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The Treatment of Conformable Systems With Second Class Constraints

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Abstract: This study develops into a conformable singular system characterized by second-class constraints. The conformable Poisson bracket (CPB) is introduced as the mathematical framework for defining the bracket operation between two functions. The Dirac theory is extended to accommodate conformable singular systems. To exemplify the practical application of the developed theory, an illustrative example is presented and solved. The obtained results align with those of Rabeı et al., validating the proposed framework, particularly when the parameter α is set to 1.

Keywords: Conformable derivative; Singular systems; Constrained system; Dirac theory; Lagrangian formulations; Hamiltonian formulations.

1 Introduction

Constrained systems, also known as singular systems, are a prominent framework of physics. When dealing with physical systems subject to certain constraints, these constraints impose limitations on the degrees of freedom, affecting the dynamics and behavior of the system, etc [13, 14, 30, 31]. In other words, the constraints impose additional relations on the system's variables, resulting in a reduction of the number of independent degrees of freedom. This reduction leads to unique phenomena in understanding the singular systems.
Singular systems find applications in various branches of physics, including classical mechanics, quantum mechanics, and field theory [11, 12, 26, 27].
In the last few decades, fractional calculus and physical applications have been developed. Particularly in Lagrangian and Hamiltonian formulations of non-conservative systems, fractional calculus has been applied [8, 9, 23, 29, 32]. The field of mathematics known as fractional calculus focuses on applying non-integer orders to differentiation and integration. This powerful mathematical framework has found wide applications in science and engineering. Due to its capacity to offer a more exact description of different physical phenomena. Besides, fractional models possess enhanced properties, such as superior memory capabilities, which render them more potent compared to conventional models [20, 22, 24]. This field has attracted a lot of interest recently: There are numerous ways to define fractional integral and derivative [21].
In [19] Khalil et al proposed a new definition which is called a conformable derivative, it has common properties with the known traditional derivative. For a given function $f(t):[0,\infty)\to R$, the conformable derivative of $f(t)$ of order α where $0<\alpha\leq 1$, denoted as $D_t^{\alpha}f(t)$ is defined as [19]:

$$D_t^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}=t^{1-\alpha}\frac{d}{dt}f(t). \tag{1}$$

In [1, 18], this conformable fractional derivative is re-investigated and new properties similar to these in traditional calculus were derived and discussed. Conformable calculus has numerous applications in many fields of physics such as

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quantum mechanics [3–6, 10], nuclear physics [15, 16], special relativity [2, 25], classical mechanics [28] and mathematical physics [7, 17].

2 Conformable singular system

Let us define the conformable Lagrangian in the following form:

$$L(t,D^{\alpha-1}q_i,D^{\alpha}q_i)=a_{ij}D^{\alpha}q_iD^{\alpha}q_j+b_jD^{\alpha}q_j+V(D^{\alpha-1}q_i), \tag{2}$$

where $0<\alpha\leq 1$.
Noting that $D^{\alpha}q_i$ is the conformable derivative of the coordinate q_i which represented the conformable velocity and $D^{\alpha-1}q_i$ is the canonical conformable conjugate coordinate. The Hessian matrix for the conformable Lagrangian is defined as

$$W_{ij}=\frac{\partial^2 L}{\partial D^{\alpha}q_i\partial D^{\alpha}q_j},\quad i,j=1,2,\ldots,n, \tag{3}$$

If this matrix has rank n, then the system is called regular and can be treated using traditional conformable mechanics and the conformable regular systems have been treated by many authors [23] and references therein, while systems with a rank less than n are called conformable singular systems. Traditional singular systems were treated using two methods: the Dirac method [13] and the canonical method [30].
The conformable singular systems are discussed by Rabei and Horani [28] using the canonical method. The conformable systems which have the rank of the Hessian matrix less than n are called conformable constrained systems. Let us assume that the rank of the Hessian matrix is r which is less than n. Following to Dirac [13], this implies the existence of $(n-r)$ conformable primary constraints. Thus, we may define the total conformable Hamiltonian as

$$H_{T\alpha}=H_{0\alpha}+v_{\mu}H'_{\mu\alpha},\quad \mu=n-r+1,\ldots,n. \tag{4}$$

and $H_{0\alpha}$ being the conformable Hamiltonian

$$H_{0\alpha}=-L(t,D^{\alpha-1}q_i,D^{\alpha}q_i)+P_{i\alpha}D^{\alpha}q_i,\quad i=1,2,\ldots,n, \tag{5}$$

Where $P_{i\alpha}$ is the conformable generalized momenta defined as

$$P_{a\alpha}=\frac{\partial L}{\partial(D^{\alpha}q_a)},\quad a=1,2,\ldots,n-r. \tag{6}$$

$$P_{\mu\alpha}=\frac{\partial L}{\partial(D^{\alpha}q_{\mu})},\quad \mu=n-p+1,2,\ldots,n. \tag{7}$$

v_{μ} are unknown coefficients and $H'_{\mu\alpha}$ are the conformable primary constraints which can be obtained using eq.(7) as follows

$$H'_{\mu\alpha}=P_{\mu\alpha}+H_{\mu}(D^{\alpha-1}q_i,D^{\alpha}q_i,P_{a\alpha}),\quad v=n-p+1,\ldots,n. \tag{8}$$

The conformable Poisson bracket (CPB) of two functions $f(D^{\alpha-1}q_i,P_{a\alpha})$ and $g(D^{\alpha-1}q_i,P_{a\alpha})$ is defined as

$$\{f,g\}_{\alpha}=\frac{\partial f}{\partial(D^{\alpha-1}q_i)}\frac{\partial g}{\partial P_{i\alpha}}-\frac{\partial f}{\partial P_{i\alpha}}\frac{\partial g}{\partial(D^{\alpha-1}q_i)}. \tag{9}$$

The total time derivative of any function g in terms of the generalized coordinates $D^{\alpha-1}q_i$ and momenta $P_{i\alpha}$ is given as:

$$\dot{g}=\{g,H_{0\alpha}\}_{\alpha}+v_{\mu}\{g,H_{\mu\alpha}\}_{\alpha}, \tag{10}$$

according to Dirac [13], the consistency conditions the total time derivative of the primary constraints H'_{μ} should be equal to zero. Thus,

$$\dot{H'}_{\mu\alpha}=\{H'_{\mu\alpha},H_{0\alpha}\}_{\alpha}+v_{\mu}\{H'_{\mu\alpha},H'_{v\alpha}\}_{\alpha}\approx 0. \tag{11}$$

This equation may be solved to obtain the unknowns v_{μ} or lead to new constraints that restrict the motion [13]. Following Dirac, the constraints that have vanishing CPB's are called Conformable First Class, and that do not have vanishing CPB's are called Conformable Second Class, constraints and the equations of motion will be proposed as,

$$D^{\alpha}q_i=\{D^{\alpha-1}q_i,H_{T\alpha}\}_{\alpha}, \tag{12}$$

$$DP_{i\alpha}=\{P_{i\alpha},H_{T\alpha}\}_{\alpha}. \tag{13}$$

3 Conformable second class constraints

To demonstrate our theory we would like to give a model of the conformable constraints of the Second Class [30]. Investigate the following Lagrangian:

$$\begin{aligned} L = &\frac{1}{2}(D^{\alpha}q_1)^2 - \frac{1}{4}((D^{\alpha}q_2)^2 - 2D^{\alpha}q_2D^{\alpha}q_3 + (D^{\alpha}q_3)^2) \\ &+ (D^{\alpha-1}q_1 + D^{\alpha-1}q_3)D^{\alpha}q_2 - (D^{\alpha-1}q_1 + D^{\alpha-1}q_2 + (D^{\alpha-1}q_3)^2). \end{aligned} \tag{14}$$

The Hessian matrix W_{33} can be constructed as follow:

$$W_{33} = \begin{bmatrix} \frac{\partial^2 L}{\partial D^{\alpha}q_1 \partial D^{\alpha}q_1} & \frac{\partial^2 L}{\partial D^{\alpha}q_1 \partial D^{\alpha}q_2} & \frac{\partial^2 L}{\partial D^{\alpha}q_1 \partial D^{\alpha}q_3} \\ \frac{\partial^2 L}{\partial D^{\alpha}q_2 \partial D^{\alpha}q_1} & \frac{\partial^2 L}{\partial D^{\alpha}q_2 \partial D^{\alpha}q_2} & \frac{\partial^2 L}{\partial D^{\alpha}q_2 \partial D^{\alpha}q_3} \\ \frac{\partial^2 L}{\partial D^{\alpha}q_3 \partial D^{\alpha}q_1} & \frac{\partial^2 L}{\partial D^{\alpha}q_3 \partial D^{\alpha}q_2} & \frac{\partial^2 L}{\partial D^{\alpha}q_3 \partial D^{\alpha}q_3} \end{bmatrix} \tag{15}$$

It is easy to show that the rank of this matrix is two. Then, the momenta read as,

$$P_{1\alpha} = \frac{\partial L}{\partial(D^{\alpha}q_1)} = D^{\alpha}q_1, \tag{16}$$

$$P_{2\alpha} = \frac{\partial L}{\partial(D^{\alpha}q_2)} = -\frac{1}{2}D^{\alpha}q_2 + \frac{1}{2}D^{\alpha}q_3 + (D^{\alpha-1}q_1 + D^{\alpha-1}q_3), \tag{17}$$

$$P_{3\alpha} = \frac{\partial L}{\partial(D^{\alpha}q_3)} = \frac{1}{2}D^{\alpha}q_2 - \frac{1}{2}D^{\alpha}q_3. \tag{18}$$

Thus, the Hamiltonian H_0 can be calculated as

$$H_{0\alpha} = \frac{1}{2}P_{1\alpha}^2 - P_{3\alpha}^2 + D^{\alpha-1}q_1 + D^{\alpha-1}q_2 + (D^{\alpha-1}q_3)^2. \tag{19}$$

Substituting eq.(18) in eq.(17) one may obtain

$$P_{2\alpha} = -P_{3\alpha} + D^{\alpha-1}q_1 + D^{\alpha-1}q_2. \tag{20}$$

The conformable momentum $P_{2\alpha}$ is not independent. Thus, following to eq.(8), these are conformable primary constraints which can be written as

$$H'_{2\alpha} = P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_2. \tag{21}$$

Making use of equation (4), the total conformable Hamiltonian reads as

$$\begin{aligned} H_{T\alpha} = &\frac{1}{2}P_{1\alpha}^2 - P_{3\alpha}^2 + D^{\alpha-1}q_1 + D^{\alpha-1}q_2 + (D^{\alpha-1}q_3)^2 \\ &+ \nu_2(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_2). \end{aligned} \tag{22}$$

Now, the total time derivative of the conformable primary constraints should be equal to zero. So, using eq.(10) we obtain

$$\begin{aligned} \dot{H}'_{2\alpha} &= \{H'_{2\alpha}, H_{\alpha T}\}_{\alpha} = \{H'_{2\alpha}, H_{0\alpha}\}_{\alpha} + \nu_2\{H'_{2\alpha}, H'_{2\alpha}\}_{\alpha} \\ &= \frac{\partial H'_{2\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H_{0\alpha}}{\partial P_{1\alpha}} - \frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H'_{2\alpha}}{\partial P_{1\alpha}} + \frac{\partial H'_{2\alpha}}{\partial D^{\alpha-1}q_2}\frac{\partial H_{0\alpha}}{\partial P_{2\alpha}} - \frac{\partial H'_{2\alpha}}{\partial P_{2\alpha}}\frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_2} \\ &+ \frac{\partial H'_{2\alpha}}{\partial D^{\alpha-1}q_3}\frac{\partial H_{0\alpha}}{\partial P_{3\alpha}} - \frac{\partial H'_{2\alpha}}{\partial P_{3\alpha}}\frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_3} \\ &= P_{1\alpha} - 1 + 2P_{3\alpha} - 2D^{\alpha-1}q_3 = 0. \end{aligned} \tag{23}$$

This consistency condition imposes a new constraint

$$H'_{1\alpha} = 2P_{3\alpha} - P_{1\alpha} - 2D^{\alpha-1}q_3 - 1. \tag{24}$$

Again the total time derivative of the new constraint should be equal to zero.

$$\begin{aligned}\dot{H}'_{1\alpha} &= \{H'_{1\alpha}, H_{\alpha T}\}_\alpha = \{H'_{1\alpha}, H_{0\alpha}\}_\alpha + \nu_2\{H'_{1\alpha}, H'_{2\alpha}\}_\alpha \\ &= \frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H_{0\alpha}}{\partial P_{1\alpha}} - \frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H'_{1\alpha}}{\partial P_{1\alpha}} + \frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_2}\frac{\partial H_{0\alpha}}{\partial P_{2\alpha}} - \frac{\partial H'_{1\alpha}}{\partial P_{2\alpha}}\frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_2} \\ &+ \frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_3}\frac{\partial H_{0\alpha}}{\partial P_{3\alpha}} - \frac{\partial H'_{1\alpha}}{\partial P_{3\alpha}}\frac{\partial H_{0\alpha}}{\partial D^{\alpha-1}q_3} + \nu_2\Big(\frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H_{2\alpha}}{\partial P_{1\alpha}} - \frac{\partial H_{2\alpha}}{\partial D^{\alpha-1}q_1}\frac{\partial H'_{1\alpha}}{\partial P_{1\alpha}} \\ &+ \frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_2}\frac{\partial H_{2\alpha}}{\partial P_{2\alpha}} - \frac{\partial H'_{1\alpha}}{\partial P_{2\alpha}}\frac{\partial H_{2\alpha}}{\partial D^{\alpha-1}q_2} + \frac{\partial H'_{1\alpha}}{\partial D^{\alpha-1}q_3}\frac{\partial H_{2\alpha}}{\partial P_{3\alpha}} - \frac{\partial H'_{1\alpha}}{\partial P_{3\alpha}}\frac{\partial H_{2\alpha}}{\partial D^{\alpha-1}q_3}\Big) \\ &= 1 + 4P_{3\alpha} - 4D^{\alpha-1}q_3 - \nu_2 = 0.\end{aligned} \tag{25}$$

Then, we arrive at the result

$$\nu_2 = -(4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1). \tag{26}$$

There is no further constraint, and the unknown ν_2 is determined and the conformable constraints $H'_{2\alpha}$ and $H'_{1\alpha}$ are second class. Thus, the total Hamiltonian read as

$$\begin{aligned}H_{T\alpha} &= \frac{1}{2}P_{1\alpha}^2 - P_{3\alpha}^2 + D^{\alpha-1}q_1 + D^{\alpha-1}q_2 + (D^{\alpha-1}q_3)^2 \\ &- (4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1)(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_3),\end{aligned} \tag{27}$$

and the equation of motion can be calculated as

$$\begin{aligned}D^\alpha q_1 &= \{D^{\alpha-1}q_1, H_{\alpha T}\}_\alpha = \{D^{\alpha-1}q_1, \frac{1}{2}P_{1\alpha}^2\}_\alpha = \frac{1}{2}P_{1\alpha}\{D^{\alpha-1}q_1, P_{1\alpha}\}_\alpha \\ &+ \{D^{\alpha-1}q_1, P_{1\alpha}\}_\alpha \frac{1}{2}P_{1\alpha} = P_{1\alpha}.\end{aligned} \tag{28}$$

$$\begin{aligned}D^\alpha q_2 &= \{D^{\alpha-1}q_2, H_{\alpha T}\}_\alpha = \nu_2\{D^{\alpha-1}q_2, P_{2\alpha}\}_\alpha = \nu_2 \\ &= 4P_{3\alpha} - 4D^{\alpha-1}q_3 + 1.\end{aligned} \tag{29}$$

$$\begin{aligned}D^\alpha q_3 &= \{D^{\alpha-1}q_3, H_{\alpha T}\}_\alpha = -\{D^{\alpha-1}q_3, P_{3\alpha}^2\}_\alpha \\ &+ \{D^{\alpha-1}q_3, 4P_{3\alpha}\}_\alpha(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_3) \\ &- (4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1)\{D^{\alpha-1}q_3, P_{3\alpha}\}_\alpha \\ &= -2P_{3\alpha} + 4(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_3) - (4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1) \\ &= 4P_{2\alpha} + 6P_{3\alpha} - 4D^{\alpha-1}q_1 - 8D^{\alpha-1}q_3 + 1.\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{P}_{1\alpha} &= \{P_{1\alpha}, H_{\alpha T}\}_\alpha = \{P_{1\alpha}, D^{\alpha-1}q_1\}_\alpha + (4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1)\{P_{1\alpha}, D^{\alpha-1}q_1\}_\alpha \\ &= -1 - (4D^{\alpha-1}q_3 - 4P_{3\alpha} - 1) \\ &= 4P_{3\alpha} - 4D^{\alpha-1}q_3.\end{aligned} \tag{31}$$

$$\dot{P}_{2\alpha} = \{P_{2\alpha}, H_{\alpha T}\}_\alpha = \{P_{1\alpha}, D^{\alpha-1}q_1\}_\alpha = -1. \tag{32}$$

$$\begin{aligned}\dot{P}_{3\alpha} &= \{P_{3\alpha}, H_{\alpha T}\}_\alpha = \{P_{3\alpha}, (D^{\alpha-1}q_3)^2\}_\alpha \\ &- 4\{P_{3\alpha}, D^{\alpha-1}q_3\}_\alpha(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_3) \\ &+ (4D^{\alpha-1}q_3 - P_{3\alpha} - 1)\{P_{3\alpha}, D^{\alpha-1}q_3\}_\alpha \\ &= -2D^{\alpha-1}q_3 + 4(P_{2\alpha} + P_{3\alpha} - D^{\alpha-1}q_1 - D^{\alpha-1}q_3) - 4D^{\alpha-1}q_3 + 4P_{3\alpha} + 1 \\ &= -10D^{\alpha-1}q_3 + 4P_{2\alpha} + 8P_{3\alpha} - 4D^{\alpha-1}q_1 + 1.\end{aligned} \tag{33}$$

Taking the total time derivative of eq.(28) we arrive

$$D^{\alpha+1}q_1 = \dot{P}_{1\alpha} = 4P_{3\alpha} - 4D^{\alpha-1}q_3. \tag{34}$$

Making use of the primary constraint, we get

$$D^{\alpha+1}q_1 = 2P_{1\alpha} + 2. \tag{35}$$

Again using the equation of motion (28), we have

$$D^{\alpha+1}q_1 - 2D^{\alpha}q_1 = 2. \tag{36}$$

One may using the conformable operator, we can rewrite the above equation in the following form

$$\ddot{q}_1 - 2\dot{q}_1 = 2t^{\alpha-1}. \tag{37}$$

This equation can be written as

$$\dot{y} - 2y = 2t^{\alpha-1}, \tag{38}$$

where $y = \dot{q}_1$. This is a non-homogeneous first-order ordinary differential equation. One may solve it to get

$$y(t) = 2e^{2t}\int e^{-2t}t^{\alpha-1}dt + Ae^{2t}. \tag{39}$$

Using the incomplete gamma function

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt. \tag{40}$$

We get

$$y(t) = -2^{1-\alpha}e^{2t}\Gamma(\alpha,2t) + Ae^{2t}. \tag{41}$$

Thus,

$$q_1 = \int y(t)dt = -2^{1-\alpha}\int e^{2t}\Gamma(\alpha,2t)dt + \frac{A}{2}e^{2t} + B, \tag{42}$$

after simple calculations, we get

$$q_1 = -2^{-\alpha}e^{2t}\Gamma(\alpha,2t) - \frac{t^{\alpha}}{\alpha} + \frac{A}{2}e^{2t} + B. \tag{43}$$

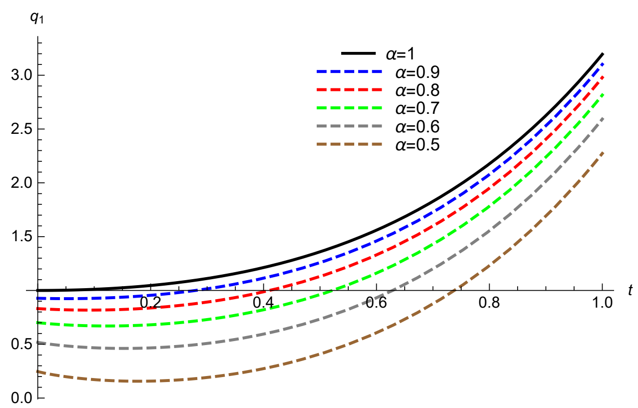

Fig. 1: The coordinate q_1 at different values of α.

Where we suppose the constants A, B are equal to 1. For $\alpha = 1$, we find

$$q_1 = -2^{-1}e^{2t}\Gamma(1,2t) - t + \frac{A}{2}e^{2t} + B$$
$$= B - \frac{1}{2} - t + \frac{A}{2}e^{2t}. \tag{44}$$

This is in agreement with Ref [30].
Taking the total time derivative of eq.(29) we have:

$$D^{\alpha+1}q_2 = 3\dot{P}_{3\alpha} - 4D^{\alpha}q_3. \tag{45}$$

Inserting eq.(30) and eq.(33) we obtain:

$$D^{\alpha+1}q_2 = 4(2P_{3\alpha} - 2D^{\alpha-1}q_3). \tag{46}$$

Using the conformable secondary constraint we can rewrite eq.(46) as

$$D^{\alpha+1}q_2 = 4D^{\alpha}q_1 + 4. \tag{47}$$

Using the definition of conformable derivative eq.(47) can be written as

$$\ddot{q}_2 = 4\dot{q}_1 + 4t^{\alpha-1}. \tag{48}$$

By integrating, we have

$$\dot{q}_2 = 4q_1 + 4\frac{t^{\alpha}}{\alpha} + C_1. \tag{49}$$

Inserting the solution of q_1, we get

$$\dot{q}_2 = -2^{2-\alpha}e^{2t}\Gamma(\alpha,2t) + 2Ae^{2t} + C_2.$$

Thus, one may obtain q_2 as follows

$$q_2 = -2^{1-\alpha}e^{2t}\Gamma(\alpha,2t) - 2\frac{t^{\alpha}}{\alpha} + Ae^{2t} + C_2t + C_3. \tag{50}$$

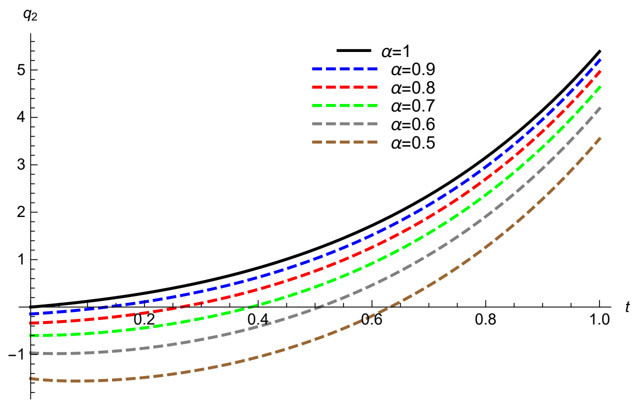

Fig. 2: The coordinate q_2 at different values of α.

Again, when $\alpha = 1$, we arrive to the same result in Ref [30].
Where we suppose the constants A, C_2, C_3 are equal to 1. Now, making use of the primary constraint (21), one may write the equation of motion (30) as

$$D^{\alpha+1}q_3 + 2D^{\alpha}q_3 = \frac{1}{2}D^{\alpha+1}q_2. \tag{51}$$

Making use of eq.(50) and the definition of conformable derivative we get:

$$\ddot{q}_3 + 2\dot{q}_3 = -2^{2-\alpha} e^{2t} \Gamma(\alpha, 2t) + 2t^{\alpha-1} + 2Ae^{2t}, \tag{52}$$

one may solve this equation to find:

$$q_3 = \frac{e^{-2t}\Gamma(\alpha,-2t)}{2^{\alpha+1}(-1)^{\alpha}} - \frac{e^{2t}}{2^{1+\alpha}}\Gamma(\alpha,2t) + \frac{A}{4}e^{2t} - \frac{B}{2}e^{-2t} + C_4. \tag{53}$$

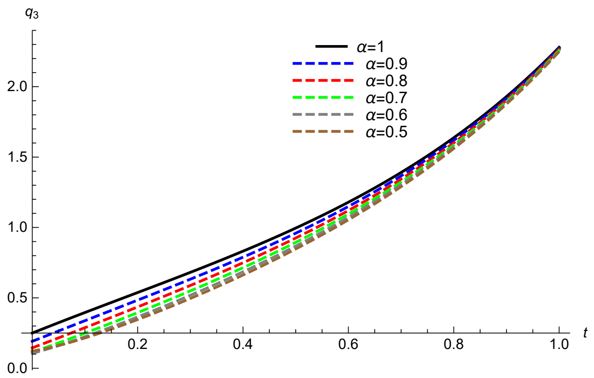

Fig. 3: The coordinate q_3 at different values of α.

where we suppose the constants A, B, C_4 are equal to 1. When $\alpha = 1$, this solution goes to the result obtained in Ref [30].

4 Conclusion

The constrained systems that involve conformable derivatives are discussed using the Dirac theory. This theory finds broad application in field theory, quantum electrodynamics theory, and quantum chromodynamics. The Dirac theory is developed to be applicable to singular systems containing conformable orders. The equations of motion are formulated using the conformable Poisson bracket and solved for an illustrative example to obtain the coordinates as functions of time. It is observed that the general solutions of the conformable singular system of second-class constraints converge to the results obtained by Rabeı et al. for the traditional second-class constraint when $\alpha = 1$. additionally, we plot the general solutions $q_1(t), q_2(t)$, and $q_3(t)$ for various values of α, setting all constants in general solutions $q_1(t), q_2(t)$, and $q_3(t)$ to 1. We demonstrate that the curve gradually converges to $\alpha = 1$, indicating a transition towards the curve traditional second-class constraints obtained by Rabei et Al.

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