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Crossover Dynamics of Lorenz Model: Numerical Simulations

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Abstract: In recent decades, there has been increased interest in the dynamic behaviors of fractional order differential systems. Many results on fractional-order chaotic systems were obtained only via analytical and numerical techniques. This paper aims to numerically study the demeanor of three different classes of piecewise chaotic systems. This system is Lorenz chaotic system. The fractional derivatives are defined in the Caputo and Riemann-Liouville senses. The deterministic model is expanded using the constant proportional Caupto operator. Grünwald−Letnikov non-standard finite difference scheme is presented to approximate the constant proportional Caupto fractional operator. Finally, numerical simulations are provided to confirm the accuracy of our research and provide comparison analyses.

Keywords: Lorenz mathematical model; Caputo proportional constant fractional derivative; nonstandard finite difference method.

1 Introduction

It is known that a chaotic demeanor has a serious impact on people in their daily lives, for example climate change. Newly, many researchers have become interested in the chaotic demeanor of fractional order dynamic systems that have been observed in several fields of science, engineering, meteorology ([1]-[7]) and hereditary properties of different materials and processes. In reality, these effects are disregarded in models with traditional integer order. The main benefit of fractional derivatives can be seen as this, and it is essential in describing the dynamics between two separate points in many other domains as well ([4]-[12]). Although fractional derivatives and integrals can be thought of as a generalization of their conventional counterparts, they are nevertheless an extraordinary and challenging topic to comprehend. Unlike widely used differential operators, it does not correspond to some significant geometrical meaning, such as the trend of functions or their convexity. Thus, this mathematical tool may occasionally be considered "far from reality". Fractional order calculus is crucial to understanding a wide range of physical phenomena since they are described in fractional orders ([4]). Fractional derivatives have several different definitions. For instance, Caputo and Riemann-Liouville introduced the concept of fractional order differentiation with power law in [5,6]. Another variant of fractional order derivative using the generalized Mittag-Leffler function with strong memory as a non-local and non-singular kernel was proposed by Atangana and Baleanu [8]. Baleanu and et al. introduce the hybrid fractional operator in [13]. A linear combination of a Riemann-Liouville integral and a Caputo derivative can express this operator. However, various research suggests that the system's memory may vary with time, place, or other circumstances. Many physical events and processes have memory and inherited qualities that can be explained by the variable order fractional(VOF) operators based on their non-stationary power-law kernel. In 1993, Samko and Ross [17] introduced the notion of variable order integral and differential as well as certain fundamental features. Thus, (VOF) calculus was a viable option for creating an efficient mathematical framework to characterize complicated physical systems and processes [16] accurately. Due to its

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suitability in modelling a wide range of phenomena across many fields of science and engineering, such as anomalous diffusion ([18]-[21]), control systems [22], petroleum engineering [23], and numerous further fields of physics and engineering, we mention some of them ([24]-[32]), VOF differential equations have subsequently drawn increasing amounts of attention.

In this work, we developed the Lorenz model by using the concept of a piecewise model. In the first interval, we defined differential equations with the correct order, the fractional order differential equations were defined in the second interval, and in the third interval the variable order differential equations were defined or vice versa. On the other hand, we will introduce modern numerical techniques. This technique is a constant proportional Cupto Grünwald−Letnikov nonstandard finite difference method (GLNFDM). Numerical simulations will be provided to demonstrate the effectiveness and broad application of the suggested strategy.

In this paper, the basic mathematical formulas are introduced in Section 2. In Section 3, the piecewise hybird fractional order derivatives for Lorenz mathematical model in the Caputo and Riemann-Liouville senses is presented. Section4 discusses equilibrium points and stability analysis. Existence and uniqueness are given in section 5. The numerical scheme for the Caputo proportional constant Grünwald−Letnikov nonstandard finite difference numerical method is presented in Section 6. Numerical simulations are discussed in Section 7. Finally, the conclusions are presented in Section 8.

2 Basic notations

In this section, we review certain crucial definitions that were utilised in the next portions of this article. Let's think about the following differential equation of the fractional order ϑ [5,6]

$$ {}_{a}^{RL}D_t^{\vartheta} y(t) = \xi(y(t),t), \quad y(0) = y_0, \quad n-1 < \vartheta \leq n. $$

Definition 1.*On the left and the right sides for a continuous function f(t), Riemann-lioville's derivatives of order ϑ are defined by [5,6]*

$$ {}_{a}^{RL}D_t^{\vartheta} y(t) = \frac{1}{\Gamma(n-\vartheta)}(\frac{d}{dt})^n \int_a^t \frac{y(\nu)}{(t-\nu)^{1-n+\vartheta}} d\nu, \ t > a, $$
$$ {}_{t}^{RL}D_b^{\vartheta} y(t) = \frac{1}{\Gamma(n-\vartheta)}(\frac{-d}{dt})^n \int_t^b \frac{y(\nu)}{(\nu-t)^{1-n+\vartheta}} d\nu, \ t < b. $$

such that

$$ -\infty < a < b < +\infty, \vartheta \in \mathbb{C}. $$

Definition 2.*On the left and the right sides for a continuous function f(t), Caputo's derivatives of order ϑ are defined by [5,6]*

$$ {}_{a}^{C}D_t^{\vartheta} y(t) = \frac{1}{\Gamma(n-\vartheta)} \int_a^t \frac{y^n(\nu)}{(t-\nu)^{1-n+\vartheta}} d\nu, \ t > a $$
$$ {}_{t}^{C}D_b^{\vartheta} y(t) = \frac{(-1)^n}{\Gamma(n-\vartheta)} \int_t^b \frac{y^n(\nu)}{(\nu-t)^{1-n+\vartheta}} d\nu, \ t < b $$

such that

$$ -\infty < a < b < +\infty, \vartheta \in \mathbb{C}. $$

Definition 3.*The Caputo proportional constant fractional order operator CPC is defined as [13]:*

$$ \begin{aligned} {}_{0}^{CPC}D_t^{\vartheta} y(t) &= (\int_0^t (k_1(\vartheta,\nu)y(\nu) + k_0(\vartheta,\nu)y^{'}(\nu))d\nu)\frac{(t-\nu)^{-\vartheta}}{\Gamma(1-\vartheta)} \\ &= \frac{t^{-\vartheta}}{\Gamma(1-\vartheta)}(y^{'}(t)k_0(\vartheta,t) + y(t)k_1(\vartheta,t)). \end{aligned} $$

Or
The CPC fractional order operator can be defined as follows [13]:

$$
\begin{aligned}
{}_{0}^{CPC}D_t^{\vartheta} y(t) &= \left(\int_0^t (t-\nu)^{-}\vartheta (k_0(\vartheta) y^{'}(\nu) + k_1(\vartheta) y(\nu) d\nu\right) \frac{1}{\Gamma(1-\vartheta)} \\
&= k_1(\vartheta) {}_{0}^{RL}I_t^{1-\vartheta} y(t) + k_0(\vartheta) {}_{0}^{c}D_t^{\vartheta} y(t),
\end{aligned}
\tag{1}
$$

where, $k_1(\vartheta), k_0(\vartheta)$ *are constants. Here we consider,* $k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta) Q^{\vartheta}$, *and* Q *is constant.*

Definition 4. *The piecewise derivative is retrieved as follows, where* $y \in C[0,T]$ *is the differentiable function [14]:*

$$
{}_{0}^{PWC}D_t^{\vartheta} y(t) = \begin{cases} y^{'}(t) & 0 < t \leq t_1 \\ {}_{0}^{C}D_t^{\vartheta} y(t) & t_1 < t \leq t_2 \\ {}_{0}^{C}D_t^{\vartheta(t)} y(t) & t_2 < t \leq T. \end{cases}
$$

In this work, we will introduce the piecewise CPC fractional derivative, which is retrieved as follows:

$$
{}_{0}^{CPC}D_t^{\vartheta} y(t) = \begin{cases} y^{'}(t) & 0 < t \leq t_1 \\ {}_{0}^{CPC}D_t^{\vartheta} y(t) & t_1 < t \leq t_2 \\ {}_{0}^{CPC}D_t^{\vartheta(t)} y(t) & t_2 < t \leq T. \end{cases}
\tag{2}
$$

Where y is the differentiable function.

3 The hybird piecewise variable order fractional for the Lorenz system

We introduce here the three various instances of the cross-over Lorenz model.

3.1 The piecewise variable order Lorenz model

Case 1: We developed the integer order Lorenz model [15] to be a piecewise model where the integer order Lorenz model is defined in the first interval $(0,t_1]$, the fractional Lorenz model is defined in the second interval $(t_1,t_2]$ and the variable order Lorenz model is defined in the third interval $(t_2,T]$. Then, the cross-over model of the Lorenz system is given as:

$$
\begin{cases} \frac{dx(t)}{dt} & = \delta(y(t) - x(t)), \\ \frac{dy(t)}{dt} & = x(t)(\gamma - z(t)) - y(t), \quad 0 < t \leq t_1, \\ \frac{dz(t)}{dt} & = x(t)y(t) - \varpi z(t), \end{cases}
\tag{3}
$$

$$
x(0) = x_0, y(0) = y_0, z(0) = z_0.
$$

$$
\begin{cases} {}_{0}^{CPC}D_t^{\vartheta} x(t) & = \delta(y(t) - x(t)), \\ {}_{0}^{CPC}D_t^{\vartheta} y(t) & = x(t)(\gamma - z(t)) - y(t), \quad t_1 < t \leq t_2, \\ {}_{0}^{CPC}D_t^{\vartheta} z(t) & = x(t)y(t) - \varpi z(t), \end{cases}
\tag{4}
$$

$$
x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1.
$$

$$\begin{cases} {}_0^{CPC}D_t^{\vartheta(t)}x(t) & = \delta(y(t)-x(t)), \\ {}_0^{CPC}D_t^{\vartheta(t)}y(t) & = x(t)(\gamma - z(t)) - y(t), \quad t_2 < t \leq T, \\ {}_0^{CPC}D_t^{\vartheta(t)}z(t) & = x(t)y(t) - \varpi z(t), \end{cases} \tag{5}$$

$$x(t_2) = x_2, y(t_2) = y_2, z(t_2) = z_2.$$

Where the Prandtl number is denoted by δ, γ is referred to as the Rayleigh number, and ϖ gives the approximate size of the region the system uses. All $\delta, \varpi, \gamma > 0$, but generally $\delta = 10$, $\gamma = 28$, and $\varpi = 8/3$. The Lorenz model presents a chaotic attractor for $\delta = 10$, $\gamma = 28$, and $\varpi = 8/3$.

Table 1: The variables in the system (3) [15].

The variable	Description
x	is proportional to the strength of the convective motion
y	is proportional to the difference in temperature between the currents that are ascending and dropping
z	is proportional to distortion of the vertical temperature profile from linearity.

Case 2: We develop here the Lorenz model [15] to be a piecewise model, where the variable order Lorenz model is defined in the first interval $(0,t_1]$, in the second interval $(t_1,t_2]$ the fractional Lorenz model is defined, and the integer order Lorenz model in the third interval $(t_2,T]$. Then, the model is given by:

$$\begin{cases} {}_0^{CPC}D_t^{\vartheta_{1(t)}}x(t) & = \delta(y(t)-x(t)), \\ {}_0^{CPC}D_t^{\vartheta_{2(t)}}y(t) & = x(t)(\gamma - z(t)) - y(t), \quad 0 < t \leq t_1, \\ {}_0^{CPC}D_t^{\vartheta_{3(t)}}z(t) & = x(t)y(t) - \varpi z(t), \end{cases} \tag{6}$$

$$x(0) = x_0, y(0) = y_0, z(0) = z_0.$$

$$\begin{cases} {}_0^{CPC}D_t^{\vartheta}x(t) & = \delta(y(t)-x(t)), \\ {}_0^{CPC}D_t^{\vartheta}y(t) & = x(t)(\rho - z(t)) - y(t), \quad t_1 < t \leq t_2, \\ {}_0^{CPC}D_t^{\vartheta}z(t) & = x(t)y(t) - \varpi z(t), \end{cases} \tag{7}$$

$$x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1.$$

$$\begin{cases} \frac{dx(t)}{dt} & = \delta(y(t)-x(t)), \\ \frac{dy(t)}{dt} & = x(t)(\gamma - z(t)) - y(t), \\ \frac{dz(t)}{dt} & = x(t)y(t) - \varpi z(t), \quad t_2 < t \leq T, \end{cases} \tag{8}$$

$$x(t_2) = x_2, y(t_2) = y_2, z(t_2) = z_2.$$

Case 3: We develop here the Lorenz model [15]to be a piecewise model, where the fractional order Lorenz model is defined in the first interval $(0,t_1]$, the integer Lorenz model is defined in the second interval $(t_1,t_2]$, and the variable order Lorenz model in the third interval $(t_2,T]$. Then, the proposed model is given by:

$$\begin{cases} {}_0^{CPC}D_t^{\vartheta_1}x(t) & = \delta(y(t)-x(t)), \\ {}_0^{CPC}D_t^{\vartheta_2}y(t) & = x(t)(\gamma - z(t)) - y(t), \quad 0 < t \leq t_1, \\ {}_0^{CPC}D_t^{\vartheta_3}z(t) & = x(t)y(t) - \varpi z(t), \end{cases} \tag{9}$$

$$x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1.$$

$$\begin{cases} {}_0^{CPC}D_t^{\vartheta_1(t)}x(t) &= \delta(y(t)-x(t)), \\ {}_0^{CPC}D_t^{\vartheta_2(t)}y(t) &= x(t)(\gamma - z(t)) - y(t), \quad t_1 < t \leq t_2, \\ {}_0^{CPC}D_t^{\vartheta_3(t)}z(t) &= x(t)y(t) - \varpi z(t), \end{cases} \tag{10}$$

$$x(0) = x_0, y(0) = y_0, z(0) = z_0.$$

$$\begin{cases} \frac{dx(t)}{dt} &= \delta(y(t)-x(t)), \\ \frac{dy(t)}{dt} &= x(t)(\gamma - z(t)) - y(t), \\ \frac{dz(t)}{dt} &= x(t)y(t) - \varpi z(t), \quad t_2 < t \leq T, \end{cases} \tag{11}$$

$$x(t_2) = x_2, y(t_2) = y_2, z(t_2) = z_2.$$

4 Equilibrium points and stability analysis:

We solved the system and then found two equilibrium points, where one is clearly in origin $\zeta_1 = (0;0;0)$, and for values of the parameters $\delta = 10, \gamma = 28$ and $\varpi = \frac{8}{3}$, are $\zeta_2 = 1.0e^-15 * (-0.2643; -0.2643; -0.2990)$. The Jacobian matrix J of the Lorenz model at the equilibrium points. (X^*, Y^*, Z^*)

$$J = \begin{pmatrix} -\delta & \delta & 0 \\ \gamma - Z^* & -1 & X^* \\ Y^* & X^* & -\varpi \end{pmatrix}, \tag{12}$$

The eigenvalues for equilibria λ_1 and λ_2 are same eigenvalues $\varsigma_1 \approx -22.8277 \varsigma_2 \approx 11.8277 \varsigma_3 \approx -2.6667$. All two equilibria are unstable.

5 Existence and uniqueness of the solution

In the following, we give the existence and uniqueness piecewisely. But to do this, we check the Lipschitz condition and the linear growth properties [36,37]. Also, let $\mathbb{B}$ is a Banach space and the norm $\parallel \varphi \parallel_\infty = \sup_{t \in D_\varphi} |\varphi(t)|$. We assume that $\forall t \in (0, t_1]$, there existence three positive constant $\parallel x \parallel_\infty < A_1, \parallel y \parallel_\infty < A_2$ *and* $\parallel z \parallel_\infty < A_3$.

$$\frac{dx}{dt} = g_1(x,y,z,t), \tag{13}$$
$$\frac{dy}{dt} = g_2(x,y,z,t), 0 < t \leq t_1, \tag{14}$$
$$\frac{dz}{dt} = g_3(x,y,z,t), \tag{15}$$

where $g_\rho(x,y,z,t), \rho = 1,2,3$, symbolize the right-hand side of the equations in the Lorenz system 3. We first verify that
$| g_\rho(x_\rho, t) |^2 < k_\rho(| x_\rho |^2 + 1)$,
$| g_\rho(x_1, t) - g_\rho(x_2, t) |^2 < k_\rho | x_1 - x_2 |^2$, where k_ρ is constant .
To substantiate the existence and uniqueness:

$$
\begin{aligned}
| g_1(x,y,z,t) |^2 =& | \delta y - \delta x |^2, \\
\leq& (| \delta y |^2 + | \delta x |^2), \\
\leq& (\delta^2 | y |^2 + \delta^2 | x |^2), \\
\leq& (\delta^2 [\sup_{t\in[0,t_1]} | y |^2 + | x |^2]), \\
\leq& (\delta^2 [||y^2||_\infty + |x|^2]), \\
\leq& \delta^2 (|x|^2 (1 + \frac{||y^2||_\infty}{|x|^2})), \\
\leq& \delta^2 (|x|^2 + 1),
\end{aligned}
$$

under the condition

$$(||y^2||_\infty) \leq 1.$$

We have

$$| g_1(x,y,z,t) |^2 < k_1(| x |^2 + 1).$$

Using the same routine,

$$
\begin{aligned}
| g_2(x,y,z,t) |^2 =& | x\gamma - \gamma z - y |^2, \\
\leq& (| x\gamma |^2 + | \gamma x |^2 + | y |^2), \\
\leq& (\sup_{t\in[0,t_1]} | x\gamma |^2 + \sup_{t\in[0,t_1]} | \gamma z |^2 + | y |^2), \\
\leq& (\gamma^2 (||x^2||_\infty + ||z^2||_\infty) + |y|^2), \\
\leq& \gamma^2 (||x^2||_\infty + ||z^2||_\infty + \frac{1}{\gamma^2} |y|^2), \\
\leq& \frac{1}{\gamma^2} (\gamma^2 |y|^2 (1 + \frac{\gamma^2 (||x^2||_\infty + ||z^2||_\infty)}{|y|^2})),
\end{aligned}
$$

under the condition

$$(\gamma^2 (||x^2||_\infty + ||z^2||_\infty)) \leq 1.$$

Then

$$| g_2(x,y,z,t) |^2 < k_2(| y |^2 + 1).$$

$$
\begin{aligned}
| g_3(x,y,z,t) |^2 =& | xy - \varpi z |^2, \\
\leq& (| x |^2 | y |^2 + \varpi^2 | z |^2), \\
\leq& (\sup_{t\in[0,t_1]} | x |^2 | y |^2 + \varpi^2 | z |^2), \\
\leq& (\| x^2 \|_\infty \| y^2 \|_\infty + \varpi^2 | z |^2), \\
\leq& \varpi^2 (|z|^2 (\frac{\| x^2 \|_\infty \| y^2 \|_\infty}{\varpi^2 |z|^2} + 1)), \\
\leq& \varpi^2 (1 + |z|^2),
\end{aligned}
$$

under the condition

$$(\frac{\| x^2 \|_\infty \| y^2 \|_\infty}{\varpi^2}) \leq 1.$$

Then

$$| g_3(x,y,z,t) |^2 < k_3(| z |^2 + 1).$$

Therefore the condition of linear growth is achieved if
$\max \{\| y^2 \|_\infty, (\gamma^2 (\| x^2 \|_\infty + \| z^2 \|_\infty)), (\frac{\|x^2\|_\infty \|y^2\|_\infty}{\varpi})\} < 1$.

6 Numerical scheme for Lorenz model

We can write the relation (1) as follows:

$$
\begin{aligned}
{}_0^{CPC}D_t^{\vartheta} y(t) &= (\Gamma(1-\vartheta))^{-1}\int_0^t (t-\nu)^{-\vartheta}(y'(\nu)K_0(\vartheta)+K_1(\vartheta)y(\nu))d\nu,\\
&= K_1(\vartheta)_0^{RL}I_t^{1-\vartheta}y(t)+K_0(\vartheta)_0^C D_t^{\vartheta}y(t),\\
&= K_1(\vartheta)_0^{RL}D_t^{\vartheta-1}y(t)+K_0(\vartheta)_0^C D_t^{\vartheta}y(t),
\end{aligned}
\tag{16}
$$

where, $K_1(\vartheta(t))$, $K_0(\vartheta(t))$are solely dependent on $\vartheta(t)$. The non-standard finite difference method (NSFDM), first presented by Mickens in [33], is a more accurate and stable approach than the standard finite difference method, see [34]. With the Grünwald−Letnikov nonstandard finite difference method (GLNSFDM), we can discretize (16) as follows:

$$
\begin{aligned}
{}_0^{CPC}D_t^{\vartheta} y(t)|_{t=t^{n_1}} = \frac{K_1(\vartheta)}{(\Theta(\Delta t))^{\vartheta-1}}\left(y_{n_1+1}+\sum_{i=1}^{1+n_1} y_{n_1+1-i}\omega_i\right)\\
+\frac{K_0(\vartheta)}{(\Theta(\Delta t))^{\vartheta}}\left(-\sum_{i=1}^{1+n_1}\mu_i y_{n1+1-i}-q_{n_1+1}y_0+y_{n_1+1}\right).
\end{aligned}
\tag{17}
$$

6.1 Numerical scheme for case 1

In the first interval $0<t\leq t_1$, the discretization scheme of (3) using the (nonlocal) GLNSFDM (17) can be expressed as follows:

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_1+1} w_j\, x_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}+\delta}\\
&\quad+\frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_1+1}\mu_j\, x_{n+1-j}-\, q_{n+1}x_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}+\delta},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
y_{n+1} &= \frac{\gamma x_n - x_n z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_1+1} w_j\, y_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}+1}\\
&\quad+\frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})(\sum_{j=1}^{n_1+1}\mu_j\, y_{n+1-j}-\, q_{n+1}y_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}+1},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
z_{n+1} &= \frac{x_n y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_1+1} w_j\, z_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta()\Delta t))^{-\vartheta}+\varpi}\\
&\quad+\frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_1+1}\mu_j\, z_{n+1-j}-\, q_{n+1}z_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}+k_0(\vartheta)(\theta()\Delta t))^{-\vartheta}+\varpi}.
\end{aligned}
\tag{20}
$$

Where $n=0,\ldots,n_1$. $k_0(\vartheta)=\vartheta Q^{(1-\vartheta)}, k_1(\vartheta)=(1-\vartheta)Q^{\vartheta}, \Delta t=\frac{T}{N_n}, N_n\in N$,
$\mu=\vartheta, q_1=\frac{1}{\Gamma(1-\vartheta)}, \mu_i=1-(\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1=1, \omega_i=1-(\frac{\vartheta}{i})\omega_{i-1}$.
The discretization of (4)in the second interval, $t_1<t\leq t_2$, is provided using the discretization of the CPC operator and

GLNSFDM (17) can be expressed as follows :

$$\begin{aligned}
x_{n+1} &= \frac{\delta y_n - \delta x_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, x_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1} \mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}}, \qquad (21)\\
y_{n+1} &= \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, y_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1} \mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})}, \qquad (22)\\
z_{n+1} &= \frac{x_n y_n - \varpi z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, z_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1} \mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}}. \qquad (23)
\end{aligned}$$

Where $n = n_1, ..., n_2$, $k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta)Q^{\vartheta}, \Delta t = \frac{T}{N_n}, N_n \in N$,
$\mu = \vartheta, q_1 = \frac{1}{\Gamma(1-\vartheta)}, \mu_i = 1-(\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1-(\frac{\vartheta}{i})\omega_{i-1}$.

The discretization of (5) in the third interval, $t_2 < t \leq T$, is provided using the discretization of the CPC operator and (GLNSFDM) (17)can be expressed as follows :

$$\begin{aligned}
x_{n+1} &= \frac{\delta y_n - \delta x_n - k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)}\sum_{j=1}^{n_3+1} w_j\, x_{n+1-j}}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}} \\
&+ \frac{k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}(\sum_{j=1}^{n_3+1} \mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}}, \qquad (24)\\
y_{n+1} &= \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)}\sum_{j=1}^{n_3+1} w_j\, y_{n+1-j}}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}} \\
&+ \frac{k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}(\sum_{j=1}^{n_3+1} \mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}}, \qquad (25)\\
z_{n+1} &= \frac{x_n y_n - \varpi z_n - k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)}\sum_{j=1}^{n_3+1} w_j\, z_{n+1-j}}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}} \\
&+ \frac{k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}(\sum_{j=1}^{n_3+1} \mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta(t))(\theta(\Delta t))^{1-\vartheta(t)} + k_0(\vartheta(t))(\theta(\Delta t))^{-\vartheta(t)}}. \qquad (26)
\end{aligned}$$

Where $n = n_2, ..., n_3$, $k_0(\vartheta(t) = \vartheta(t)Q^{(1-\vartheta(t))}, k_1(\vartheta(t)) = (1-\vartheta(t))Q^{\vartheta(t)}, \Delta t = \frac{T}{N_n}, N_n \in N$,
$\mu = \vartheta(t), q_1 = \frac{1}{\Gamma(1-\vartheta(t))}, \mu_i = 1-(\frac{\vartheta(t)+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1-(\frac{\vartheta(t)}{i})\omega_{i-1}$.

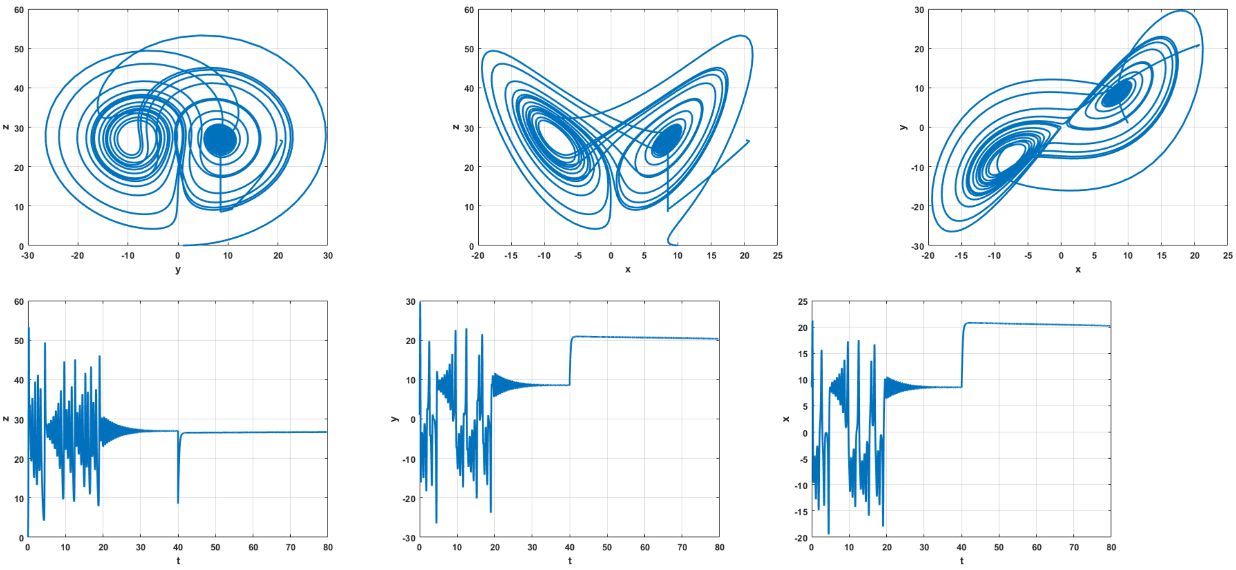

Fig. 1: Numerical simulation for case1 at $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0.95,, \vartheta_1(t) = \vartheta_2(t) = \vartheta_3(t) = 0.9 - 0.001t$.

6.2 Numerical scheme for case 2

In the first interval $0 < t \leq t_1$, the discretization scheme of (5) is provided using the discretization of CPC operator and (GLNSFDM) (17)can be expressed as follows :

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - \delta x_n - k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)} \sum_{j=1}^{n_1+1} w_j \, x_{n+1-j}}{k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)} + k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)}} \\
&+ \frac{k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)} (\sum_{j=1}^{n_1+1} \mu_j \, x_{n+1-j} - \, q_{n+1} x_0)}{k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)} + k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)}}, \qquad (27) \\
y_{n+1} &= \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)} \sum_{j=1}^{n_1+1} w_j \, y_{n+1-j}}{k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)} + k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)}} \\
&+ \frac{k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)} (\sum_{j=1}^{n_1+1} \mu_j \, y_{n+1-j} - \, q_{n+1} y_0)}{k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)} + k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)}}, \qquad (28) \\
z_{n+1} &= \frac{x_n y_n - \varpi z_n - k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)} \sum_{j=1}^{n_1+1} w_j \, z_{n+1-j}}{k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)} + k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)}} \\
&+ \frac{k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)} (\sum_{j=1}^{n_1+1} \mu_j \, z_{n+1-j} - \, q_{n+1} z_0)}{k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)} + k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)}}. \qquad (29)
\end{aligned}
$$

Where $n = 0, ..., n_1$, $k_0(\vartheta(t) = \vartheta(t) Q^{(1-\vartheta(t))}, k_1(\vartheta(t)) = (1-\vartheta(t)) Q^{\vartheta(t)}, \Delta t = \frac{T}{N_n}, N_n \in N,$ $\mu = \vartheta(t), q_1 = \frac{1}{\Gamma(1-\vartheta(t))}, \mu_i = 1 - (\frac{\vartheta(t)+1}{i}) \mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta(t)}{i}) \omega_{i-1}.$

The discretization of (4)in the second interval, $t_1 < t \leq t_2$, is provided using the discretization of the CPC operator, and

GLNSFDM (17) can be expressed as follows :

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - \delta x_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, x_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1}\mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}}, \qquad (30)\\
y_{n+1} &= \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, y_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1}\mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})}, \qquad (31)\\
z_{n+1} &= \frac{x_n y_n - \varpi z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_2+1} w_j\, z_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_2+1}\mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}}. \qquad (32)
\end{aligned}
$$

Where $n = n_1, ..., n_2$, $k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta)Q^{\vartheta}, \Delta t = \frac{T}{N_n}, N_n \in N$, $\mu = \vartheta, q_1 = \frac{1}{\Gamma(1-\vartheta)}, \mu_i = 1 - (\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta}{i})\omega_{i-1}$.

The discretization of (5) in the third interval, $t_2 < t \le T$, is provided using the (nonlocal) GLNSFDM (17) can be expressed as follows:

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, x_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + \delta} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_3+1}\mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + \delta}, \qquad (33)\\
y_{n+1} &= \frac{\gamma x_n - x_n z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, y_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + 1} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})(\sum_{j=1}^{n_3+1}\mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + 1}, \qquad (34)\\
z_{n+1} &= \frac{x_n y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, z_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta()\Delta t))^{-\vartheta} + \varpi} \\
&\quad + \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_3+1}\mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta()\Delta t))^{-\vartheta} + \varpi}. \qquad (35)
\end{aligned}
$$

Where $n = n_2, ..., n_3$, $k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta)Q^{\vartheta}, \Delta t = \frac{T}{N_n}, N_n \in N$, $\mu = \vartheta, q_1 = \frac{1}{\Gamma(1-\vartheta)}, \mu_i = 1 - (\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta}{i})\omega_{i-1}$.

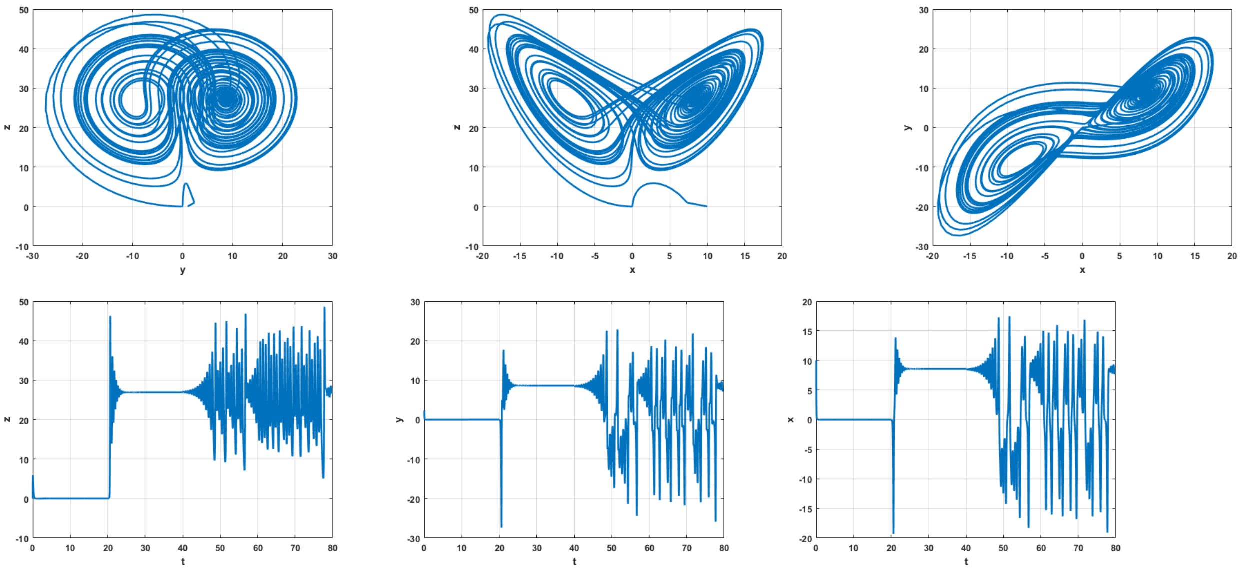

Fig. 2: Numerical simulation for case2 at $\vartheta_1 = \vartheta_2 = \vartheta_3 = .85,\ \vartheta_1(t) = 0.9 - 0.001t,\ \vartheta_2(t) = 0.98 - 0.001t,\ \vartheta_3(t) = 0.97 - 0.001t$.

6.3 Numerical scheme for case 3

We can formulate the discretization scheme of (4) in the first interval $0 < t \le t_1$, using the discretization of the CPC operator and GLNSFDM (17) as follows :

$$x_{n+1} = \frac{\delta y_n - \delta x_n - k_1(\vartheta_1)(\theta(\Delta t))^{1-\vartheta_1} \sum_{j=1}^{n_1+1} w_j\, x_{n+1-j}}{k_1(\vartheta_1)(\theta(\Delta t))^{1-\vartheta_1} + k_0(\vartheta_1)(\theta(\Delta t))^{-\vartheta_1}} + \frac{k_0(\vartheta_1)(\theta(\Delta t))^{-\vartheta_1}(\sum_{j=1}^{n_1+1} \mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta_1)(\theta(\Delta t))^{1-\vartheta_1} + k_0(\vartheta_1)(\theta(\Delta t))^{-\vartheta_1}}, \tag{36}$$

$$y_{n+1} = \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta_2)(\theta(\Delta t))^{1-\vartheta_2} \sum_{j=1}^{n_1+1} w_j\, y_{n+1-j}}{k_1(\vartheta_2)(\theta(\Delta t))^{1-\vartheta_2} + k_0(\vartheta_2)(\theta(\Delta t))^{-\vartheta_2})} + \frac{k_0(\vartheta_2)(\theta(\Delta t))^{-\vartheta_2}(\sum_{j=1}^{n_1+1} \mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta_2)(\theta(\Delta t))^{1-\vartheta_2} + k_0(\vartheta_2)(\theta(\Delta t))^{-\vartheta_2})}, \tag{37}$$

$$z_{n+1} = \frac{x_n y_n - \varpi z_n - k_1(\vartheta_3)(\theta(\Delta t))^{1-\vartheta_3} \sum_{j=1}^{n_1+1} w_j\, z_{n+1-j}}{k_1(\vartheta_3)(\theta(\Delta t))^{1-\vartheta_3} + k_0(\vartheta_3)(\theta(\Delta t))^{-\vartheta_3}} + \frac{k_0(\vartheta_3)(\theta(\Delta t))^{-\vartheta_3}(\sum_{j=1}^{n_1+1} \mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta_3)(\theta(\Delta t))^{1-\vartheta_3} + k_0(\vartheta_3)(\theta(\Delta t))^{-\vartheta_3}}. \tag{38}$$

Where $n = 0,...,n_1,\ k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta)Q^{\vartheta}, \Delta t = \frac{T}{N_n}, N_n \in N,$
$\mu = \vartheta, q_1 = \frac{1}{\Gamma(1-\vartheta)}, \mu_i = 1 - (\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta}{i})\omega_{i-1}.$

The discretization of (5)in the second interval, $t_1 < t \le t_2$, is provided using the discretization of the CPC operator, and GLNSFDM (17) can be formulated as follows :

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - \delta x_n - k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)}\sum_{j=1}^{n_2+1} w_j\, x_{n+1-j}}{k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)} + k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)}} \\
&+ \frac{k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)}(\sum_{j=1}^{n_2+1} \mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta_1(t))(\theta(\Delta t))^{1-\vartheta_1(t)} + k_0(\vartheta_1(t))(\theta(\Delta t))^{-\vartheta_1(t)}}, \quad (39) \\
y_{n+1} &= \frac{\gamma x_n - y_n - x_n z_n - k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)}\sum_{j=1}^{n_2+1} w_j\, y_{n+1-j}}{k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)} + k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)}} \\
&+ \frac{k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)}(\sum_{j=1}^{n_2+1} \mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta_2(t))(\theta(\Delta t))^{1-\vartheta_2(t)} + k_0(\vartheta_2(t))(\theta(\Delta t))^{-\vartheta_2(t)}}, \quad (40) \\
z_{n+1} &= \frac{x_n y_n - \varpi z_n - k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)}\sum_{j=1}^{n_2+1} w_j\, z_{n+1-j}}{k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)} + k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)}} \\
&+ \frac{k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)}(\sum_{j=1}^{n_2+1} \mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta_3(t))(\theta(\Delta t))^{1-\vartheta_3(t)} + k_0(\vartheta_3(t))(\theta(\Delta t))^{-\vartheta_3(t)}}. \quad (41)
\end{aligned}
$$

(42)

(43)

Where $n = n_1, ..., n_2$, $k_0(\vartheta(t) = \vartheta(t)Q^{(1-\vartheta(t))}, k_1(\vartheta(t)) = (1-\vartheta(t))Q^{\vartheta(t)}, \Delta t = \frac{T}{N_n}, N_n \in N$,
$\mu = \vartheta(t), q_1 = \frac{1}{\Gamma(1-\vartheta(t))}, \mu_i = 1 - (\frac{\vartheta(t)+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta(t)}{i})\omega_{i-1}$.

We can write the discretization of (3) in the third interval $t_2 < t \leq T$, using the (nonlocal) GLNSFDM (17) as follows:

$$
\begin{aligned}
x_{n+1} &= \frac{\delta y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, x_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + \delta} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_3+1} \mu_j\, x_{n+1-j} - q_{n+1}x_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + \delta}, \quad (44) \\
y_{n+1} &= \frac{\gamma x_n - x_n z_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, y_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + 1} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta})(\sum_{j=1}^{n_3+1} \mu_j\, y_{n+1-j} - q_{n+1}y_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta(\Delta t))^{-\vartheta} + 1}, \quad (45) \\
z_{n+1} &= \frac{x_n y_n - k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta}\sum_{j=1}^{n_3+1} w_j\, z_{n+1-j}}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta()\Delta t))^{-\vartheta} + \varpi} \\
&+ \frac{k_0(\vartheta)(\theta(\Delta t))^{-\vartheta}(\sum_{j=1}^{n_3+1} \mu_j\, z_{n+1-j} - q_{n+1}z_0)}{k_1(\vartheta)(\theta(\Delta t))^{1-\vartheta} + k_0(\vartheta)(\theta\Delta t))^{-\vartheta} + \varpi}. \quad (46)
\end{aligned}
$$

Where $n = n_2, ..., n_3$, $k_0(\vartheta) = \vartheta Q^{(1-\vartheta)}, k_1(\vartheta) = (1-\vartheta)Q^{\vartheta}, \Delta t = \frac{T}{N_n}, N_n \in N$,
$\mu = \vartheta, q_1 = \frac{1}{\Gamma(1-\vartheta)}, \mu_i = 1 - (\frac{\vartheta+1}{i})\mu_{i-1}, \omega_1 = 1, \omega_i = 1 - (\frac{\vartheta}{i})\omega_{i-1}$.

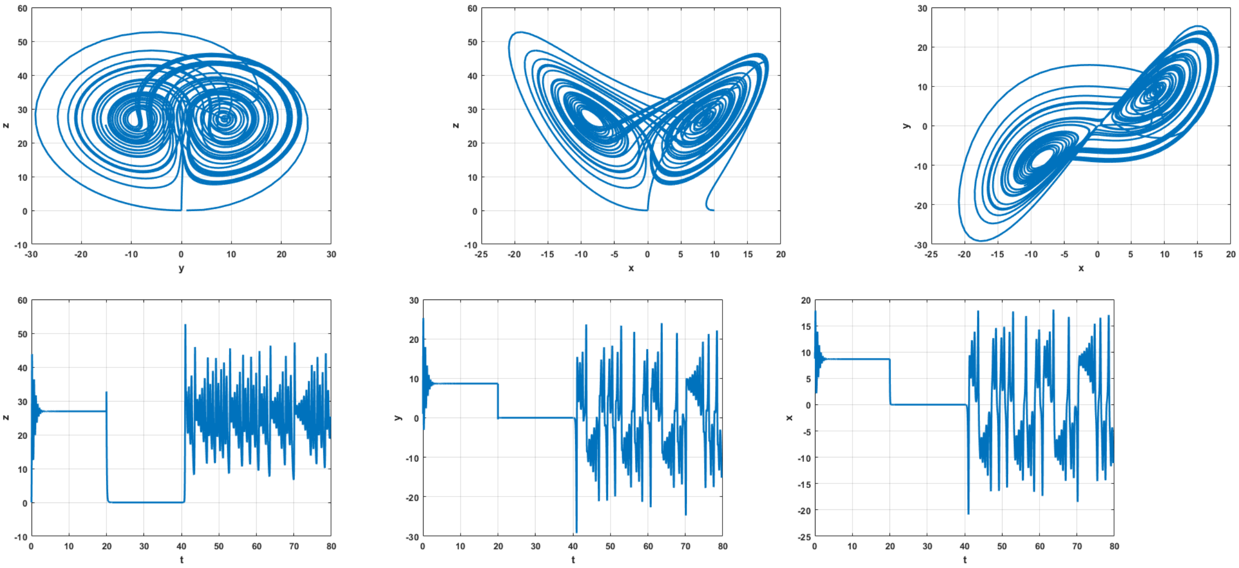

Fig. 3: Numerical simulation for case3 at $\vartheta_1 = 1, \vartheta_2 = .9, \vartheta_3 = .85,\ \vartheta_1(t) = 0.9-0.001t, \vartheta_2(t) = 0.98-0.001t,\ \vartheta_3(t) = 0.97-0.001t$.

Stability of CPC−GLNSFDM:

Consider the following fractional-order Lorenz system in the general form [35] :

$$ {}^{CPC}_{0}D^{\vartheta}_{t} y_s(t) = \Lambda_s(y_1, y_2, ..., y_k) - y_s\Theta(y_1, y_2, ..., y_k), \qquad y_s(t_0) = y_{s,0}, s = 1, ..., k. \tag{47} $$

Where Λ_s is continuous function on $\mathbb{R}^k$.

Theorem 1. *The CPC-GLNSFDM is a stable method.*

Proof. Using CPC-GLNSFDM to approximate (6.3) we have:

$$ ll_1 y_{s,n+1} + ll_1 \sum_{i=1}^{n+1} y_{s,n+1-i}\omega_i + ll_2 y_{s,n+1} - ll_2 \sum_{i=1}^{n+1} \mu_i y_{s,n+1-i} - l_2 q_{s,n+1} y_{s,0} = \Lambda_s(y_{1,n}, y_{2,n}, \dots, y_{k,n}) - y_{s,n+1}\Theta_s(y_{1,n+1}, y_{2,n+1}, \dots, y_{k,n+1}), \tag{48} $$

where $ll_1 = \frac{(1-\vartheta)Q^{\vartheta}}{\Delta t^{\vartheta-1}}, ll_2 = \frac{\vartheta Q^{(1-\vartheta)}}{\Delta t^{\vartheta}}$.

Then we have :

$$ y_{s,n+1} = \frac{-ll_1 \sum_{i=1}^{n+1} y_{s,n+1-i}\omega_i + ll_2 \sum_{i=1}^{n+1} \mu_i y_{s,n+1-i} + ll_2 q_{s,n+1} y_{s,0} + \Lambda_s(y_{1,n}, y_{2,n}, \dots, y_{k,n})}{l_1 + ll_2 + \Theta_s(y_{1,n+1}, y_{2,n+1}, \dots, y_{k,n+1})}, \tag{49} $$

since

$ll_1 + ll_2 + \Theta_s(y_{1,n+1}, y_{2,n+1}, ..., y_{k,n+1}) > 1$,
then

$y_{s,1<y_{s,0}}$, and $y_{s,n+1} < y_{s,n} < y_{s,n-1} \dots < y_{s,1} < y_{s,0}$.

So the proposed scheme is stable.

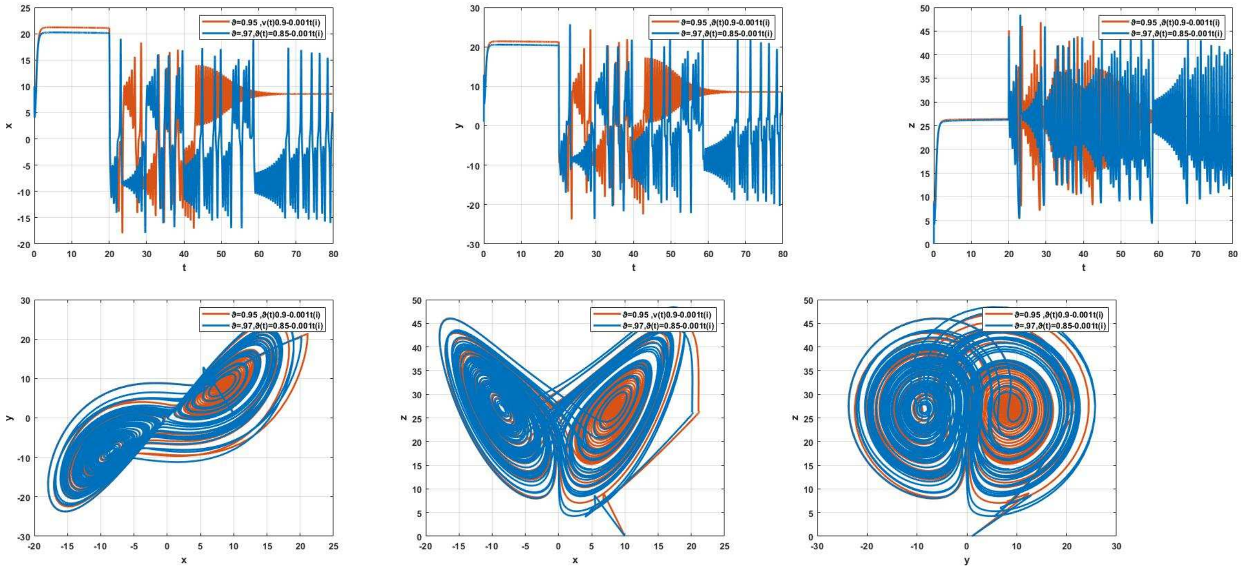

Fig. 4: Numerical simulation for case 3.1 at $\vartheta(t) = 0.9 - 0.001t$ in case of orange color and $\vartheta = 0.85 - 0.001t$ in case of blue color.

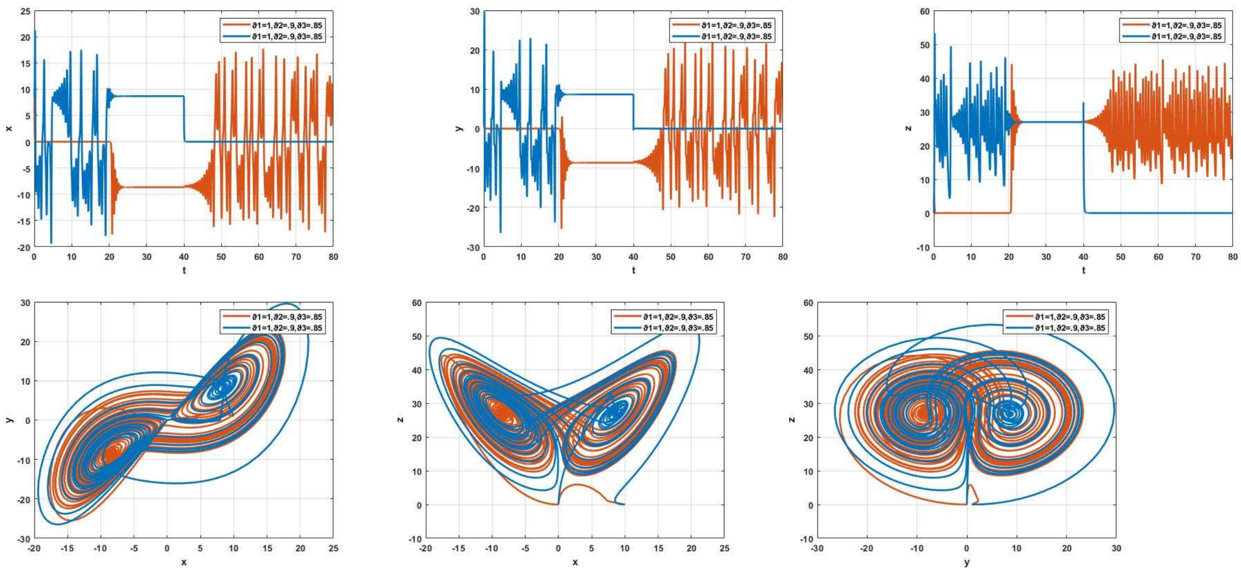

Fig. 5: Numerical simulation for case(3.1) $\vartheta_1(t) = 0.99 - 0.001t, \vartheta_2(t) = 0.98 - 0.001t, \vartheta_3(t) = 0.97 - 0.001t$ with different order

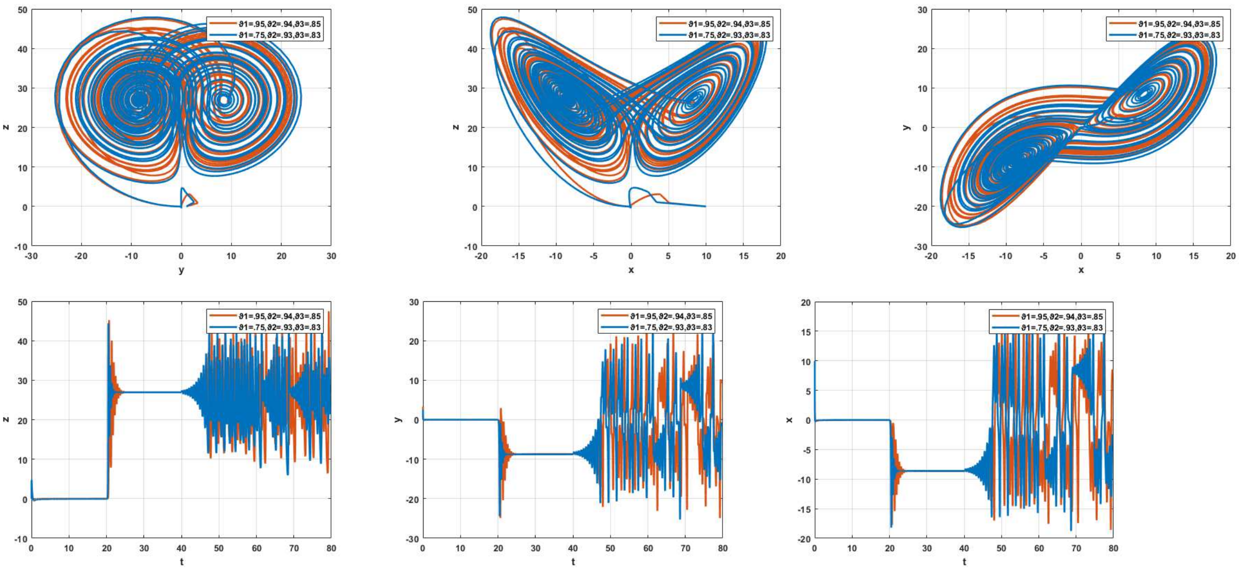

Fig. 6: Numerical simulation for case(3.1)at $\vartheta_1(t) = 0.99 - 0.001t, \vartheta_2(t) = 0.9 - 0.002t, \vartheta_3(t) = 0.9 - 0.005t$ in case of orange color and $\vartheta_1(t) = 0.9 - 0.002t, \vartheta_2(t) = .99 - .02cost^2t, \vartheta_3(t) = .99 - .02sin^2t$ in case of blue color

7 Numerical simulations

Here, we take into account the simulation of the Lorenz model. The following are the parameter values. :$\delta = 10, \gamma = 28$ and $\varpi = \frac{8}{3}$, initial conditions used as follow: $X(0) = 10, Y(0) = 1$ and $Z(0) = 0$. The numerical results of the Lorenz model for the three cases are presented graphically at various values of $0 < \vartheta \leq 1$. The simulations were executed using CPC-GLNSFDM. Figure (1) is the representation of the dynamical behaviour of case (3.1); the ϑ values used are $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0.95, \vartheta_1(t) = \vartheta_2(t) = \vartheta_3(t) = 0.9 - 0.001t$ .Figure (2) is the representation of the dynamical behaviour of case (3.1); the ϑ values used are $\vartheta_1 = \vartheta_2 = \vartheta_3 = .85, \vartheta_1(t) = 0.9 - 0.001t, \vartheta_2(t) = 0.98 - 0.001t, \vartheta_3(t) = 0.97 - 0.001t$. Also, figure (3) is the representation of the dynamical behaviour of case (3.1); the ϑ values used are $\vartheta_1 = 1, \vartheta_2 = .9, \vartheta_3 = .85, \vartheta_1(t) = 0.9 - 0.001t, \vartheta_2(t) = 0.98 - 0.001t\vartheta_3(t) = 0.97 - 0.001t$. Figure(4) shows how the solutions change with changing values of ϑ for case(3.1), the values used in case of blue colour are $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0.97, \vartheta_1(t) = \vartheta_2(t) = \vartheta_3(t) = 0.85 - 0.001t$, and the values used in case of orange color are $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0.95, \vartheta_1(t) = \vartheta_2(t) = \vartheta_3(t) = 0.9 - 0.001t$. The order used in the two colors is fractional order in the first region, integer order in the second region, and variable order in the third region. Figure(5) shows how the solutions change with changing order for case(3.1) with the same values of ϑ; the order used in blue colour is integer order in the first region, fractional order in the second region and variable order in the third region. The order used in orange colour is variable in the first region, fractional order in the second region and integer order in the third region. Figure(6) for (3.1) show how the solutions change with changing the values of ϑ, the values used in case of blue color are $\vartheta_1 = .75, \vartheta_2 = .93, \vartheta_3 = 0.83, \vartheta_1(t) = 0.9 - 0.002t(i), \vartheta_2(t) = .99 - .02cos^2t(i), \vartheta_3(t) = .99 - .02sin^2t(i)$, and the values used in case of orange color are $\vartheta_1 = .95, \vartheta_2 = .94, \vartheta_3 = 0.85, \vartheta_1(t) = 0.9 - 0.001t(i), \vartheta_2(t) = 0.9 - 0.002t(i), \vartheta_3(t) = 0.9 - 0.005t(i)$, these illustrations showcase the chaotic model's new dynamic aspects.

8 Conclusion

In this paper, a hybrid fractional piecewise Lorenz mathematical model has been developed. The CPC operator is one of the most dependable and efficient, and it is also more versatile than the Caputo fractional operator. The deterministic model is expanded using the CPC operator. For the approximation CPC fractional operator, GLNSFDM is used. For solving these proposed systems, have great stability qualities, and precise approximations are given. Graphical outcomes for suggested models are shown. Finally, utilizing the piecewise hybrid mathematical model of Lorenz, we have gotten more realistic and general results.

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