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A Pathway Idea for Model Building

A.M. Mathai[1] *and Panagis Moschopoulos*[2]

[1] McGill University, Canada and Centre for Mathematical Sciences, India
[2] University of Texas at El Paso, Dept. of Mathematical Sciences, El Paso, TX 79968, USA

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Abstract: Models, mathematical or stochastic, which move from one functional form to another through pathway parameters, so that in between stages can be captured, are examined in this article. Models which move from generalized type-1 beta family to type-2 beta family, to generalized gamma family to generalized Mittag-Leffler family to Lévy distributions are examined here. It is known that one can likely find an approximate model for the data at hand whether the data are coming from biological, physical, engineering, social sciences or other areas. Different families of functions are connected through the pathway parameters and hence one will find a suitable member from within one of the families or in between stages of two families. Graphs are provided to show the movement of the different models showing thicker tails, thinner tails, right tail cut off etc.

Keywords: Generalized gamma, Mittag-Leffler distributions.

1. Introduction

When fitting a mathematical model for data coming from physical, social and engineering sciences one often picks a possible function from a family of functions such as a gamma family for positive observations. In this case the selected function is

$$f_1(x) = c_1 x^{\alpha-1} e^{-x/\beta}, x \geq 0, \alpha > 0, \beta > 0. \qquad (1.1)$$

In some situations there may be a right tail or the right tail is cut off or the data can never be higher than a threshold number. In that case one often looks into a generalized type-1 beta family of functions such as

$$f_2(x) = c_2 x^{\alpha-1}[1 - ax^{\delta}]^{\gamma}, 0 \leq x \leq a^{-1/\delta}, a > 0,$$
$$\alpha > 0, \delta > 0, \gamma > 0. \qquad (1.2)$$

where $a^{-1/\delta}$ will determine the threshold number. Then the fitting is done by selecting the parameters $\alpha, a, \delta, \gamma$. But there may be a situation where the underlying model may be in between $f_1(x)$ and $f_2(x)$. In some physical situations the stable situation may be an exponential form such as a special case of (1.1), or a Maxwell-Boltzmann distribution, which is a special case of (1.1), may be the stable situation. The data at hand may be describing some disturbance from this stable situation or in a neighborhood or path leading to the stable situation. In order to cover the stable as well as the transitional stages a pathway model was introduced by Mathai (2005). The model introduced therein is for a pathway describing transitions of rectangular matrix-variate distributions in the real case and the corresponding pathway model in the complex domain was given in Mathai and Provost (2007). The pathway model for the real positive scalar case is the following:

$$f_3(x) = c_3 x^{\alpha-1}[1 - (1-q)ax^{\delta}]^{\eta/(1-q)}, a > 0,$$
$$\delta > 0, \eta > 0, 1 - (1-q)ax^{\delta} > 0. \qquad (1.3)$$

Observe that (1.3) for $q < 1$ stays in the generalized type-1 beta family of densities. When $q > 1$, writing $1 - q = -(q-1)$ we have

$$f_4(x) = c_4 x^{\alpha-1}[1 + (q-1)ax^{\delta}]^{-\eta/(q-1)},$$
$$q > 1, a > 0, \delta > 0, \eta > 0, x \geq 0. \qquad (1.4)$$

The function $f_4(x)$ is the generalized type-2 beta family of densities. Now, let us see what happens when $q \to 1$, from the left or from the right.

$\lim_{q\to 1_+} f_4(x) = \lim_{q\to 1_-} f_3(x) = f_5(x)$

* Corresponding author: e-mail: pmoschopoulos@utep.edu

where

$$f_5(x) = c_5 x^{\alpha-1} e^{-a\eta x^{\delta}}, x \geq 0, a > 0, \eta > 0, \delta > 0. \tag{1.5}$$

That is, $f_3(x)$ and $f_4(x)$ go to $f_5(x)$, which is the generalized gamma family. Thus, if $f_5(x)$ or its particular case is the stable density in a physical system then unstable neighborhoods or paths leading to $f_5(x)$ are described by $f_3(x)$ and $f_4(x)$. Then q there can be called the pathway parameter connecting the three functional forms $f_3(x), f_4(x)$ and $f_5(x)$.

1.1. The normalizing constants

The normalizing constants c_3, c_4 and c_5 can be computed by making the substitutions $u = (1-q)ax^{\delta}, v = (q-1)ax^{\delta}, \omega = a\eta x^{\delta}$ and then making use of the type-1 beta, type-2 beta and gamma integrals respectively. The final results are the following:

$$c_3 = \frac{\delta \Gamma[\alpha/\delta + \eta/(1-q) + 1][a(1-q)]^{(\alpha/\delta)}}{\Gamma(\alpha/\delta)\Gamma[\eta/(1-q)+1]}, \tag{1.6}$$

$q < 1, R(\alpha) > 0, \delta > 0, \eta > 0, a > 0$

.

$$c_4 = \frac{\delta \Gamma[\eta/(q-1)][a(q-1)]^{\alpha/\delta}}{\Gamma(\alpha/\delta)\Gamma[\eta/(q-1) - \alpha/\delta]}, \tag{1.7}$$

$q > 1, R(\alpha) > 0, \delta > 0, \eta > 0, a > 0, R[\eta/(q-1) - \alpha/\delta] > 0$

$$c_5 = \frac{\delta (a\eta)^{\alpha/\delta}}{\Gamma(\alpha/\delta)}, R(\alpha) > 0, \delta > 0, \eta > 0, a > 0, \tag{1.8}$$

where $R(\cdot)$ denotes the real part of $(\cdot)$. In model building situations, usually all the parameters are real.

1.2. Graphs of the pathway model

The graphs of f_3, f_4, and f_5 illustrate the versatility of the models and the fact that, as the parameter q approaches 1 from the left or from the right the limiting forms of the densities f_3 and f_4 are indeed f_5 (see attached).

2. The Pathway Idea

The mathematical property of (1.3) and (1.4) going to (1.5) is a technique in the theory of special functions. When $\delta = 1$ the functions in (1.3) and (1,4), excluding $x^{\alpha-1}$ are binomial functions or in the language of hypergeometric functions they are ${}_1F_0$ functions. But $f_5(x)$ for $\delta = 1$ and excluding $x^{\alpha-1}$, is the exponential function ${}_0F_0$. Hence the transition is the case of ${}_1F_0$ going to ${}_0F_0$. The technique of getting rid off parameters from a hypergeometric function is an age-old technique. This technique was successfully adapted by Mathai in the 1970's for population studies and it was introduced and elaborated in Mathai (2005) due to the emergence of the new branch of non-extensive statistical mechanics, initiated by Tsiallis (1988). The mathematical technique behind this whole branch is ${}_1F_0$ going to ${}_0F_0$ in the language of hypergeometric functions. The formulation of Tsallis (1988) is in the form of generalized entropy, which is one of the $\alpha-$ generalized entropies discussed in Mathai and Rathie (1975). It is shown in Mathai and Haubold (2007) that there can be entropic, distributional or differential pathways, or the same idea can be elucidated in terms of entropies or information measures or statistical distributions or differential equations. These are the three types of pathways for the same idea.

2.1. Tsallis' Statistics, Superstatistics and Mittag-Leffler Distributions

During the past ten years there have been invigorated activities in the area of astrophysics claiming that Tsallis' statistics can describe most of physical situations or Tsallis' model can describe most physical situations deviating from the Maxwell-Boltzman stable distribution, thus extending statistical mechanics beyond Boltzman and Gibbs. Another claim is made in introducing what is known as super-statistics by Beck and Cohen (2003) and Beck (2006) that super-statistics is the right candidate to describe deviations from Maxwell-Boltzman distribution. It is pointed out in Mathai and Haubold

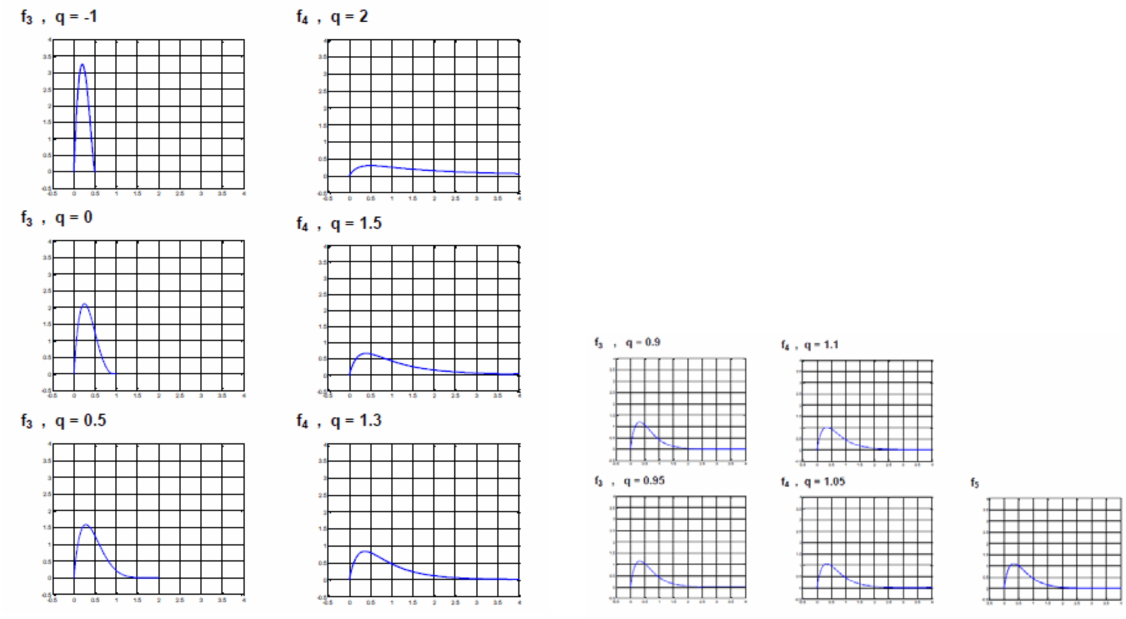

Figure 1 The graphs of $f_3(x)$, $f_4(x)$ and $f_5(x)$. The values of the parameters are: $a = 1, \alpha = 2, \delta = 1, \eta = 3$

(2007) that from a statistical point of view super-statistics is nothing but the unconditional distribution in Bayesian statistical analysis when the conditional density and the prior density both belong to generalized gamma densities of the form (1.5). Further, that Tsallis' statistics can describe more situations than the ones covered by super-statistics. Seybold et al. (2005) made a comparison of the various models, with reference to astrophysics problems, and came to the conclusion that a Mittag-Leffler model may be a better candidate compared to Tsallis' statistics, super-statistics and stretched exponential. Mittag-Leffler fits in nicely in many types of data. A recent survey on Mittag-Leffler function, Mittag-Leffler density, their properties and applications, is done by Haubold, Mathai and Saxena (2009). Mittag-Leffler function is a particular case of Wright function which is a particular case of the H-function. The theory and applications of H-function may be seen from the recent books Mathai and Haubold (2008) and Mathai, Saxena and Haubold (2009).
The pathway idea will be illustrated here in terms of a particular case of the H-function. This aspect is not examined by anyone so far. Let us take an ${}_1F_1$-type particular case of an H-function. Consider the standard notation of an H-function

$$H^{1,1}_{1,2}[x\,|^{(1-a,\alpha)}_{(0,1),(1-b,\beta)}] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(a-\alpha s)}{\Gamma(b-\beta s)}x^{-s}ds \tag{2.1}$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(a+\alpha k)}{\Gamma(b+\beta k)}\frac{(-x)^k}{k!} \tag{2.2}$$

where $i = \sqrt{-1}, \alpha > 0, \beta > 0$ are real; a,b are complex numbers, c is real and $0 < c < R(a/\alpha), a, b \neq 0, -1, -2, ...$ where $R(\cdot)$ denotes the real part of $(\cdot)$. Consider

$$\frac{\Gamma(b)}{\Gamma(a)}H^{1,1}_{1,2}[x\,|^{(1-a,\alpha)}_{(0,1),(1-b,\beta)}]$$

and let $|a| \to \infty$. Then by using Stirling's approximation for the gamma function, namely,

$$\Gamma(z+c) \approx \sqrt{2\pi}z^{z+c-1/2}e^{-z} \tag{2.3}$$

for $|z| \to \infty$ and c a bounded quantity, we have

$$\frac{\Gamma(a-\alpha s)}{\Gamma(a)}a^{\alpha s} \approx \frac{\sqrt{2\pi}a^{a-\alpha s-1/2}e^{-a}a^{\alpha s}}{\sqrt{2\pi}a^{a-1/2}e^{-a}} = 1. \tag{2.4}$$

Therefore

$$\lim_{a\to\infty}\frac{\Gamma(b)}{\Gamma(a)}H_{1,2}^{1,1}\left[\frac{x}{a^{\alpha}}\ \middle|_{(0,1),(1-b,\beta)}^{(1-a,\alpha)}\right]=\Gamma(b)H_{0,2}^{1,0}[x\mid(0,1),(1-b,\beta)]$$

$$=\Gamma(b)\sum_{k=0}^{\infty}\frac{x^k}{k!\Gamma(b+\beta k)}=\Gamma(b)E_{\beta,b}(-x) \tag{2.5}$$

where $E_{\beta,b}(x)$ is the Mittag-Lefler function. When $\alpha=1$, (2.2) reduces to

$$\frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+\beta k)}\frac{(-x)^k}{k!}=\Gamma(b)\sum_{k=0}^{\infty}\frac{(a)_k}{\Gamma(b+\beta k)}\frac{(-x)^k}{k!}=\Gamma(b)E_{\beta,b}^{a}(-x) \tag{2.6}$$

where $(a)_k$ is the Pochhammer symbol $(a)_k=a(a+1)\cdots(a+k-1), (a)_0=1, a\neq 0$ where $E_{\beta,b}^{a}(-x)$ is the generalized Mittag-Leffler function. When $a=1$, we have $E_{\beta,b}^{1}(-x)=E_{\beta,b}(-x)$.

$$E_{\beta,1}^{1}(-x)=E_{\beta,1}(-x)=E_{\beta}(-x) \tag{2.7}$$

where $E_{\beta}(x), E_{\beta,b}(x), E_{\beta,b}^{a}(x)$ are various forms of the Mittag-Leffler function and these are particular cases of a Wright's function

$${}_p\Psi_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)}\frac{x^k}{k!}. \tag{2.8}$$

2.2. Movement towards Tsallis' Statistics and Superstatistics

In (2.5) we have considered the case of getting rid off an upper parameter from an H-function. Suppose we want to get rid of the denominator gamma from (2.1). Going through parallel steps we see that

$$\lim_{b\to\infty}\frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(s)\Gamma(a-\alpha s)}{\Gamma(b-\beta s)}(b^{\beta}x)^{-s}=\frac{1}{\Gamma(a)}\Gamma(s)\Gamma(a-\alpha s)x^{-s}. \tag{2.9}$$

Then the function reduces to

$$\frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(a-\alpha s)x^{-s}ds=\frac{1}{\Gamma(a)}\sum_{k=0}^{\infty}\Gamma(a+\alpha k)\frac{(-x)^k}{k!}. \tag{2.10}$$

For $\alpha=1$ this is the binomial series

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}(-x)^k=(1+x)^{-a}, \mid x\mid<1. \tag{2.11}$$

When x is replaced by $(q-1)x$ and a is replaced by $(q-1)^{-1}$ one has Tsallis' statistics. Under these replacements for $q>1$ and when (2.11) is multiplied by a factor x^{γ} one has the super-statistics of Beck and Cohen (2003). Here one may also replace x by x^{δ} to give other forms of super-statistics. Thus when we get rid off the numerator gamma from the H-function in (2.1) we end up in the very particular Mittag-Leffler function. When we get rid off the denominator gamma in (2.1) we end up in Tsallis' statistics, super-statistics, and particular cases of the pathway model in Mathai (2005).

These various forms of the Mittag-Leffler functions appear naturally in the solutions of fractional order differential equations. For example to a simple integer order differential equation

$$\frac{df(x)}{dx}=-\rho f(x)\Rightarrow f(x)=f_0e^{-\rho x}, \tag{2.12}$$

the solution is the exponential function, while to a fractional order differential or integral equation

$$f(x)-c={}_0D_x^{-\alpha}f(t), c=\text{constant}, \tag{2.13}$$

the solution is available in terms of a Mittag-Leffler function, where the left-sided Riemann-Liouville fractional integral is defined as

$${}_0D_x^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, R(\alpha)>0. \tag{2.14}$$

A large number of applications of fractional calculus to reaction rate theory, diffusion and reaction-diffusion problems are given in a series of papers by Haubold, Mathai and Saxena, a summary of which is available in Mathai and Haubold (2008).

3. Mittag-Leffler Density

Let us consider a function $f(x)$ associated with the Mittag-Leffler function. Let, for $\beta > 0$ real, $a, b \neq 0, -1, -2, ...$

$$f(x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+\beta k)} \frac{(-x^{\beta})^k}{k!} x^{b-1} \tag{3.1}$$

and let us consider the Laplace transform $L_f(t)$, where

$$L_f(t) = \int_0^{\infty} e^{-tx} f(x) dx = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+\beta k)} \frac{(-1)^k}{k!} \int_0^{\infty} x^{b+\beta k-1} e^{-tx} dx$$

$$= \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \Gamma(a+k) \frac{(-1)^k}{k!} t^{-b-\beta k} \tag{3.2}$$

$$= \Gamma(b) \sum_{k=0}^{\infty} (a)_k \frac{(-1)^k}{k!} t^{-b-\beta k} = \Gamma(b) t^{-b} [1 + t^{-\beta}]^{-a} \text{for} |t^{-\beta}| < 1$$

$$= \Gamma(b) t^{-b+a\beta} (1 + t^{\beta})^{-a} = \Gamma(b)(1 + t^{\beta})^{-a} \tag{3.3}$$

for $b = a\beta$. Observe that the integration inside the series is valid in (3.1) Thus for $b = a\beta$

$$f(x) = \frac{1}{\Gamma(a)} x^{a\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a\beta+k\beta)} \frac{(-x^{\beta})^k}{k!}. \tag{3.4}$$

The series can be represented in terms of a Mellin-Barnes representation. That is

$$f(x) = \frac{1}{\Gamma(a)} x^{a\beta-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a\beta-\beta s)} x^{-\beta s} ds. \tag{3.5}$$

Let us make a change of variable. let
$a\beta - 1 - s\beta = -s_1 \Rightarrow s = a - 1/\beta + s_1/\beta$.
Under the transformation we may rewrite (3.5) as follows:

$$f(x) = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(a-1/\beta+s_1/\beta)\Gamma(1/\beta-s_1/\beta)}{\beta\Gamma(1-s_1)} x^{-s_1} ds_1$$

$$= \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(a-1/\beta+s_1/\beta)\Gamma(1+1/\beta-s_1/\beta)}{\Gamma(2-s_1)} x^{-s_1} ds_1 \tag{3.6}$$

for $-R(a\beta) < R(s) < R(\beta) < 1$. Since (3.6) can be looked upon as an inverse Mellin transform we have

$$\int_0^{\infty} x^{s-1} f(x) dx = \frac{\Gamma(a-1/\beta+s/\beta)\Gamma(1+1/\beta-s/\beta)}{\Gamma(a)\Gamma(2-s)} = 1 \text{ for } s = 1. \tag{3.7}$$

Since $f(x) \geq 0$ for all x and since $\int_0^{\infty} x^0 f(x) dx = \int_0^{\infty} f(x) dx = 1$ this $f(x)$ is a density. Hence the expected value of x^{s-1} for this density, that is,

$$E(x^{s-1}) = \frac{\Gamma(a-1/\beta+s/\beta)\Gamma(1+1/\beta-s/\beta)}{\Gamma(a)\Gamma(2-s)}, -R(a\beta) < R(s) < \beta < 1. \tag{3.8}$$

Thus, the Mittag-Leffler density is given by

$$f(x) = \frac{x^{a\beta-1}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a\beta+\beta k)} \frac{(-x^{\beta})^k}{k!} \tag{3.9}$$

$$= x^{a\beta-1} \sum_{k=0}^{\infty} \frac{(a)_k}{\Gamma(a\beta+\beta k)} \frac{(-x^{\beta})^k}{k!}, \beta > 0, a > 0, x \geq 0. \tag{3.10}$$

One can even have a scaling factor for x. In this case $f(x)$ will become

$$g(x) = \frac{x^{a\beta-1}}{\delta^a} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \frac{(-x^{\beta})^k}{\delta^k \Gamma(a\beta+\beta k)}, 0 \leq x < \infty, \delta > 0, \beta > 0, a > 0. \tag{3.11}$$

3.1. Limiting Form of the Mittag-Leffler Density

Following through the earlier steps we can compute the Laplace transform of the density in (3.11),

$$L_g(t) = [1 + \delta t^{\beta}]^{-\alpha} \tag{3.12}$$

If $x_1, \ldots, x_n$ are independent and identically distributed Mittag-Leffler random variables with the density as in (3.11), then for the Laplace transform of the sum $x = x_1 + \cdots + x_n$ denoting it by $L_x(t)$, we have

$$L_x(t) = [1 + \delta t^{\beta}]^{-na}, \tag{3.13}$$

which shows that x is again a Mittag-Leffler variable, which also indicates that the Mittag-Leffler variable is infinitley divisible. In (3.12) if δ is replaced by $q-1$ and a by $(q-1)^{-1}, q > 1$ and then take the limit as $q \to 1^{+}$ then we have,

$$\lim_{q\to 1^+} L_g(t) = \lim_{q\to 1^+} [1 + (q-1)t^{\beta}]^{-\frac{1}{q-1}} = e^{-t^{\beta}} \tag{3.14}$$

which is the Laplace transform of the Lévy distribution. Thus, one interesting aspect is that the Mittag-Leffler density, in the general form, goes to the Lévy density. The path here leads to a thick-tailed Lévy density rather than the generalized gamma density. The Mittag-Leffler density in its simplest form is available for $\delta = 1, a = 1$ in (3.11). The Lévy density and Linnik density are very often used to describe non-Gaussian stochastic processes and non-Gaussian time series.

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