)} * \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^3)}\\ &* \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^4)} * \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^5)}\\ &* \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_2,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^6)} *\cdots* \frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^{3j})}\end{aligned} \tag{36}$$

$$
\begin{aligned}
\Xi(\rho_\pi, \rho_{\pi+3j+2}, \theta\wp) &\leq \Xi\left(\rho_0, \rho_{3j+2}, \frac{\wp}{\theta^{\pi-1}}\right) \\
&\leq \Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta\Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^2\Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^3\Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^4\Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^5\Xi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^6\Xi\left(\rho_0, \rho_2, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \cdots \circ \theta^{3j}\Xi\left(\rho_0, \rho_2, \frac{\wp}{4\theta^{\pi-1}}\right)
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
\Phi(\rho_\pi, \rho_{\pi+3j+2}, \theta\wp) &\leq \Phi\left(\rho_0, \rho_{3j+2}, \frac{\wp}{\theta^{\pi-1}}\right) \\
&\leq \Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta\Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^2\Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^3\Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^4\Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^5\Phi\left(\rho_0, \rho_1, \frac{\wp}{4\theta^{\pi-1}}\right) \\
&\circ \theta^6\Phi\left(\rho_0, \rho_2, \frac{\wp}{4\theta^{\pi-1}}\right) \circ \cdots \circ \theta^{3j}\Phi\left(\rho_0, \rho_2, \frac{\wp}{4\theta^{\pi-1}}\right).
\end{aligned} \tag{38}
$$

From 23, we have

$$
\begin{aligned}
\Gamma(\rho_0, \rho_6, \wp) &\geq \Gamma\left(\rho_0, \rho_1, \frac{\wp}{4}\right) * \Gamma\left(\rho_1, \rho_2, \frac{\wp}{4}\right) * \Gamma\left(\rho_2, \rho_3, \frac{\wp}{4}\right) * \Gamma\left(\rho_3, \rho_6, \frac{\wp}{4}\right) \\
&\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta)} * \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})} + (1-\theta^2)} \\
&* \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4})} + (1-\theta^3)},
\end{aligned}
$$

$$
\begin{aligned}
\Xi(\rho_0, \rho_6, \wp) &\leq \Xi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \Xi\left(\rho_1, \rho_2, \frac{\wp}{4}\right) \circ \Xi\left(\rho_2, \rho_3, \frac{\wp}{4}\right) \circ \Xi\left(\rho_3, \rho_6, \frac{\wp}{4}\right) \\
&\leq \Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^1\Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^2\Xi(\rho_0, \rho_1, \frac{\wp}{4}) \circ \theta^3\Xi\left(\rho_0, \rho_3, \frac{\wp}{4}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\rho_0, \rho_6, \wp) &\leq \Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \Phi\left(\rho_1, \rho_2, \frac{\wp}{4}\right) \circ \Phi\left(\rho_2, \rho_3, \frac{\wp}{4}\right) \circ \Phi\left(\rho_3, \rho_6, \frac{\wp}{4}\right) \\
&\leq \Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta\Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta^2\Phi\left(\rho_0, \rho_1, \frac{\wp}{4}\right) \circ \theta^3\Phi\left(\rho_0, \rho_3, \frac{\wp}{4}\right).
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\Gamma(\rho_0,\rho_9,\wp) &\geq \Gamma\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Gamma\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Gamma\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Gamma\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\
&*\Gamma\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Gamma\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Gamma\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\
&\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}}*\frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)}*\frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\
&*\frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^3)}*\frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^4)}\\
&*\frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^5)}*\frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4})}+(1-\theta^6)},
\end{aligned}$$

$$\begin{aligned}
\Xi(\rho_0,\rho_9,\wp) &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)*\Xi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)*\Xi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)*\Xi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\
&*\Xi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)*\Xi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)*\Xi\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\
&\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\
&\circ\Xi\theta^3\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\
&\circ\theta^6\Xi\left(\rho_0,\rho_3,\frac{\wp}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\rho_0,\rho_9,\wp) &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\Phi\left(\rho_1,\rho_2,\frac{\wp}{4}\right)\circ\Phi\left(\rho_2,\rho_3,\frac{\wp}{4}\right)\circ\Phi\left(\rho_3,\rho_4,\frac{\wp}{4}\right)\\
&\circ\Phi\left(\rho_4,\rho_5,\frac{\wp}{4}\right)\circ\Phi\left(\rho_5,\rho_6,\frac{\wp}{4}\right)\circ\Phi\left(\rho_6,\rho_9,\frac{\wp}{4}\right)\\
&\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\
&\circ\theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\circ\theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right)\\
&\circ\theta^6\Phi\left(\rho_0,\rho_3,\frac{\wp}{4}\right).
\end{aligned}$$

For each $j = 1,2,3,....$, we get

$$\begin{aligned}
\Gamma(\rho_0,\rho_{3j+3},\wp) &\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}}*\frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta)}*\frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^2)}\\
&*\frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^3)}*\frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^4)}\\
&*\frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4})}+(1-\theta^5)}*\frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4})}+(1-\theta^6)}\\
&*\cdots*\frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4})}+(1-\theta^{3j})}
\end{aligned}$$

$$\begin{aligned}\Xi(\rho_0,\rho_{3j+3},\wp) \leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \\ \circ \theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \\ \circ \theta^6\Xi\left(\rho_0,\rho_3,\frac{\wp}{4}\right) \circ \cdots \circ \theta^{3j}\Xi\left(\rho_0,\rho_3,\frac{\wp}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho_0,\rho_{3j+3},\wp) \leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \\ \circ \theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \circ \theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4}\right) \\ \circ \theta^6\Phi\left(\rho_0,\rho_3,\frac{\wp}{4}\right) \circ \cdots \circ \theta^{3j}\Phi\left(\rho_0,\rho_3,\frac{\wp}{4}\right).\end{aligned}$$

Now, from 23, we get

$$\begin{aligned}\Gamma(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\geq \Gamma\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right) \\ &\geq \frac{1}{\frac{1}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}} * \frac{1}{\frac{\theta}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta)} \\ &* \frac{1}{\frac{\theta^2}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^2)} * \frac{1}{\frac{\theta^3}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^3)} \\ &* \frac{1}{\frac{\theta^4}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^4)} * \frac{1}{\frac{\theta^5}{\Gamma(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^5)} \\ &* \frac{1}{\frac{\theta^6}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^6)} \\ &* \cdots * \frac{1}{\frac{\theta^{3j}}{\Gamma(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}})}+(1-\theta^{3j})},\end{aligned} \tag{39}$$

$$\begin{aligned}\Xi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\leq \Xi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right) \\ &\leq \Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \\ &\circ \theta^2\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^3\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \\ &\circ \theta^4\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \circ \theta^5\Xi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right) \\ &\circ \theta^6\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}}\right) \circ \cdots \circ \theta^{3j}\Xi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}}\right)\end{aligned} \tag{40}$$

and

$$\begin{aligned}\Phi(\rho_\pi,\rho_{\pi+3j+3},\theta\wp) &\leq \Phi\left(\rho_0,\rho_{3j+3},\frac{\wp}{\theta^{\pi-1}}\right)\\ &\leq \Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^2\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^3\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^4\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\theta^5\Phi\left(\rho_0,\rho_1,\frac{\wp}{4\theta^{\pi-1}}\right)\\ &\circ\theta^6\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}}\right)\circ\cdots\circ\theta^{3j}\Phi\left(\rho_0,\rho_3,\frac{\wp}{4\theta^{\pi-1}}\right).\end{aligned} \tag{41}$$

Using (33)-(41), for each case $\pi\to\mathbb{R}^+$, we deduce that

$$\begin{aligned}\lim_{\pi\to\mathbb{R}^+}\Gamma(\rho_\pi,\rho_{\pi+\mathrm{q}},\wp) &= 1*1*\cdots*=1,\\ \lim_{\pi\to\mathbb{R}^+}\Xi(\rho_\pi,\rho_{\pi+\mathrm{q}},\wp) &= 0\circ0\circ\cdots\circ0=0,\end{aligned}$$

and

$$\lim_{\pi\to\mathbb{R}^+}\Phi(\rho_\pi,\rho_{\pi+\mathrm{q}},\wp)=0\circ0\circ\cdots\circ0=0$$

Therefore, $\{\rho_\pi\}$ is an $\mathscr{O}$-Cauchy sequence. Since $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$, there exists

$$\lim_{\pi\to\mathbb{R}^+}\rho_\pi=\rho.$$

Applying (v), (x) and (xv), we get

$$\begin{aligned}\frac{1}{\Gamma(\omega\rho_\pi,\omega\rho,\wp)}-1 &\leq \theta\left[\frac{1}{\Gamma(\rho_\pi,\rho,\wp)}-1\right]=\frac{\theta}{\Gamma(\rho_\pi,\rho,\wp)}-\theta\\ &\Rightarrow \frac{1}{\frac{\theta}{\Gamma(\rho_\pi,\rho,\wp)}+(1-\theta)}\leq\Gamma(\omega\rho_\pi,\omega\rho,\wp).\end{aligned}$$

Using the above inequality, we obtain that

$$\begin{aligned}\Gamma(\rho,\omega\rho,\wp) &\geq \Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4}\right)*\Gamma\left(\rho_{\pi+1},\omega\rho_{\pi+2},\frac{\wp}{4}\right)*\Gamma\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\ &\geq \Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\omega\rho_{\pi-1},\omega\rho_\pi,\frac{\wp}{4}\right)*\Gamma\left(\omega\rho_\pi,\omega\rho_{\pi+1},\frac{\wp}{4}\right)*\Gamma\left(\omega\rho_{\pi+1},\omega\rho,\frac{\wp}{4}\right)\\ &\geq \Gamma\left(\rho,\rho_\pi,\frac{\wp}{4}\right)*\frac{1}{\frac{\theta^{\pi-1}}{\Gamma(\rho_{\pi-1},\rho_\pi,\frac{\wp}{4})}+(1-\theta^{\pi-1})}*\frac{1}{\frac{\theta^{\pi}}{\Gamma(\rho_\pi,\rho_{\pi+1},\frac{\wp}{4})}+(1-\theta^{\pi})}\\ &*\frac{1}{\frac{\theta}{\Gamma(\rho_{\pi+1},\rho,\frac{\wp}{4})}+(1-\theta)}\\ &\to 1*1*1*1=1 \text{ as } \pi\to\mathbb{R}^+,\end{aligned}$$

$$\begin{aligned}\Xi(\rho,\omega\rho,\wp) &\leq \Xi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi},\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi+1},\omega\rho_{\pi+2},\frac{\wp}{4}\right)\circ\Xi\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\Xi\left(\omega\rho_{\pi-1},\omega\rho_{\pi},\frac{\wp}{4}\right)\circ\Xi\left(\omega\rho_{\pi},\omega\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Xi\left(\omega\rho_{\pi+1},\omega\rho,\frac{\wp}{4}\right)\\ &\leq \Xi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\theta^{\pi-1}\Xi\left(\rho_{\pi-1},\rho_{\pi},\frac{\wp}{4}\right)\circ\theta^{\pi}\Xi\left(\rho_{\pi},\rho_{\pi+1},\frac{\wp}{4}\right)\\ &\circ\theta\Xi\left(\rho_{\pi+1},\rho,\frac{\wp}{4}\right)\\ &\to 0\circ 0\circ 0\circ 0\circ 0 = 0 \text{ as } \pi\to\mathbb{R}^{+}\end{aligned}$$

and

$$\begin{aligned}\Phi(\rho,\omega\rho,\wp) &\leq \Phi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi},\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi+1},\rho_{\pi+2},\frac{\wp}{4}\right)\circ\Phi\left(\rho_{\pi+2},\omega\rho,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\Phi\left(\omega\rho_{\pi-1},\omega\rho_{\pi},\frac{\wp}{4}\right)\circ\Phi\left(\omega\rho_{\pi},\omega\rho_{\pi+1},\frac{\wp}{4}\right)\circ\Phi\left(\omega\rho_{\pi},\omega\rho,\frac{\wp}{4}\right)\\ &\leq \Phi\left(\rho,\rho_{\pi},\frac{\wp}{4}\right)\circ\theta^{\pi-1}\Phi\left(\rho_{\pi-1},\rho_{\pi},\frac{\wp}{4}\right)\circ\theta^{\pi}\Phi\left(\rho_{\pi},\rho_{\pi+1},\frac{\wp}{4}\right)\circ\theta\Phi\left(\rho_{\pi},\rho,\frac{\wp}{4}\right)\\ &\to 0\circ 0\circ 0\circ 0\circ 0 = 0 \text{ as } \pi\to\mathbb{R}^{+}.\end{aligned}$$

Hence, $\omega\rho=\rho$. For the uniqueness, let $\tilde{n}$ be a another fixed point of ω such that $\omega\tilde{n}=\tilde{n}\neq\rho=\omega\rho$. Then

$$\rho\perp\tilde{n}.$$

Since, ω is an $\perp$-preserving, we have

$$\omega\rho\perp\omega\tilde{n}.$$

Then,

$$\begin{aligned}\frac{1}{\Gamma(\rho,\tilde{n},\wp)}-1 &= \frac{1}{\Gamma(\omega\rho,\omega\tilde{n},\wp)}-1\\ &\leq \theta\left[\frac{1}{\Gamma(\rho,\tilde{n},\wp)}-1\right] < \frac{1}{\Gamma(\rho,\tilde{n},\wp)}-1,\end{aligned}$$

which is a contradiction.

$$\Xi(\rho,\tilde{n},\wp)=\Xi(\omega\rho,\omega\tilde{n},\wp)\leq\theta\Xi(\rho,\tilde{n},\wp)<\Xi(\rho,\tilde{n},\wp),$$

which is a contradiction again, and

$$\Phi(\rho,\tilde{n},\wp)=\Phi(\omega\rho,\omega\tilde{n},\wp)\leq\theta\Phi(\rho,\tilde{n},\wp)<\Phi(\rho,\tilde{n},\wp),$$

It is a contradiction. Hence, we must have $\Gamma(\rho,\tilde{n},\wp)=1, \Xi(\rho,\tilde{n},\wp)=0$ and $\Phi(\rho,\tilde{n},\wp)=0$, therefore, $\rho=\tilde{n}$.

*Example 2.*Let $\Lambda=[0,1]$ and define a binary relation $\perp$ by

$$\upsilon\perp\Omega \text{ iff } \upsilon,\Omega\geq 0.$$

Define $\Gamma,\Xi,\Phi:\Lambda\times\Lambda\times(0,+\infty)\to[0,1]$ as

$$\begin{aligned}\Gamma(\rho,\Omega,\wp) &= \frac{\wp}{\wp+|\rho-\Omega|},\\ \Xi(\rho,\Omega,\wp) &= \frac{|\rho-\Omega|}{\wp+|\rho-\Omega|},\\ \Phi(\rho,\Omega,\wp) &= \frac{|\rho-\Omega|}{\wp}.\end{aligned}$$

Clearly ω is an $\perp$-preserving mapping. Then, $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$ with continuous $t-$norm $\rho*\tau=\rho\tau$ and continuous t-co-norm $\rho\circ\tau=\max\{\rho,\tau\}$. Define $\omega\colon\Lambda\to\Lambda$ by $\omega(\rho)=\frac{1-3^{-\rho}}{12}$ and take $\theta\in[\frac{1}{2},1)$, then

$$
\begin{aligned}
\Gamma(\omega\rho,\omega\Omega,\theta\wp) &= \Gamma\left(\frac{1-3^{-\rho}}{12},\frac{1-3^{-\Omega}}{12},\theta\wp\right)\\
&= \frac{\theta\wp}{\theta\wp+\left|\frac{1-3^{-\rho}}{12}-\frac{1-3^{-\Omega}}{12}\right|} = \frac{\theta\wp}{\theta\wp+\frac{|3^{-\rho}-3^{-\Omega}|}{12}}\\
&\geq \frac{\theta\wp}{\theta\wp+\frac{|\rho-\Omega|}{12}} = \frac{12\theta\wp}{12\theta\wp+|\rho-\Omega|} \geq \frac{\wp}{\wp+|\rho-\Omega|} = \Gamma(\rho,\Omega,\wp),
\end{aligned}
$$

$$
\begin{aligned}
\Xi(\omega\rho,\omega\Omega,\theta\wp) &= \Xi\left(\frac{1-3^{-\rho}}{12},\frac{1-3^{-\Omega}}{12},\theta\wp\right)\\
&= \frac{\left|\frac{1-3^{-\rho}}{12}-\frac{1-3^{-\Omega}}{12}\right|}{\theta\wp+\left|\frac{1-3^{-\rho}}{12}-\frac{1-3^{-\Omega}}{12}\right|} = \frac{\frac{|3^{-\rho}-3^{-\Omega}|}{12}}{\theta\wp+\frac{|3^{-\rho}-3^{-\Omega}|}{12}}\\
&= \frac{|3^{-\rho}-3^{-\Omega}|}{12\theta\wp+|3^{-\rho}-3^{-\Omega}|} \leq \frac{|\rho-\Omega|}{12\theta\wp+|\rho-\Omega|} \leq \frac{|\rho-\Omega|}{\wp+|\rho-\Omega|} = \Xi(\rho,\Omega,\wp)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\omega\rho,\omega\Omega,\theta\wp) &= \Phi\left(\frac{1-3^{-\rho}}{12},\frac{1-3^{-\Omega}}{12},\theta\wp\right)\\
&= \frac{\left|\frac{1-3^{-\rho}}{12}-\frac{1-3^{-\Omega}}{12}\right|}{\theta\wp} = \frac{\frac{|3^{-\rho}-3^{-\Omega}|}{12}}{\theta\wp}\\
&= \frac{|3^{-\rho}-3^{-\Omega}|}{12\theta\wp} \leq \frac{|\rho-\Omega|}{12\theta\wp} \leq \frac{|\rho-\Omega|}{\wp} = \Phi(\rho,\Omega,\wp).
\end{aligned}
$$

As a result, all of the conditions of Theorem 1 are satisfied, and 0 is the only fixed point for ω.

4 Application to Fredholm integral equation

Suppose $\Lambda=\mathscr{C}([\aleph,\mathfrak{a}],\mathbb{R})$ is the set of real value continuous functions defined on $[\aleph,\mathfrak{a}]$.
Suppose the integral equation below:

$$
\rho(\varpi)=\wedge(\varpi)+\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u \quad \text{for} \quad \varpi,u\in[\aleph,\mathfrak{a}] \tag{42}
$$

where $\delta>0,\wedge(u)$ is a fuzzy function of $u\colon u\in[\aleph,\mathfrak{a}]$ and $\mho\colon\mathscr{C}([\aleph,\mathfrak{a}]\times\mathbb{R})\to\mathbb{R}^+$. Define a binary relation $\perp$ by

$$
\upsilon\perp\Omega \text{ iff } \upsilon,\Omega\geq 0.
$$

Define Γ, Ξ and Φ by means of

$$
\begin{aligned}
\Gamma(\rho(\varpi),\Omega(\varpi),\wp) &= \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|} \quad \text{for all} \quad \rho,\Omega\in\Lambda \text{ and } \wp>0,\\
\Xi(\rho(\varpi),\Omega(\varpi),\wp) &= 1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|} \quad \text{for all} \quad \rho,\Omega\in\Lambda \text{ and } \wp>0,
\end{aligned}
$$

and

$$\Phi(\rho(\varpi),\Omega(\varpi),\wp)=\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\rho(\varpi)-\Omega(\varpi)|}{\wp}\quad\text{for all}\quad \rho,\Omega\in\Lambda \text{ and } \wp>0,$$

with continuous $t-$norm and continuous $t-$co-norm define by $\rho*\tau=\rho\cdot\tau$ and $\rho\circ\tau=\max\{\rho,\tau\}$. Clearly ω is an $\perp$-preserving mapping. Then $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$.

Theorem 3. *Assume that the following conditions are fulfilled:*

(i) for all $\rho,\Omega\in\Lambda$,

$$|\mho(\varpi,u)\rho(\varpi)-\mho(\varpi,u)\Omega(\varpi)|\leq|\rho(\varpi)-\Omega(\varpi)|,$$

(ii) for all $\varpi,u\in[\aleph,\mathfrak{a}]$,

$$\mho(\varpi,u)(\delta\int_{\aleph}^{\mathfrak{a}}du)\leq\theta<1,$$

where $\theta\in(0,1)$.

Then, the integral Equation (42) has a unique solution.

Proof. Define an operator $\omega:\Lambda\to\Lambda$ by

$$\omega\rho(\varpi)=\wedge(\varpi)+\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u\quad\text{for all}\quad\varpi,u\in[\aleph,\mathfrak{a}].$$

Now, for all $\rho,\Omega\in\Lambda$, we deduce that

$$\begin{aligned}
&\Gamma(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp)\\
&=\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\omega\rho(\varpi)-\omega\Omega(\varpi)|}\\
&=\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\wedge(\varpi)+\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\wedge(\varpi)-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}\\
&=\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}\\
&=\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\mho(\varpi,u)\rho(\varpi)-\mho(\varpi,u)\Omega(\varpi)|(\delta\int_{\aleph}^{\mathfrak{a}}\beta u)}\\
&\geq\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\\
&\geq\Gamma(\rho(\varpi),\Omega(\varpi),\wp),
\end{aligned}$$

$$\begin{aligned}
&\Xi(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp)\\
&=1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\omega\rho(\varpi)-\omega\Omega(\varpi)|}\\
&=1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\wedge(\varpi)+\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\wedge(\varpi)-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}\\
&=1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}\\
&=1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\theta\wp}{\theta\wp+|\mho(\varpi,u)\rho(\varpi)-\mho(\varpi,u)\Omega(\varpi)|(\delta\int_{\aleph}^{\mathfrak{a}}\beta u)}\\
&\leq1-\sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\\
&\leq\Xi(\rho(\varpi),\Omega(\varpi),\wp),
\end{aligned}$$

and

$$\begin{aligned}\Phi(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp) &= \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\omega\rho(\varpi)-\omega\Omega(\varpi)|}{\theta\wp}\\ &= \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\wedge(\varpi)+\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\wedge(\varpi)-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}{\theta\wp}\\ &= \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u-\delta\int_{\aleph}^{\mathfrak{a}}\mho(\varpi,u)\rho(\varpi)\beta u|}{\theta\wp}\\ &= \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\mho(\varpi,u)\rho(\varpi)-\mho(\varpi,u)\Omega(\varpi)|(\delta\int_{\aleph}^{\mathfrak{a}}\beta u)}{\theta\wp}\\ &\leq \sup_{\varpi\in[\aleph,\mathfrak{a}]}\frac{|\rho(\varpi)-\Omega(\varpi)|}{\wp}\\ &\leq \Phi(\rho(\varpi),\Omega(\varpi),\wp),\end{aligned}$$

As a result, all of the conditions of Theorem (1) are satisfied and operator ω has a unique fixed point.

The example below support Theorem 3:

*Example 3.*Assume the following non-linear integral equation:

$$\rho(\varpi)=|\cos\varpi|+\frac{1}{13}\int_0^1 u\rho(u)\beta u,\ \forall\, u\in[0,1]$$

Then, it has a solution in Λ.

*Proof.*Let $\omega:\Lambda\to\Lambda$ be defined by

$$\omega\rho(\varpi)=|\cos\varpi|+\frac{1}{13}\int_0^1 u\rho(u)\beta u,$$

and set $\mho(\varpi,u)\rho(\varpi)=\frac{1}{13}u\rho(u)$ and $\mho(\varpi,u)\Omega(\varpi)=\frac{1}{13}u\Omega(u)$, where $\rho,\Omega\in\Lambda$, and $\forall\,\varpi,u\in[0,1]$. Then we get

$$\begin{aligned}|\mho(\varpi,u)\rho(\varpi)-\mho(\varpi,u)\Omega(\varpi)| &= |\frac{1}{13}u\rho(u)-\frac{1}{13}u\Omega(u)|\\ &= \frac{u}{13}|\rho(u)-\Omega(u)|\leq|\rho(u)-\Omega(u)|.\end{aligned}$$

Additionally, see that $\frac{1}{13}\int_0^1 u\beta u=\frac{1}{13}\big(\frac{(1)^2}{2}-\frac{(0)^2}{2}\big)=\frac{1}{13}=\theta\leq 1$, where $\delta=\frac{1}{13}$. Then, it is clear that all other conditions of the above application are easy to examine and the above problem has a solution in Λ.

5 Application to fractional differential equations

Firstly, we remember some fundamental definitions from the theory of fractional calculus.
For a function $\rho\in\mathscr{C}[0,1]$, the Reiman-Liouville fractional derivative of order $\delta>0$ is given by

$$\frac{1}{\Gamma(\pi-\delta)}\frac{d^\pi}{d\varpi^\pi}\int_0^\varpi\frac{\rho(\aleph)d\aleph}{(\varpi-\aleph)^{\delta-\pi+1}}=\mathscr{D}^\delta\rho(\varpi),$$

provided that the right hand side is pointwise defined on $[0,1]$, where $[\delta]$ is the integer part of the number δ,Γ is the Euler gamma function.
Let as consider the fractional differential equation

$$\begin{aligned}&{}^{\aleph}\mathscr{D}^\xi\rho(\varpi)+\chi(\varpi,\rho(\varpi))=0,\quad 1\leq\varpi\leq 0,\quad 2\leq\xi>1;\\ &\rho(0)=\rho(1)=0,\end{aligned}\tag{43}$$

where χ is a continuous function from $[0,1]\times\mathbb{R}$ to $\mathbb{R}$ and ${}^{\aleph}\mathscr{D}^{\xi}$ represents the Caputo fractional derivative of order ξ and it is defined by

$$^{\aleph}\mathscr{D}^{\xi}=\frac{1}{\Gamma(\pi-\xi)}\int_{0}^{\varpi}\frac{\rho^{\pi}(\aleph)d\aleph}{(\varpi-\aleph)^{\xi-\pi+1}}.$$

The given fractional differential equation (43) is equivalent

$$\rho(\varpi)=\int_{0}^{1}\Omega(\varpi,\aleph)\chi(\varpi,\rho(\aleph))d\aleph,$$

for all $\rho\in\mathscr{Y}$ and $\varpi\in[0,1]$, where

$$\Omega(\varpi,\aleph)=\begin{cases}\frac{[\varpi(1-\aleph)]^{\xi-1}-(\varpi-\aleph)^{\xi-1}}{\Gamma(\xi)}, & 0\leq\aleph\leq\varpi\leq1,\\ \frac{[\varpi(1-\aleph)]^{\xi-1}}{\Gamma(\xi)}, & 0\leq\varpi\leq\aleph\leq1.\end{cases}$$

Let $\mathscr{C}([0,1],\mathbb{R})=\Lambda$ be the space of all continuous functions defined on $[0,1]$. Define a binary relation $\perp$ by

$$\upsilon\perp\Omega \text{ iff } \upsilon,\Omega\geq0.$$

Define Γ, Ξ and Φ by means of

$$\Gamma(\rho(\varpi),\Omega(\varpi),\wp)=\sup_{\varpi\in[0,1]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\quad\text{for all}\quad\rho,\Omega\in\Lambda\text{ and }\wp>0,$$

$$\Xi(\rho(\varpi),\Omega(\varpi),\wp)=1-\sup_{\varpi\in[0,1]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\quad\text{for all}\quad\rho,\Omega\in\Lambda\text{ and }\wp>0$$

and

$$\Phi(\rho(\varpi),\Omega(\varpi),\wp)=\sup_{\varpi\in[0,1]}\frac{|\rho(\varpi)-\Omega(\varpi)|}{\wp}\quad\text{for all}\quad\rho,\Omega\in\Lambda\text{ and }\wp>0,$$

with continuous t-norm and continuous t-co-norm define by $\rho*\tau=\rho\cdot\tau$ and $\rho\circ\tau=\max\{\rho,\tau\}$. Clearly, ω is an $\perp$-preserving mapping. Then, $(\Lambda,\Gamma,\Xi,\Phi,*,\circ,\perp)$ is a complete $\mathscr{ONPMS}$.

Theorem 4. *Let the nonlinear fractional differential equation (43). Assume that the given assertions are satisfied:*

(i) there exists $\varpi\in[0,1]$ and $\rho,\Omega\in\mathscr{C}([0,1],\mathbb{R})$ s.t.

$$|\chi(\varpi,\rho)-\chi(\varpi,\Omega)|\leq|\rho(\varpi)-\Omega(\varpi)|;$$

(ii)

$$\sup_{\varpi\in[0,1]}\int_{0}^{1}\Omega(\varpi,\aleph)\beta\varpi\leq\theta<1.$$

Then the fractional differential equation (43) has a unique solution in Λ.

Proof. Let the map $\omega:\Lambda\to\Lambda$ defined as

$$\Lambda\rho(\varpi)=\int_{0}^{1}\Omega(\varpi,\aleph)\chi(\varpi,\rho(\aleph))d\aleph.$$

It is easy to note that if $\rho^{*}\in\Lambda$ is a fixed point of ω then ρ^{*} is a solution of the problem (43).
Now, for all $\rho,\Omega\in\Lambda$, we deduce that

$$\begin{aligned}\Gamma(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp)&=\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+|\omega\rho(\varpi)-\omega\Omega(\varpi)|}\\&=\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+|\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\rho(\aleph))d\aleph-\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\Omega(\aleph))d\aleph|}\\&=\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+\int_0^1\Omega(\varpi,\aleph)|\chi(\varpi,\rho(\aleph))-\chi(\varpi,\Omega(\aleph))|d\aleph}\\&\geq\sup_{\varpi\in[0,1]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\\&\geq\Gamma(\rho(\varpi),\Omega(\varpi),\wp),\end{aligned}$$

$$\begin{aligned}\Xi(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp) &= 1-\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+|\omega\rho(\varpi)-\omega\Omega(\varpi)|}\\ &= 1-\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+|\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\rho(\aleph))d\aleph-\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\Omega(\aleph))d\aleph|}\\ &= 1-\sup_{\varpi\in[0,1]}\frac{\theta\wp}{\theta\wp+\int_0^1\Omega(\varpi,\aleph)|\chi(\varpi,\rho(\aleph))-\chi(\varpi,\Omega(\aleph))|d\aleph}\\ &\leq 1-\sup_{\varpi\in[0,1]}\frac{\wp}{\wp+|\rho(\varpi)-\Omega(\varpi)|}\\ &\leq \Xi(\rho(\varpi),\Omega(\varpi),\wp)\end{aligned}$$

and

$$\begin{aligned}\Phi(\omega\rho(\varpi),\omega\Omega(\varpi),\theta\wp) &= \sup_{\varpi\in[0,1]}\frac{|\omega\rho(\varpi)-\omega\Omega(\varpi)|}{\theta\wp}\\ &= \sup_{\varpi\in[0,1]}\frac{|\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\rho(\aleph))d\aleph-\int_0^1\Omega(\varpi,\aleph)\chi(\varpi,\Omega(\aleph))d\aleph|}{\theta\wp}\\ &= \sup_{\varpi\in[0,1]}\frac{\int_0^1\Omega(\varpi,\aleph)|\chi(\varpi,\rho(\aleph))-\chi(\varpi,\Omega(\aleph))|d\aleph}{\theta\wp}\\ &\leq \sup_{\varpi\in[0,1]}\frac{|\rho(\varpi)-\Omega(\varpi)|}{\wp}\\ &\leq \Phi(\rho(\varpi),\Omega(\varpi),\wp).\end{aligned}$$

As a result, all of the conditions of Theorem (1) are fulfilled and operator ω has a unique fixed point.

6 Conclusion

We have explored the concept of $\mathscr{ONPMS}$ (orthogonal neutrosophic pentagonal metric space) and presented novel types of fixed-point theorems. Our results serve to unify and generalize numerous articles within this domain. Furthermore, we have supplemented our work with an application that demonstrates the superiority of our developed approach over existing methods found in the literature. It is shown that, the utilization of fixed point theory is paramount in establishing the existence and uniqueness of a wide range of problems modeled by nonlinear relations. Particularly, in the study of fractional differential equations, fixed point theory plays a crucial role in addressing the issues of existence and uniqueness. This theory has been continuously developed for nearly a century and remains a subject of active research due to its widespread applications. Moreover, its applicability extends to various spatial configurations, encompassing metric, abstract, and Sobolev spaces. Given the significance of fractional ordinary, partial differential, and difference equations in describing real-world scientific problems, fixed point theory proves to be exceptionally valuable in investigating these matters.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

: The authors declare that they have no conflicts of interest.

Author's contributions

All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

References

[1] A.H. Abdel-Aty, M.M.A. Khater, R.A.M. Attia, M. Abdel-Aty, H. Eleuch, On the new explicit solutions of the fractional nonlinear space-time nuclear model, *Fractals*, 2020 : DOI 10.1142/S0218348X20400356.
[2] W. Gao, J.L.G. Guirao, M. Abdel-Aty, W. Xi, An independent set degree condition for fractional critical deleted graphs, *Discrete and Continuous Dynamical Systems-Series S*, **12(4-5)**, (2019), 877-886. DOI 10.3934/dcdss.2019058
[3] X.J. Yang, M. Abdel-Aty, C. Cattani, A New general fractional-order derivative with rabotnov fractional-exponential kernel applied to model the anomalous heat transfer, *Thermal Sci.*, **23(3)**, (2019), 1677–1681. DOI 10.2298/TSCI180320239Y
[4] G.M. Ismail, H.R. Abdl-Rahim, A.H. Abdel-Aty, W. .Alharbi, M. Abdel-Aty, An analytical solution for fractional oscillator in a resisting medium, *Chaos, Solitons & Fract.*, **130**, (2020), 109395. DOI 10.1016/j.chaos.2019.109395
[5] A.H. Bhrawy, M.A. Zaky, M. Abdel-Aty, A fast and precise numerical algorithm for a class of variable-order fractional differential equations, *Proc. Romanian Academy Series A−Math. Phy. Technical Sci. Information Sci.*, 18(1), (2017), 17–24. DOI 10.1016/j.chaos.2020.110048
[6] Q. Zhang, W. Xiaopeng, An Efficient approach for DNA fractal-based image encryption, *Appl. Math. Inf. Sci.*, **5(3)**, (2011), 445-459.
[7] H.W. Tin, S.W. Leu, H. Sasaki, S.H. Chang, A Novel fractal block coding method by using new shape-based descriptor, *Appl. Math. Inf. Sci.,* **8(2),** (2014), 849-855.
[8] R. Uthayakumar, A. Gowrisankar, Generation of fractals via self-similar group of Kannan iterated function system, *Appl. Math. Inf. Sci.*, **9(6)**, (2015), 3245-3250.
[9] E. Wolfgang, Lorenz, J. Andres, G. Franck, Fractal aesthetics in architecture, *Appl. Math. Inf. Sci.*, **11(4)**, (2017), 971-981. doi:10.18576/amis/110404
[10] M.A. Noor, K.I. Noor, S. Iftikhar, M.U. Awan, Fractal integral inequalities for Harmonic convex functions, *Appl. Math. Inf. Sci.,* **12(4)**, (2018), 831-839. Doi:10.18576/amis/120418
[11] H.A. Hammad, M. De la Sen, Analytical solution of Urysohn integral equations by fixed point technique in complex valued metric spaces, *Mathematics*, **7**, 2019, 852. https://doi.org/10.3390/math7090852
[12] H.A. Hammad, H. Aydi, H. Işik, M. De la Sen, Existence and stability results for a coupled system of impulsive fractional differential equations with Hadamard fractional derivatives, *AIMS Math.*, **8,** (2023), 6913–6941. Doi:10.3934/math.2023350
[13] H.A. Hammad, R.A. Rashwan, A. Nafea, M.E. Samei, S. Noeiaghdam, Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions, *J. Vibration Control,* **0(0)**, (2023) doi:10.1177/10775463221149232
[14] H.A. Hammad, M. De la Sen, Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales, *Fractal Fract.*, **7** (2023), 92. https://doi.org/10.3390/fractalfract7010092
[15] H.A. Hammad, M. De la Sen, H. Aydi, Generalized dynamic process for an extended multi-valued $F-$contraction in metric-like spaces with applications, *Alex. Eng. J.*, **59** (2020), 3817–3825. https://doi.org/10.1016/j.aej.2020.06.037
[16] V. Dinesh, G. Murugesan, A CPW-Fed Hexagonal Antenna with fractal elements for UWB applications, *Appl. Math. Inf. Sci*. **13(1)**, (2019), 73-79. Doi:10.18576/amis/130110
[17] R. Uthayakumar, A. Gowrisankar, Generalized fractal dimensions in image thresholding technique, *Inf. Sci. Lett.* **3(3)**, (2014), 125-134. http://dx.doi.org/10.12785/isl/030306
[18] A.B. Shaalan, N.F. Habubi, S.S. Chiad, Z.A. Toma, New design of hairpin-koch fractal filter for suppression of spurious band, *Int. J. Thin Film Sci. Tech.* 2(3), (2013), 217-221.
[19] T. Said, A. Chouikh, M. Bennai, Two-Step scheme for implementing N two-qubit quantum logic gates via cavity QED, **Appl. Math. Inf. Sci.,** 12(4), (2018), 699-704. Doi:10.18576/amis/120404
[20] N. Nikolai, Jr. Bogolyubov, V. Andrey, Soldatov, Time-convolutionless Master equation for multi-level open quantum systems with initial system-environment correlations, *Appl. Math. Inf. Sci.,* **13(5)**, (2019), 725-734. Doi:10.18576/amis/130504
[21] A. Redwan, A.H. Abdel-Aty, N. Zidan, Dynamics of classical and quantum information on spin-chains with multiple, *Inf. Sci. Lett.* **7(2),** (2018), 29-33. http://dx.doi.org/10.12785/isl/070201
[22] R. Sridevi, P. Philominathan, Quantum colour image encryption algorithm based on DNA and unified logistic tent map, *Inf. Sci. Lett.*, **9(3)**, (2020), 219-231. http://dx.doi.org/10.18576/isl/090309
[23] L.A. Zadeh, Fuzzy sets. *Inf. Control*, **8**, (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X
[24] Soong, Cai-Juan, Rosshairy Abd Rahman, Razamin Ramli, Mohammed Suhaimee Abd Manaf, and Chek-Choon Ting. An Evolutionary Algorithm: An Enhancement of Binary Tournament Selection for Fish Feed Formulation. Complexity, **2022** Article ID 7796633 (2022): 1 -15. https://doi.org/10.1155/2022/7796633
[25] B. Schweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.*, **10**, (1960), 314-334.
[26] I. Kramosil, J. Michlek, Fuzzy metric and statistical metric spaces, *Kybernetika*, **11**, (1975), 336-344.
[27] M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets Syst.*, **27**, (1988), 385-389.
[28] S.U. Rehman, S. Jabeen, S.U. Khan, M.M.M. Jaradat, Some $\alpha-\phi$-fuzzy cone contraction results with integral type application, *J. Math.,* **2021**, (2021), 1566348. https://doi.org/10.1155/2021/1566348
[29] J.H. Park, Intuitionistic fuzzy metric spaces, *Chaos Solitons Fractals*, **22**, (2004), 1039-1046. https://doi.org/10.1016/j.chaos.2004.02.051
[30] N. Konwar, Extension of fixed results in intuitionistic fuzzy $b-$metric spaces, *J. Intell. Fuzzy Syst.,* **39**, (2020), 7831-7841.
[31] M. Kirişci, N. Simsek, Neutrosophic metric spaces, *Math. Sci.,* **14**, (2020), 241-248. https://doi.org/10.1007/s40096-020-00335-8
[32] N. Simsek, M. Kirişci, Fixed point theorems in neutrosophic metric spaces, *Sigma J. Eng. Nat. Sci.,* **10**, (2019), 221-230.

[33] S. Sowndrarajan, M. Jeyarama, F. Smarandache, Fixed point results for contraction theorems in neutrosophic metric spaces, *Neutrosophic Sets Syst.,* **36**, (2020), 23.
[34] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.,* **67**, (1979), 261-273.
[35] N. Mlaiki, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6**, (2018), 194.
[36] M.S. Sezen, Controlled fuzzy metric spaces and some related fixed point results, *Numer. Methods Partial Differ. Equ.,* **37**, (2020), 583-593.
[37] N. Saleem, H. Isik, S. Furqan, C. Park, Fuzzy double controlled metric spaces, *J. Intell. Fuzzy Syst.*, **40**, (2021), 9977-9985.
[38] M. Rafi, M.S.M. Noorani, Fixed theorems on intuitionistic fuzzy metric space, *Iran. J. Fuzzy Syst.,* **3**, (2006), 23-29.
[39] W. Sintunavarat, P. Kumam, Fixed theorems for a generalized intuitionistic fuzzy contraction in intuitionistic fuzzy metric spaces, *Thai J. Math.,* **10**, (2012), 123-135.
[40] C. Alaca, D. Turkoglu, C. Yildiz, Fixed points in intuitionistic fuzzy metric spaces, *Chaos Solitons Fract.,* **29**, (2006), 1073-1078.
[41] H.A. Hammad, H. Aydi, Y.U. Gaba, Exciting fixed point results on a novel space with supportive applications, *J. Function Spaces*, **2021**, (2021), 1-12. https://doi.org/10.1155/2021/6613774
[42] A. Mohamad, Fixed-point theorems in intuitionistic fuzzy metric spaces, *Chaos Solitons Fract.,* **34**, (2007), 1689-1695.
[43] D. Dey, M. Saha, An extension of Banach fixed point theorem in fuzzy metric space, *Bol. Soc. Parana. Math.,* **32**, (2014), 299-304.
[44] Humaira, H.A. Hammad, M. Sarwar, M. De la Sen, Existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces, *Adv. Differ. Equ.,* **2021**, (2021), 242. https://doi.org/10.1186/s13662-021-03401-0
[45] M.E. Gordji, M. Ramezani, M. De La Sen, and Y. J. Cho, On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory (FPT),* **18(2)**, (2017) 569-578.
[46] M. Gunaseelan, G.A. Joseph, P. Choonkil, Y. Sungsik, Orthogonal O-contractions on b-complete metric space, *AIMS Math.,* **6(8)**, (2021), 8315-8330.
[47] P.S. Kumar, G.A. Joseph, M. Gunaseelan, S. Kumar, Solving differential equation via orthogonal branciari metric spaces, *J. Function Spaces*, **2023**, (2023), 1-10.
[48] H.A. Hammad, M. Zayed, Solving systems of coupled nonlinear Atangana-Baleanu-type fractional differential equations, *Bound. Value Probl.*, **2022**, (2022), 101. https://doi.org/10.1186/s13661-022-01684-0
[49] G. Janardhanan, G. Mani, O. Ege, V. Vidhya, R. George, Orthogonal neutrosophic 2-metric spaces, *J Inequal Appl.,* **2023**, (2023), 112. https://doi.org/10.1186/s13660-023-03024-x
[50] H.A. Hammad, M. De la Sen, Fixed-point results for a generalized almost $(s,q)-$Jaggi $F-$contraction-type on $b-$metric-like spaces, *Mathematics*, **8**, (2020), 63. https://doi.org/10.3390/math8010063
[51] E. Ameer, H. Aydi, H.A. Hammad, W. Shatanawi, N. Mlaiki, On $(\phi,\psi)-$metric spaces with applications, Symmetry, **12**, (2020), 1459. https://doi.org/10.3390/sym12091459