Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110205

Computational Analysis of Monkeypox Disease with Incident Infection Rate by using Fractal Operator

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Received: 2 Feb. 2024, Revised: 18 Jun. 2024, Accepted: 20 Aug. 2024
Published online: 1 Apr. 2025

Abstract: In this work, we proposed a fractional order Monkeypox virus model with Caputo Fabrizio fractional operator to investigate the dynamical transmission of the Monkeypox virus and its effects on society. Qualitative analysis of the model is examined such as the existence and uniqueness of Lipschitz conditions, including analysis of the endemic equilibrium and the disease-free and epidemic equilibrium points. Laplace transform with the Adomian decomposition method is used to construct the iterative scheme of the model. Self mapping with a unique solution with a fractional Lagrange multiplier is used for Picard stability under Banach space theory for the iterative scheme. It will be demonstrated through some numerical comparisons that the findings produced by the fractional order model are substantially more accurate than those of the integer order model when compared to some genuine data. In the end, we have also determined the numerical results proposed model utilizing the Laplace transform method having the fastest convergence approach to steady state point for such an epidemic model.

Keywords: Monkeypox; Laplace transform; symmetry; existence; uniqueness; Picard stability; exponential kernel.

1 Introduction

The contagious viral illness Monkeypox, also known as mpox by the WHO, may affect both humans and certain other animals. Fever, enlarged lymph nodes, and a rash that boils and then crusts over are all symptoms. Five to 21 days pass from exposure to the beginning of symptoms. Symptoms last between two and four weeks on average. The potential for further transmission, the absence of adequate surveillance, and the recent apparent increase in human Monkeypox cases across a vast geographic area, according to Petersen et al. [1], have all raised the degree of worry for this new zoonosis. In collaboration with the Centers for Disease Control and Prevention, the World Health Organization held an informal consultation on Monkeypox in November 2017 with researchers, partners in global health, ministries of health, and experts in the orthopoxvirus to review and discuss human Monkeypox in African nations. The Orthopox viral genus includes the variola virus, which causes smallpox, the vaccinia virus used in the smallpox vaccine, and the cowpox virus, which was employed in an earlier vaccine. Although it happens frequently, the Monkeypox virus mainly spreads to humans through wild animals like rats and primates. A skin lesion on an infected person, respiratory droplets, contact with body fluids, a contaminated patient's surroundings, and contaminated patient's goods have all been related to human to human transmission [2]. The Monkeypox virus has replaced smallpox as the most prevalent orthopaedic virus [3]. Symptoms of Monkeypox include fever, headache, muscular pains, backaches, enlarged lymph nodes, chills, and exhaustion in some people. Up to 10% of people who with Monkeypox pass away, with children under the age of puberty accounting for

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the majority of fatalities [4]. Monkeypox was later recognised in the United Kingdom and Israel as a result of its arrival. Mortality rates varied from 1% to 10%, with the majority of deaths occurring in younger age groups [5]. Monkeypox takes around six to sixteen days to incubate, although it can take anywhere between 5 and 21 days. The infectious phase comprises two parts, the first of which lasts 5 days and is characterized by fever, lymph node swelling, back pain, acute headache, muscle soreness, and significant energy depletion. A fat based skin rash starts one to three days after the onset of the fever and progresses to small fuid filled blisters that become pus-filled and crust up in 10 days [6].

Monkeypox infection currently has no effective therapies, however new antivirals like Brincindofovir, Tecovirimat, and vaccinia immune globulin can help contain the disease's transmission. The last ten years have seen a sharp rise in Monkeypox cases, which have been linked to a loss of smallpox herd immunity. Monkeypox can be prevented by smallpox vaccine to an extent of 85%, however it is no longer widely accessible owing to smallpox's global elimination. Post-exposure immunisation may assist to prevent or lessen the severity of the illness, according to [7,8]. This method enhanced the Schmidt orthogonalization approach and Sobolev space-based solutions, which, given the arbitrary kernel functions satisfy Robin's homogeneous conditions, may be directly employed to produce Fourier expansion at a high convergence rate [14,15]. Because of a characterization of the memory and hereditary properties in [17], Ordinary integer order cannot always explain real-world situations as clearly as fractional order, which incorporates integration and transects differentiation with the aid of fractional calculus. Fractional order may also aid in the modelling of real occurrences.

The Caputo derivative is expressed as a fractional type in the model. The recommended settings, we examine the proposed model's chaotic behavior with different values of the fractional order parameter. Numerical approach is used to provide the graphical results for the Caputo operator derivative. Additionally, in order to investigate for the provided set of parameters, we display the graphical results. Generalized models, also known as fractional models, are essential in the mathematical modeling of real world issues. The fractional order models are known as generalized models since they may be examined in non-integer cases in addition to arbitrary cases. Due to its memory and heredity characteristics, fractional order models are preferred over integer order. In addition, fractional modeling is the sole way to adequately study the crossover behavior in nonlinear epidemic models. Additionally, we show the usefulness and application of a numerical approach that includes estimating partial derivatives when working with Caputo fractional operators. In order to demonstrate how different kernels for the fractional operator may be utilized to better correctly characterize the process, an application of the fractional derivative is then explained [18,19].

In this study, we examine a FD of the type Caputo Fabrizio with respect to another function. Additionally, we demonstrate the effectiveness and application of a numerical technique for dealing with the Caputo fractional operator that involves estimating the fractional derivative [20,21]. The application of the fractional derivative is then offered, using the population growth simulate to demonstrate with different kernels, such as the Power law kernel, Mittag Leffler law kernel, and exponential law kernel for the fractional operator, may be used to model the process more precisely. In this work, the connection between the immune system and cancer cells is investigated using fractal fractional operators in the Caputo and Caputo-Fabrizio senses [22,23]. The numerical and theoretical analysis of the singular and non-singular fractal fractional operators has focused on the Monkeypox model. The model under the Caputo Fabrizio fractal fractional operator has been shown to exist and be unique using fixed point theorems. Under the Caputo-Fabrizio case, it has been determined that a singular solution exists. The dynamics of the Monkeypox model with fractional derivative are explored in the current work. The Caputo-Fabrizio fractional derivative is used to formulate the model. Also with recommended settings, we examine the proposed model's chaotic behaviour. Results for the fixed points' stability are displayed. With different values of the fractional order parameter, a numerical approach is used to generate the graphical results for the Caputo-Fabrizio derivative. Additionally, we provide the graphical results to investigate how the model responds to the given set of parameters in periodic and quasi-periodic limit cycles. Fractional models, often referred to as generalised models, are important in the mathematical modelling of everyday issues. We explored some of the favourable properties of the new derivative and applied them to solve the fractional heat transfer model. It is suggested that new derivatives with nonlocal and nonsingular kernels be used in this work [24]. The exponential kernel, Caputo-Fabrizio derivative, and Hilfer fractional derivative are employed in a novel fractional model for the human liver [25,26].

Because of their limitations at birth, working with nonsingular kernels is a difficult undertaking, as Refai and Baleanu discovered. In this brief study, we propose an extension of the fractional operator that permits an integrable singular kernel at the origin by involving the Mittag-Leffler kernel. New solutions to the associated differential equations were presented along with various modeling-related viewpoints [27]. By replacing this $f'(t)$ with a more generic proportional derivative, Baleanu et al. [28] develop a new fractional operator. This novel operator can also be expressed as a linear combination of a Riemann-Liouville integral and a Caputo derivative in some significant particular instances, or as a Riemann-Liouville integral of a proportionate derivative. For neural networks with ring or hub architectures, the stability area of a steady state has been extensively characterised, and the critical fractional order values for which Hopf bifurcations may occur have been discovered [29]. An example of fractional-order neural networks with mixed delays that exhibit periodic oscillation caused by delays in [30]. The dynamics of the model have changed, according to simulations. With the aid of fractional values and discoveries from multiple dimensions, the findings of the nonlinear system memory were also identified.

Without imposing any more requirements, it provides a better technique for how you want to manage the sickness. The numerical results show how the dynamics in the different fractional orders behave [31,32]. Some application of modified Atangana-Baleanu [33], piecewise fractional analysis [34], intravenous drug model [35], piecewise constant for chemo-immunotherapy [36], SEIQR model [37] and HIV/AIDS with new fractional techniques in [38].

The remaining portions of this study article are as continues to follow: The complete introduction of the recommended model is described in detail in the section 1. Some basic fractional order derivatives are provided in the section 2 that may be utilised to help address the epidemiological suggested model. The generalised form, singularity and existence of the model are all described in the section 3. In section "4", stability results of the solutions gained by the iterative Laplace transform technique are produced by applying the Picard successive approximation methodology and the fixed point theory coming from Banach. In sections 5 and 6, the recommended approach is applied to a fractional Monkeypox model, and numerical simulations are visually displayed. Section 7 discuss the results and the conclusion, respectively.

2 Fundamental Fractional Operator Concepts

In [9,16,12], we obtained a number of significant and useful dynamic behavior and current calculus results.

Definition 1. *The Caputo-Fabrizio fractional differential operator is described as if* $\Psi(t) \in \mathscr{H}^1(0,b), b>0,\ 0<\nu<1$.

$$ {}^{\mathscr{CF}}\mathscr{D}_t^{\nu}\Psi(t) = \frac{(2-\nu)\mathscr{M}(\nu)}{2(1-\nu)}\int_0^t exp\left\{-\frac{\nu(t-s)}{1-\nu}\right\}\Psi'(\rho)d\rho, \tag{1}$$

where the normalisation function $\mathscr{M}(\nu)$ *relies on* ν *such that* $\mathscr{M}(o)=\mathscr{M}(1)=1$ *and* $t\geq 0,\ \ 0<\nu<1$.

Definition 2. *This formula yields the Caputo-Fabrizio fractional integral operator of order* $0<\nu<1$.

$$ {}^{\mathscr{CF}}\mathscr{J}_t^{\nu}\Psi(t) = \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\Psi(t) + \frac{2\nu}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \Psi(\rho)d\rho. \tag{2}$$

This new operator gives ${}^{\mathscr{CF}}\mathscr{D}_t^{\nu}\Psi(t)=0$, *which is similar to the conventional Caputo derivative,* $\Psi(t)$ *has a constant value.*

The lack of the $s=t$ *singularity in the new kernel is the fundamental advantage of the Caputo-Fabrizio operator over the original Caputo operator.*

Definition 3. *The following Laplace transform applies to the Caputo-Fabrizio fractional operator of order* $0<\nu\leq 1$ *and* $\eta\in N$

$$\begin{aligned} \mathscr{L}({}^{\mathscr{CF}}\mathscr{D}_t^{\nu}\Psi(t)) &= \frac{1}{1-\nu}\mathscr{L}(\Psi^{(1+\eta)}(t))\mathscr{L}\left\{exp\left(-\frac{\nu}{1-\nu}t\right)\right\}, \\ &= \frac{s^{\eta+1}\mathscr{L}(\Psi(t)) - s^{\eta}\Psi(0) - s^{\eta-1}\Psi'(0) - \cdots - \Psi^{\eta}(0)}{s+\nu(1-s)}, \end{aligned} \tag{3}$$

Specifically, we have

$$\mathscr{L}({}^{\mathscr{CF}}\mathscr{D}_t^{\nu}\Psi(t)) = \frac{s\mathscr{L}(\Psi(t))}{s+\nu(1-s)}, \qquad \eta = 0. \tag{4}$$

$$\mathscr{L}({}^{\mathscr{CF}}\mathscr{D}_t^{\nu}\Psi(t)) = \frac{s^2\mathscr{L}(\Psi(t)) - s\Psi(0) - \Psi'(0)}{s+\nu(1-s)}, \tag{5}$$

where $\eta = 1$.

3 Monkeypox Mathematical Model with Fractional Operator

We offer a deterministic model of the dynamics of Monkeypox transmission that takes into consideration the populations of people and rodents. The rodent population is separated into three sections while the human population is divided into five compartments. Researchers are looking at the causes and recurrence of epidemics of Monkeypox. The full list of parameters for the developed framework is presented in table 1. Now let us look at some key aspects of the compartmental mathematical epidemic model for viral transmission put out by Peter et al. [11]. A specific time t, the suggested system splits the whole population $\mathbb{N}$ into eight divisions.

Table 1: The compartment and parameters of the proposed model are described.

$S_h(t)$	Susceptible Humans Group
$E_h(t)$	Exposed Humans Group
$I_h(t)$	Infected Humans Group
$Q_h(t)$	Isolated Humans Group
$R_h(t)$	Recovered Humans Group
$S_r(t)$	Susceptible Rodents Group
$E_r(t)$	Exposed Rodents Group
$I_r(t)$	Infected Rodents Group
θ_r	Recruitment Rate for Rodents
θ_h	Recruitment Rate for Humans
β_1	Rodent Contact Rate to Humans
β_2	Human to Humans Contact Rate
β_3	Rodent to Rodent Contact Rate
α_1	Proportion of (Exposed to Infected) Humans
α_2	Proportion Identifed as Suspected Case
τ	Progression from Isolated to Recovered Group
φ	Proportion not Detected after Diagnosis
γ	Humans Recovery Rate
μ_h	Natural Death Rate of Human
δ_h	Disease Induced Death Rate for Humans
μ_r	Natural Death Rate of Rodents
δ_r	Disease Induced Death Rate for Rodents

The following set of nonlinear ordinary differential equations serves as a representation of the epidemic Monkeypox model

$$\begin{cases} {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} S_h = \theta_h - \frac{(\beta_1 I_r + \beta_2 I_h) S_h}{N_h} - \mu_h S_h + \varphi Q_h, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} E_h = \frac{(\beta_1 I_r + \beta_2 I_h) S_h}{N_h} - (\alpha_1 + \alpha_2 + \mu_h) E_h, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} I_h = \alpha_1 E_h - (\mu_h + \delta_h + \gamma) I_h, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} Q_h = \alpha_2 E_h - (\varphi + \tau + \delta_h + \mu_h) Q_h, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} R_h = \gamma I_h + \tau Q_h - \mu_h R_h, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} S_r = \theta_r - \frac{\beta_3 S_r I_r}{N_r} - \mu_r S_r, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} E_r = \frac{\beta_3 S_r I_r}{N_r} - (\mu_r + \alpha_3) E_r, \\ {}^{\mathscr{CF}}\mathbb{D}_t^{\nu} I_r = \alpha_3 E_r - (\mu_r + \delta_r) I_r. \end{cases} \tag{6}$$

$$S_h(0) = S_h^0, E_h(0) = E_h^0, Q_h(0) = Q_h^0, I_h(0) = I_h^0, R_h(0) = R_h^0, S_r(0) = S_r^0, I_r(0) = I_r^0, E_r(0) = E_r^0. \tag{7}$$

3.1 System Qualitative Analysis

By setting the left side to zero, we get to an equilibrium point that is free of sickness and endemic. The following equation is used to compute the disease-free equilibrium

$$E^\star = (S_h^\star, I_h^\star, E_h^\star, R_h^\star, Q_h^\star, S_r^\star, E_r^\star, I_r^\star)$$

and endemic equilibrium point is given as

$$E_h^* = -\frac{\Theta_1}{\Theta_2} \tag{8}$$

$$\begin{aligned}\Theta_1 &= N_h\gamma\mu_h^3 + N_h\gamma\mu_h^2\tau + N_h\gamma\mu_h^2\varphi + N_h\gamma\mu_h^2\alpha_1 + N_h\gamma\mu_h^2\alpha_2 + N_h\gamma\mu_h^2\delta_h + N_h\gamma\mu_h\tau\alpha_1\\ &+ N_h\gamma\mu_h\tau\alpha_2 + N_h\gamma\mu_h\varphi\alpha_1 + N_h\gamma\mu_h\varphi\alpha_2 + N_h\gamma\mu_h\alpha_1\delta_h + N_h\gamma\mu_h\alpha_2\delta_h + N_h\mu_h^4 + N_h\mu_h^3\tau\\ &+ N_h\mu_h^3\varphi + N_h\mu_h^3\alpha_1 + N_h\mu_h^3\alpha_2 + 2N_h\mu_h^3\delta_h + N_h\mu_h^2\tau\alpha_1 + N_h\mu_h^2\tau\alpha_2 + N_h\mu_h^2\tau\delta_h\\ &+ N_h\mu_h^2\varphi\alpha_1 + N_h\mu_h^2\varphi\alpha_2 + N_h\mu_h^2\varphi\delta_h + 2N_h\mu_h^2\alpha_1\delta_h + 2N_h\mu_h^2\alpha_2\delta_h + N_h\mu_h^2\delta_h^2 + N_h\mu_h\tau\alpha_1\delta_h\\ &+ N_h\mu_h\tau\alpha_2\delta_h + N_h\mu_h\varphi\alpha_1\delta_h + N_h\mu_h\varphi\alpha_2\delta_h + N_h\mu_h\alpha_1\delta_h^2 + N_h\mu_h\alpha_2\delta_h^2 - \beta_2\mu_h\alpha_1\theta_h - \beta_2\tau\alpha_1\theta_h\\ &- \beta_2\varphi\alpha_1\theta_h - \beta_2\alpha_1\delta_h\theta_h\end{aligned}$$

$$\Theta_2 = \alpha_1\left(\mu_h^2 + \mu_h\tau + \mu_h\varphi + \mu_h\alpha_1 + \mu_h\alpha_2 + \mu_h\delta_h + \tau\alpha_1 + \tau\alpha_2 + \varphi\alpha_1 + \alpha_1\delta_h + \alpha_2\delta_h\right)\beta_2$$

$$E_r^* = 0 \tag{9}$$

$$I_h^* = -\frac{\Theta_3}{\Theta_4} \tag{10}$$

$$\begin{aligned}\Theta_3 &= N_h\gamma\mu_h^3 + N_h\gamma\mu_h^2\tau + N_h\gamma\mu_h^2\varphi + N_h\gamma\mu_h^2\alpha_1 + N_h\gamma\mu_h^2\alpha_2 + N_h\gamma\mu_h^2\delta_h + N_h\gamma\mu_h\tau\alpha_1\\ &+ N_h\gamma\mu_h\tau\alpha_2 + N_h\gamma\mu_h\varphi\alpha_1 + N_h\gamma\mu_h\varphi\alpha_2 + N_h\gamma\mu_h\alpha_1\delta_h + N_h\gamma\mu_h\alpha_2\delta_h + N_h\mu_h^4 + N_h\mu_h^3\tau\\ &+ N_h\mu_h^3\varphi + N_h\mu_h^3\alpha_1 + N_h\mu_h^3\alpha_2 + 2N_h\mu_h^3\delta_h + N_h\mu_h^2\tau\alpha_1 + N_h\mu_h^2\tau\alpha_2 + N_h\mu_h^2\tau\delta_h\\ &+ N_h\mu_h^2\varphi\alpha_1 + N_h\mu_h^2\varphi\alpha_2 + N_h\mu_h^2\varphi\delta_h + 2N_h\mu_h^2\alpha_1\delta_h + 2N_h\mu_h^2\alpha_2\delta_h + N_h\mu_h^2\delta_h^2\\ &+ N_h\mu_h\tau\alpha_1\delta_h + N_h\mu_h\tau\alpha_2\delta_h + N_h\mu_h\varphi\alpha_1\delta_h + N_h\mu_h\varphi\alpha_2\delta_h + N_h\mu_h\alpha_1\delta_h^2 + N_h\mu_h\alpha_2\delta_h^2\\ &- \beta_2\mu_h\alpha_1\theta_h - \beta_2\tau\alpha_1\theta_h - \beta_2\varphi\alpha_1\theta_h - \beta_2\alpha_1\delta_h\theta_h\end{aligned}$$

$$\begin{aligned}\Theta_4 &= \beta_2(\gamma\mu_h^2 + \gamma\mu_h\tau + \gamma\mu_h\varphi + \gamma\mu_h\alpha_1 + \gamma\mu_h\alpha_2 + \gamma\mu_h\delta_h + \gamma\tau\alpha_1 + \gamma\tau\alpha_2 + \gamma\varphi\alpha_1 + \gamma\alpha_1\delta_h\\ &+ \gamma\alpha_2\delta_h + \mu_h^3 + \mu_h^2\tau + \mu_h^2\varphi + \mu_h^2\alpha_1 + \mu_h^2\alpha_2 + 2\mu_h^2\delta_h + \mu_h\tau\alpha_1 + \mu_h\tau\alpha_2 + \mu_h\tau\delta_h + \mu_h\varphi\alpha_1\\ &+ \mu_h\varphi\delta_h + 2\mu_h\alpha_1\delta_h + 2\mu_h\alpha_2\delta_h + \mu_h\delta_h^2 + \tau\alpha_1\delta_h + \tau\alpha_2\delta_h + \varphi\alpha_1\delta_h + \alpha_1\delta_h^2 + \alpha_2\delta_h^2)\end{aligned}$$

$$I_r^* = 0 \tag{11}$$

$$Q_h^* = -\frac{\Theta_5}{\Theta_6} \tag{12}$$

$$\begin{aligned}\Theta_5 &= \alpha_2(N_h\gamma\mu_h^2 + N_h\gamma\mu_h\alpha_1 + N_h\gamma\mu_h\alpha_2 + N_h\mu_h^3 + N_h\mu_h^2\alpha_1 + N_h\mu_h^2\alpha_2 + N_h\mu_h^2\delta_h + N_h\mu_h\alpha_1\delta_h\\ &+ N_h\mu_h\alpha_2\delta_h - \beta_2\alpha_1\theta_h)\end{aligned}$$

$$\Theta_6 = \alpha_1(\mu_h^2 + \mu_h\tau + \mu_h\varphi + \mu_h\alpha_1 + \mu_h\alpha_2 + \mu_h\delta_h + \tau\alpha_1 + \tau\alpha_2 + \varphi\alpha_1 + \alpha_1\delta_h + \alpha_2\delta_h)\beta_2$$

$$R_h^* = -\frac{\Theta_7}{\Theta_8} \tag{13}$$

$$\begin{aligned}\Theta_7 &= N_h\gamma^2\mu_h^3\alpha_1 + N_h\gamma^2\mu_h^2\tau\alpha_1 + N_h\gamma^2\mu_h^2\tau\alpha_2 + N_h\gamma^2\mu_h^2\varphi\alpha_1 + N_h\gamma^2\mu_h^2\alpha_1^2 + N_h\gamma^2\mu_h^2\alpha_1\alpha_2\\ &+ N_h\gamma^2\mu_h^2\alpha_1\delta_h + N_h\gamma^2\mu_h\tau\alpha_1^2 + 2N_h\gamma^2\mu_h\tau\alpha_1\alpha_2 + N_h\gamma^2\mu_h\tau\alpha_2^2 + N_h\gamma^2\mu_h\varphi\alpha_1^2 + N_h\gamma^2\mu_h\varphi\alpha_1\alpha_2\\ &+ N_h\gamma^2\mu_h\alpha_1^2\delta_h + N_h\gamma^2\mu_h\alpha_1\alpha_2\delta_h + N_h\gamma\mu_h^4\alpha_1 + N_h\gamma\mu_h^3\tau\alpha_1 + 2N_h\gamma\mu_h^3\tau\alpha_2 + N_h\gamma\mu_h^3\varphi\alpha_1\\ &+ N_h\gamma\mu_h^3\alpha_1^2 + N_h\gamma\mu_h^3\alpha_1\alpha_2 + 2N_h\gamma\mu_h^3\alpha_1\delta_h + N_h\gamma\mu_h^2\tau\alpha_1^2 + 3N_h\gamma\mu_h^2\tau\alpha_1\alpha_2 + N_h\gamma\mu_h^2\tau\alpha_1\delta_h\\ &+ 2N_h\gamma\mu_h^2\tau\alpha_2^2 + 2N_h\gamma\mu_h^2\tau\alpha_2\delta_h + N_h\gamma\mu_h^2\varphi\alpha_1^2 + N_h\gamma\mu_h^2\varphi\alpha_1\alpha_2 + N_h\gamma\mu_h^2\varphi\alpha_1\delta_h + 2N_h\gamma\mu_h^2\alpha_1^2\delta_h\\ &+ 2N_h\gamma\mu_h^2\alpha_1\alpha_2\delta_h + N_h\gamma\mu_h^2\alpha_1\delta_h^2 + N_h\gamma\mu_h\tau\alpha_1^2\delta_h + 3N_h\gamma\mu_h\tau\alpha_1\alpha_2\delta_h + 2N_h\gamma\mu_h\tau\alpha_2^2\delta_h\\ &+ N_h\gamma\mu_h\varphi\alpha_1^2\delta_h + N_h\gamma\mu_h\varphi\alpha_1\alpha_2\delta_h + N_h\gamma\mu_h\alpha_1^2\delta_h^2 + N_h\gamma\mu_h\alpha_1\alpha_2\delta_h^2 + N_h\mu_h^4\tau\alpha_2 + N_h\mu_h^3\tau\alpha_1\alpha_2\\ &+ N_h\mu_h^3\tau\alpha_2^2 + 2N_h\mu_h^3\tau\alpha_2\delta_h + 2N_h\mu_h^2\tau\alpha_1\alpha_2\delta_h + 2N_h\mu_h^2\tau\alpha_2^2\delta_h + N_h\mu_h^2\tau\alpha_2\delta_h^2 + N_h\mu_h\tau\alpha_1\alpha_2\delta_h^2\\ &+ N_h\mu_h\tau\alpha_2^2\delta_h^2 - \beta_2\gamma\mu_h\alpha_1^2\theta_h - \beta_2\gamma\tau\alpha_1^2\theta_h - \beta_2\gamma\tau\alpha_1\alpha_2\theta_h - \beta_2\gamma\varphi\alpha_1^2\theta_h - \beta_2\gamma\alpha_1^2\delta_h\theta_h\\ &- \beta_2\mu_h\tau\alpha_1\alpha_2\theta_h - \beta_2\tau\alpha_1\alpha_2\delta_h\theta_h\end{aligned}$$

$$\Theta_8 = \mu_h\Big(\mu_h^2 + \mu_h\tau + \mu_h\varphi + \mu_h\alpha_1 + \mu_h\alpha_2 + \mu_h\delta_h + \tau\alpha_1 + \tau\alpha_2 + \varphi\alpha_1 + \alpha_1\delta_h + \alpha_2\delta_h\Big)$$
$$\times\,\alpha_1(\mu_h + \delta_h + \gamma)\beta_2$$

$$S_h^* = \frac{N_h(\gamma\mu_h + \gamma\alpha_1 + \gamma\alpha_2 + \mu_h^2 + \mu_h\alpha_1)}{\beta_2\alpha_1} + \frac{N_h(\mu_h\alpha_2 + \mu_h\delta_h + \alpha_1\delta_h + \alpha_2\delta_h)}{\beta_2\alpha_1} \tag{14}$$

$$S_r^* = \frac{\theta_t}{\mu_r} \tag{15}$$

The proposed model [11] reproduction number is reported as

$$\mathbb{R}_o = \frac{\alpha_1\beta_1}{(\alpha_1 + \alpha_2 + \mu_h)(\mu_h + \delta_h + \gamma)} \tag{16}$$

3.2 Analysis of the Proposed System

In this section, the fixed point hypothesis (6) is used to explain the existence and uniqueness of the system's solution.
Taking into account equation (2), we have

$$S_h(t) = S_h(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\theta_h - \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h(t)}{N_h} - \mu_h S_h(t) + \varphi Q_h(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)} \times \int_0^t \Big\{\theta_h - \frac{(\beta_1 I_r(\rho) + \beta_2 I_h(\rho))S_h(\rho)}{N_h} - \mu_h S_h(\rho) + \varphi Q_h(\rho)\Big\}d\rho,$$
$$E_h(t) = E_h(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h(t)}{N_h} - (\alpha_1 + \alpha_2 + \mu_h)E_h(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)} \times \int_0^t \Big\{\frac{(\beta_1 I_r(\rho) + \beta_2 I_h(\rho))S_h(\rho)}{N_h} - (\alpha_1 + \alpha_2 + \mu_h)E_h(\rho)\Big\}d\rho,$$
$$I_h(t) = I_h(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\{\alpha_1 E_h(t) - (\mu_h + \delta_h + \gamma)I_h(t)\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \{\alpha_1 E_h(\rho) - (\mu_h + \delta_h + \gamma)I_h(\rho)\}\,d\rho,$$
$$Q_h(t) = Q_h(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\alpha_2 E_h(t) - (\varphi + \tau + \delta_h + \mu_h)Q_h(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \{\alpha_2 E_h(\rho) - (\varphi + \tau + \delta_h + \mu_h)Q_h(\rho)\}\,d\rho,$$
$$R_h(t) = R_h(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\gamma I_h(t) + \tau Q_h(t) - \mu_h R_h(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \{\gamma I_h(\rho) + \tau Q_h(\rho) - \mu_h R_h(\rho)\}\,d\rho,$$
$$S_r(t) = S_r(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\theta_r - \frac{\beta_3 S_r(t)I_r(t)}{N_r} - \mu_r S_r(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \Big\{\theta_r - \frac{\beta_3 S_r(\rho)I_r(\rho)}{N_r} - \mu_r S_r(\rho)\Big\}\,d\rho,$$
$$E_r(t) = E_r(0) + \frac{2(1-\nu)}{2\nu\mathscr{N}(\nu)}\Big\{\frac{\beta_3 S_r(t)I_r(t)}{N_r} - (\mu_r + \alpha_3)E_r(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \Big\{\frac{\beta_3 S_r(\rho)I_r(\rho)}{N_r} - (\mu_r + \alpha_3)E_r(\rho)\Big\}\,d\rho,$$
$$I_r(t) = I_r(0) + \frac{2(1-\nu)}{2\nu\mathscr{M}(\nu)}\Big\{\alpha_3 E_r(t) - (\mu_r + \delta_r)I_r(t)\Big\}$$
$$+ \frac{2(1-\nu)}{(2-\nu)\mathscr{M}(\nu)}\int_0^t \{\alpha_3 E_r(\rho) - (\mu_r + \delta_r)I_r(\rho)\}\,d\rho,$$

Now, let's think about the subsequent kernels

$$\begin{cases} \phi_1 = \theta_h - \frac{(\beta_1 I_r + \beta_2 I_h)S_h}{N_h} - \mu_h S_h + \varphi Q_h, \\ \phi_2 = \frac{(\beta_1 I_r + \beta_2 I_h)S_h}{N_h} - (\alpha_1 + \alpha_2 + \mu_h)E_h, \\ \phi_3 = E_h\alpha_1 - (\delta_h + \mu_h + \gamma)I_h, \\ \phi_4 = E_h\alpha_2 - (\varphi + \delta_h + \mu_h + \tau)Q_h, \\ \phi_5 = \gamma I_h + \tau Q_h - \mu_h R_h, \\ \phi_6 = \theta_r - \frac{\beta_3 S_r I_r}{N_r} - \mu_r S_r, \\ \phi_7 = \frac{\beta_3 S_r I_r}{N_r} - (\mu_r + \alpha_3)E_r, \\ \phi_8 = \alpha_3 E_r - (\mu_r + \delta_r)I_r. \end{cases} \tag{17}$$

Theorem 1. *The kernels $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7$ and ϕ_8 given in (17) satisfy the Lipschitz axioms if the aforementioned disparity persists*

$$0 < \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8 < 1 \tag{18}$$

Proof. Let S_h^1 and S_h^2, for the kernel ϕ_1, E_h^1 and E_h^2, for the kernel ϕ_2, I_h^1 and I_h^2, for the kernel ϕ_3, Q_h^1 and Q_h^2, for the kernel ϕ_4, R_h^1 and R_h^2, for the kernel ϕ_5, S_r^1 and S_r^2, for the kernel ϕ_6, E_r^1 and E_r^2, for the kernel ϕ_7 and I_r^1 and I_r^2, for the kernel ϕ_8 are corresponding functions that go with ϕ_8

$$\begin{aligned} &\|\phi_1(t, S_h^1(t)) - \phi_1(t, S_h^2(t))\| \\ &= \left\| \left\{ \theta_h - \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h^1(t)}{N_h} - \mu_h S_h^1(t) + \varphi Q_h(t) \right\} \right. \\ &\left. - \left\{ \theta_h - \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h^2(t)}{N_h} - \mu_h S_h^2(t) + \varphi Q_h(t) \right\} \right\| \\ &= \left\| \theta_h - \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h^1(t)}{N_h} - \mu_h S_h^1(t) + \varphi Q_h(t) \right. \\ &\left. - \theta_h + \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))S_h^2(t)}{N_h} + \mu_h S_h^2(t) - \varphi Q_h(t) \right\|, \end{aligned}$$

$$\|\phi_1(t, S_h^1(t)) - \phi_1(t, S_h^2(t))\| = \left\| - \left\{ \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))}{N_h} + \mu_h \right\} (S_h^1(t) - S_h^2(t)) \right\|, \tag{19}$$

and similarly

$$\|\phi_2(t, E_h^1(t)) - \phi_2(t, E_h^2(t))\| = \left\| - \{\alpha_1 + \alpha_2 + \mu_h\} (E_h^1(t) - E_h^2(t)) \right\|, \tag{20}$$

$$\|\phi_3(t, I_h^1(t)) - \phi_3(t, I_h^2(t))\| = \left\| - \{\mu_h + \delta_h + \gamma\} (I_h^1(t) - I_h^2(t)) \right\|, \tag{21}$$

$$\|\phi_4(t, Q_h^1(t)) - \phi_4(t, Q_h^2(t))\| = \left\| - \{\varphi + \tau + \delta_h + \mu_h\} (Q_h^1(t) - Q_h^2(t)) \right\|, \tag{22}$$

$$\|\phi_5(t, R_h^1(t)) - \phi_5(t, R_h^2(t))\| = \left\| - \{\mu_h\} (R_h^1(t) - R_h^2(t)) \right\|, \tag{23}$$

$$\|\phi_6(t, S_r^1(t)) - \phi_6(t, S_r^2(t))\| = \left\| - \left\{ \frac{\beta_3 I_r}{N_r} + \mu_r \right\} (S_r^1(t) - S_r^2(t)) \right\|, \tag{24}$$

$$\|\phi_7(t, E_r^1(t)) - \phi_7(t, E_r^2(t))\| = \left\| - \{\mu_r + \alpha_3\} (E_r^1(t) - E_r^2(t)) \right\|, \tag{25}$$

$$\|\phi_8(t, I_r^1(t)) - \phi_8(t, I_r^2(t))\| = \left\| - \{\mu_r + \delta_r\} (I_r^1(t) - I_r^2(t)) \right\|. \tag{26}$$

Now we suppose

$$\|\phi_1(t, S_h^1(t)) - \phi_1(t, S_h^2(t))\| = \left\| - \left\{ \frac{(\beta_1 I_r(t) + \beta_2 I_h(t))}{N_h} + \mu_h \right\} (S_h^1(t) - S_h^2(t)) \right\|,$$

$$\|\phi_1(t,S_h^1(t))-\phi_1(t,S_h^2(t))\|\leq\left\{\frac{\beta_1\|I_r\|+\beta_2\|I_h\|}{N_h}+\mu_h\right\}\left\|(S_h^1(t)-S_h^2(t))\right\|,$$

$$\|\phi_1(t,S_h^1(t))-\phi_1(t,S_h^2(t))\|\leq\left\{\frac{\beta_1 i_r+\beta_2 i_h}{N_h}+\mu_h\right\}\left\|(S_h^1(t)-S_h^2(t))\right\|,$$

$$\|\phi_1(t,S_h^1(t))-\phi_1(t,S_h^2(t))\|\leq\varpi_1\left\|(S_h^1(t)-S_h^2(t))\right\|, \tag{27}$$

where $\varpi_1=\frac{\beta_1 i_r+\beta_2 i_h}{N_h}+\mu_h$, $i_h=\max_{t\in\Upsilon}\|I_h(t)\|$ and $i_r=\max_{t\in\Upsilon}\|I_r(t)\|$ are bounded functions. Similarly, we can discover

$$\|\phi_2(t,E_h^1(t))-\phi_2(t,E_h^2(t))\|\leq\varpi_2\left\|(E_h^1(t)-E_h^2(t))\right\|, \tag{28}$$

$$\|\phi_3(t,I_h^1(t))-\phi_3(t,I_h^2(t))\|\leq\varpi_3\left\|(I_h^1(t)-I_h^2(t))\right\|, \tag{29}$$

$$\|\phi_4(t,Q_h^1(t))-\phi_4(t,Q_h^2(t))\|\leq\varpi_4\left\|(Q_h^1(t)-Q_h^2(t))\right\|, \tag{30}$$

$$\|\phi_5(t,R_h^1(t))-\phi_5(t,R_h^2(t))\|\leq\varpi_5\left\|(R_h^1(t)-R_h^2(t))\right\|, \tag{31}$$

$$\|\phi_6(t,S_r^1(t))-\phi_6(t,S_r^2(t))\|\leq\varpi_6\left\|(S_r^1(t)-S_r^2(t))\right\|, \tag{32}$$

$$\|\phi_7(t,E_r^1(t))-\phi_7(t,E_r^2(t))\|\leq\varpi_7\left\|(E_r^1(t)-E_r^2(t))\right\|, \tag{33}$$

$$\|\phi_8(t,I_r^1(t))-\phi_8(t,I_r^2(t))\|\leq\varpi_8\left\|(I_r^1(t)-I_r^2(t))\right\|. \tag{34}$$

where

$$\begin{aligned}
&\varpi_2=\alpha_1+\alpha_2+\mu_h,\\
&\varpi_3=\mu_h+\delta_h+\gamma,\\
&\varpi_4=\varphi+\tau+\delta_h+\mu_h,\\
&\varpi_5=\mu_h,\\
&\varpi_6=\tfrac{\beta_3 I_r}{N_r}+\mu_r,\\
&\varpi_7=\mu_r+\alpha_3,\\
&\varpi_8=\mu_r+\delta_r.
\end{aligned}$$

The following recursive procedure yields

$$\begin{cases}
S_h^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_1(t,S_h^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_1(\rho,S_h^{n-1}(\rho))d\rho,\\
E_h^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_2(t,E_h^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_2(\rho,E_h^{n-1}(\rho))d\rho,\\
I_h^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_3(t,I_h^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_3(\rho,I_h^{n-1}(\rho))d\rho,\\
Q_h^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_4(t,Q_h^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_4(\rho,Q_h^{n-1}(\rho))d\rho,\\
R_h^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_5(t,R_h^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_5(\rho,R_h^{n-1}(\rho))d\rho,\\
S_r^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_6(t,S_r^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_6(\rho,S_r^{n-1}(\rho))d\rho,\\
E_r^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_7(t,E_r^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_7(\rho,E_r^{n-1}(\rho))d\rho,\\
I_r^n(t)=\frac{2(1-\nu)}{(2-\nu).\mathscr{N}(\nu)}\phi_8(t,I_r^{n-1}(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_8(\rho,I_r^{n-1}(\rho))d\rho.
\end{cases} \tag{35}$$

Triangular inequality is also used to acquire

$$\begin{cases} \|M_1^n\| = \|S_h^n(t) - S_h^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_1(t, S_h^{n-1}(t)) - \phi_1(t, S_h^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_1(\rho, S_h^{n-1}(\rho)) - \phi_1(\rho, S_h^{n-2}(\rho))\} d\rho \|, \\ \|M_2^n\| = \|E_h^n(t) - E_h^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_2(t, E_h^{n-1}(t)) - \phi_2(t, E_h^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_2(\rho, E_h^{n-1}(\rho)) - \phi_2(\rho, E_h^{n-2}(\rho))\} d\rho \|, \\ \|M_3^n\| = \|I_h^n(t) - I_h^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_3(t, I_h^{n-1}(t)) - \phi_3(t, I_h^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_3(\rho, I_h^{n-1}(\rho)) - \phi_3(\rho, I_h^{n-2}(\rho))\} d\rho \|, \\ \|M_4^n\| = \|Q_h^n(t) - Q_h^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_4(t, Q_h^{n-1}(t)) - \phi_4(t, Q_h^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_4(\rho, Q_h^{n-1}(\rho)) - \phi_4(\rho, Q_h^{n-2}(\rho))\} d\rho \|, \\ \|M_5^n\| = \|R_h^n(t) - R_h^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_5(t, R_h^{n-1}(t)) - \phi_5(t, R_h^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_5(\rho, R_h^{n-1}(\rho)) - \phi_5(\rho, R_h^{n-2}(\rho))\} d\rho \|, \\ \|M_6^n\| = \|S_r^n(t) - S_r^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_6(t, S_r^{n-1}(t)) - \phi_6(t, S_r^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_6(\rho, S_r^{n-1}(\rho)) - \phi_6(\rho, S_r^{n-2}(\rho))\} d\rho \|, \\ \|M_7^n\| = \|E_r^n(t) - E_r^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_7(t, E_r^{n-1}(t)) - \phi_7(t, E_r^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_7(\rho, E_r^{n-1}(\rho)) - \phi_7(\rho, E_r^{n-2}(\rho))\} d\rho \|, \\ \|M_8^n\| = \|I_r^n(t) - I_r^{n-1}(t)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \|\phi_8(t, I_r^{n-1}(t)) - \phi_8(t, I_r^{n-2}(t))\| \\ + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \| \int_0^t \{\phi_8(\rho, I_r^{n-1}(\rho)) - \phi_8(\rho, I_r^{n-2}(\rho))\} d\rho \|. \end{cases} \tag{36}$$

$$\begin{aligned} S_h^n(t) &= \sum_{m=0}^{\infty} M_1^m(t), \; E_h^n(t) = \sum_{m=0}^{\infty} M_2^m(t), \; I_h^n(t) = \sum_{m=0}^{\infty} M_3^m(t), \; Q_h^n(t) = \sum_{m=0}^{\infty} M_4^m(t), \\ R_h^n(t) &= \sum_{m=0}^{\infty} M_5^m(t), \; S_r^n(t) = \sum_{m=0}^{\infty} M_6^m(t), \; E_r^n(t) = \sum_{m=0}^{\infty} M_7^m(t), \; I_r^n(t) = \sum_{m=0}^{\infty} M_8^m(t). \end{aligned} \tag{37}$$

Since the kernels ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, ϕ_6, ϕ_7, and ϕ_8 holds the Lipschitz condition, we obtain

$$\begin{cases} \|M_1^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_1 \|S_h^{n-1}(t) - S_h^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_1 \int_0^t \|S_h^{n-1}(\rho) - S_h^{n-2}(\rho)\| d\rho, \\ \|M_2^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_2 \|E_h^{n-1}(t) - E_h^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_2 \int_0^t \|E_h^{n-1}(\rho) - E_h^{n-2}(\rho)\| d\rho, \\ \|M_3^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_3 \|I_h^{n-1}(t) - I_h^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_3 \int_0^t \|I_h^{n-1}(\rho) - I_h^{n-2}(\rho)\| d\rho, \\ \|M_4^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_4 \|Q_h^{n-1}(t) - Q_h^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_4 \int_0^t \|Q_h^{n-1}(\rho) - Q_h^{n-2}(\rho)\| d\rho, \\ \|M_5^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_5 \|R_h^{n-1}(t) - R_h^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_5 \int_0^t \|R_h^{n-1}(\rho) - R_h^{n-2}(\rho)\| d\rho, \\ \|M_6^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_6 \|S_r^{n-1}(t) - S_r^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_6 \int_0^t \|S_r^{n-1}(\rho) - S_r^{n-2}(\rho)\| d\rho, \\ \|M_7^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_7 \|E_r^{n-1}(t) - E_r^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_7 \int_0^t \|E_r^{n-1}(\rho) - E_r^{n-2}(\rho)\| d\rho, \\ \|M_8^n\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)} \psi_8 \|I_r^{n-1}(t) - I_r^{n-2}(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)} \psi_8 \int_0^t \|I_r^{n-1}(\rho) - I_r^{n-2}(\rho)\| d\rho. \end{cases} \tag{38}$$

Which proves the result.

3.2.1 Existence of the Suggested System's Solution

Theorem 2. *The system described in (6) has a solution.*

Proof. We have found the following using the recursive formula and the results of (38).

$$\begin{cases} \|M_1^n\| \leq \|S_h(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_1\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_1^t\right)^n \right\}, \\ \|M_2^n\| \leq \|E_h(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_2\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_2^t\right)^n \right\}, \\ \|M_3^n\| \leq \|I_h(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_3\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_3^t\right)^n \right\}, \\ \|M_4^n\| \leq \|Q_h(0)\| + \left\{ \left(\frac{2(1-\nu)}{\mathscr{M}(\nu)(2-\nu)}\psi_4\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_4^t\right)^n \right\}, \\ \|M_5^n\| \leq \|R_h(0)\| + \left\{ \left(\frac{2(1-\nu)}{\mathscr{M}(\nu)(2-\nu)}\psi_5\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_5^t\right)^n \right\}, \\ \|M_6^n\| \leq \|S_r(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_6\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_6^t\right)^n \right\}, \\ \|M_7^n\| \leq \|E_r(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_7\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_7^t\right)^n \right\}, \\ \|M_8^n\| \leq \|I_r(0)\| + \left\{ \left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_8\right)^n + \left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_8^t\right)^n \right\}. \end{cases} \tag{39}$$

Consequently, (39) exists. Additionally, we demonstrate the system of solutions in (39) and (6) that the functions.

$$\begin{aligned} &S_h(t) = S_h^n(t) - \Delta_{1(n)}(t), \;\; E_h(t) = E_h^n(t) - \Delta_{2(n)}(t), \;\; I_h(t) = I_h^n(t) - \Delta_{3(n)}(t), \\ &Q_h(t) = Q_h^n(t) - \Delta_{4(n)}(t), \;\; R_h(t) = R_h^n(t) - \Delta_{5(n)}(t), \;\; S_r(t) = S_r^n(t) - \Delta_{6(n)}(t), \\ &E_r(t) = E_r^n(t) - \Delta_{7(n)}(t), \;\; I_r(t) = I_r^n(t) - \Delta_{8(n)}(t). \end{aligned} \tag{40}$$

where $\Delta_{1(n)}(t)$, $\Delta_{2(n)}(t)$, $\Delta_{3(n)}(t)$, $\Delta_{4(n)}(t)$, $\Delta_{5(n)}(t)$, $\Delta_{6(n)}(t)$, $\Delta_{7(n)}(t)$ and $\Delta_{8(n)}(t)$ remaining conditions of the solution. Hence, we get

$$\begin{cases} S_h(t) - S_h^{n-1}(t) = \frac{(1-\nu)2}{\mathscr{M}(\nu)(2-\nu)}\phi_1(t, S_h(t) - \Delta_{1(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_1(\rho, S_h(\rho) - \Delta_{1(n)}(\rho)), \\ E_h(t) - E_h^{n-1}(t) = \frac{(1-\nu)2}{\mathscr{M}(\nu)(2-\nu)}\phi_2(t, E_h(t) - \Delta_{1(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_2(\rho, E_h(\rho) - \Delta_{2(n)}(\rho)), \\ I_h(t) - I_h^{n-1}(t) = \frac{2(1-\nu)}{\mathscr{M}(\nu)(2-\nu)}\phi_3(t, I_h(t) - \Delta_{1(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_3(\rho, I_h(\rho) - \Delta_{3(n)}(\rho)), \\ Q_h(t) - Q_h^{n-1}(t) = \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_4(t, Q_h(t) - \Delta_{4(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_4(\rho, Q_h(\rho) - \Delta_{4(n)}(\rho)), \\ R_h(t) - R_h^{n-1}(t) = \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_5(t, R_h(t) - \Delta_{5(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_5(\rho, R_h(\rho) - \Delta_{5(n)}(\rho)), \\ S_r(t) - S_r^{n-1}(t) = \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_6(t, S_r(t) - \Delta_{6(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_6(\rho, S_r(\rho) - \Delta_{6(n)}(\rho)), \\ E_r(t) - E_r^{n-1}(t) = \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_7(t, E_r(t) - \Delta_{7(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_7(\rho, E_r(\rho) - \Delta_{7(n)}(\rho)), \\ I_r(t) - I_r^{n-1}(t) = \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_8(t, I_r(t) - \Delta_{8(n)}(t)) + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t \phi_8(\rho, I_r(\rho) - \Delta_{8(n)}(\rho)). \end{cases} \tag{41}$$

Utilizing the Lipschitz property and the norm, we have

$$\begin{cases} S_h(t)-S_h(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_1(t,S_h(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_1(\rho,S_h(\rho))d\rho \\ \leq \|\Delta_{1(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_1\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_1^t\right)\right\}, \\ E_h(t)-E_h(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_2(t,E_h(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_2(\rho,E_h(\rho)d\rho) \\ \leq \|\Delta_{2(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_2\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_2^t\right)\right\}, \\ I_h(t)-I_h(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_3(t,I_h(t))-\frac{2\nu}{(2-\nu).\mathscr{N}(\nu)}\int_0^t\phi_3(\rho,I_h(\rho))d\rho \\ \leq \|\Delta_{3(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_3\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_3^t\right)\right\}, \\ Q_h(t)-Q_h(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_4(t,Q_h(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_4(\rho,Q_h(\rho))d\rho \\ \leq \|\Delta_{4(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_4\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_4^t\right)\right\}, \\ R_h(t)-R_h(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_5(t,R_h(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_5(\rho,R_h(\rho))d\rho \\ \leq \|\Delta_{5(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_5\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_5^t\right)\right\}, \\ S_r(t)-S_r(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_6(t,S_r(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_6(\rho,S_r(\rho))d\rho \\ \leq \|\Delta_{6(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_6\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_6^t\right)\right\}, \\ E_r(t)-E_r(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_7(t,E_r(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_7(\rho,E_r(\rho))d\rho \\ \leq \|\Delta_{7(n)}(t)\|\left\{1+\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_7\right)+\left(\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_7^t\right)\right\}, \\ I_r(t)-I_r(0)-\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_8(t,I_r(t))-\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_8(\rho,I_r(\rho))d\rho \\ \leq \|\Delta_{8(n)}(t)\|\left\{\left(\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_8\right)+1+\left(\frac{2\psi_8^t\nu}{\mathscr{M}(\nu)(2-\nu)}\right)\right\}. \end{cases} \tag{42}$$

Now attempting to take $\lim_{n\to\infty}$ in the Equation (42), we get $\|\Delta_{i(n)}\|\to 0$, $i=1,2,\cdots,8$. Thus, we get

$$\begin{cases} S_h(t)=S_h(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_1(t,S_h(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_1(\rho,S_h(\rho))d\rho, \\ E_h(t)=E_h(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_2(t,E_h(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_2(\rho,E_h(\rho))d\rho, \\ I_h(t)=I_h(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_3(t,I_h(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_3(\rho,I_h(\rho))d\rho, \\ Q_h(t)=Q_h(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_4(t,Q_h(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_4(\rho,Q_h(\rho))d\rho, \\ R_h(t)=R_h(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_5(t,R_h(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_5(\rho,R_h(\rho))d\rho, \\ S_r(t)=S_r(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_6(t,S_r(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_6(\rho,S_r(\rho))d\rho, \\ E_r(t)=E_r(0)+\frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\phi_7(t,E_r(t))+\frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\int_0^t\phi_7(\rho,E_r(\rho))d\rho, \\ I_r(t)=I_r(0)+\frac{2(1-\nu)\phi_8(t,I_r(t))}{\mathscr{M}(\nu)(-\nu+2)}+\frac{2\nu}{\mathscr{M}(\nu)(-\nu+2)}\int_0^t\phi_8(\rho,I_r(\rho))d\rho. \end{cases} \tag{43}$$

Similarly, as limit $\lim_{n\to\infty}$, we get $\|\Delta_{i(n)}(t)\|\to 0$, $i=1,2,\cdots,8$. As a result, there are work around to (6) like (43).

3.2.2 Solution for the Proposed System's Uniqueness

Theorem 3. *The system mentioned in (6) has a clear solution.*

Proof. Now we suppose there is another solution of the system (6), say $S_h^\star$, $E_h^\star$, $I_h^\star$, $Q_h^\star$, $R_h^\star$, $S_r^\star$, $E_r^\star$ and $I_r^\star$, then we get

$$\begin{cases} S_h(t) - S_h^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_1(t, S_h(t)) - \phi_1(t, S_h^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_1(\rho, S_h(\rho)) - \phi_1(\rho, S_h^\star(\rho)) \right\} d\rho, \\ E_h(t) - E_h^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_2(t, E_h(t)) - \phi_2(t, E_h^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_2(\rho, E_h(\rho)) - \phi_2(\rho, E_h^\star(\rho)) \right\} d\rho, \\ I_h(t) - I_h^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_3(t, I_h(t)) - \phi_3(t, I_h^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_3(\rho, I_h(\rho)) - \phi_3(\rho, I_h^\star(\rho)) \right\} d\rho, \\ Q_h(t) - Q_h^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_4(t, Q_h(t)) - \phi_4(t, Q_h^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_4(\rho, Q_h(\rho)) - \phi_4(\rho, Q_h^\star(\rho)) \right\} d\rho, \\ R_h(t) - R_h^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_5(t, R_h(t)) - \phi_5(t, R_h^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_5(\rho, R_h(\rho)) - \phi_5(\rho, R_h^\star(\rho)) \right\} d\rho, \\ S_r(t) - S_r^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_6(t, S_r(t)) - \phi_6(t, S_r^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_6(\rho, S_r(\rho)) - \phi_6(\rho, S_r^\star(\rho)) \right\} d\rho, \\ E_r(t) - E_r^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_7(t, E_r(t)) - \phi_7(t, E_r^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_7(\rho, E_r(\rho)) - \phi_7(\rho, E_r^\star(\rho)) \right\} d\rho, \\ I_r(t) - I_r^\star(0) = \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \left\{ \phi_8(t, I_r(t)) - \phi_8(t, I_r^\star(t)) \right\} \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \left\{ \phi_8(\rho, I_r(\rho)) - \phi_8(\rho, I_r^\star(\rho)) \right\} d\rho. \end{cases} \tag{44}$$

Again by using norm on (44), we obtain

$$\begin{cases} \|S_h(t) - S_h^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_1(t, S_h(t)) - \phi_1(t, S_h^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_1(\rho, S_h(\rho)) - \phi_1(\rho, S_h^\star(\rho))\| d\rho, \\ \|E_h(t) - E_h^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_2(t, E_h(t)) - \phi_2(t, E_h^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_2(\rho, E_h(\rho)) - \phi_2(\rho, E_h^\star(\rho))\| d\rho, \\ \|I_h(t) - I_h^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_3(t, I_h(t)) - \phi_3(t, I_h^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_3(\rho, I_h(\rho)) - \phi_3(\rho, I_h^\star(\rho))\| d\rho, \\ \|Q_h(t) - Q_h^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_4(t, Q_h(t)) - \phi_4(t, Q_h^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_4(\rho, Q_h(\rho)) - \phi_4(\rho, Q_h^\star(\rho))\| d\rho, \\ \|R_h(t) - R_h^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_5(t, R_h(t)) - \phi_5(t, R_h^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_5(\rho, R_h(\rho)) - \phi_5(\rho, R_h^\star(\rho))\| d\rho, \\ \|S_r(t) - S_r^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_6(t, S_r(t)) - \phi_6(t, S_r^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_6(\rho, S_r(\rho)) - \phi_6(\rho, S_r^\star(\rho))\| d\rho, \\ \|E_r(t) - E_r^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_7(t, E_r(t)) - \phi_7(t, E_r^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_7(\rho, E_r(\rho)) - \phi_7(\rho, E_r^\star(\rho))\| d\rho, \\ \|I_r(t) - I_r^\star(0)\| \leq \frac{2(1-v)}{(2-v).\mathscr{M}(v)} \|\phi_8(t, I_r(t)) - \phi_8(t, I_r^\star(t))\| \\ + \frac{2v}{(2-v).\mathscr{M}(v)} \int_0^t \|\phi_8(\rho, I_r(\rho)) - \phi_8(\rho, I_r^\star(\rho))\| d\rho. \end{cases} \tag{45}$$

Taking into consideration above Theorems 1 and 2, the outcomes attained are

$$\begin{cases} \|S_h(t)-S_h^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_1\|S_h(t)-S_h^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_1^t\|S_h(\rho)-S_h^\star(\rho)\|, \\ \|E_h(t)-E_h^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_2\|E_h(t)-E_h^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_2^t\|E_h(\rho)-E_h^\star(\rho)\|, \\ \|I_h(t)-I_h^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_3\|I_h(t)-I_h^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_3^t\|I_h(\rho)-I_h^\star(\rho)\|, \\ \|Q_h(t)-Q_h^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_4\|Q_h(t)-Q_h^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_4^t\|Q_h(\rho)-Q_h^\star(\rho)\|, \\ \|R_h(t)-R_h^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_5\|R_h(t)-R_h^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_5^t\|R_h(\rho)-R_h^\star(\rho)\|, \\ \|S_r(t)-S_r^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_6\|S_r(t)-S_r^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_6^t\|S_r(\rho)-S_r^\star(\rho)\|, \\ \|E_r(t)-E_r^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_7\|E_r(t)-E_r^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_7^t\|E_r(\rho)-E_r^\star(\rho)\|, \\ \|I_r(t)-I_r^\star(0)\| \le \frac{2(1-\nu)}{(2-\nu).\mathscr{M}(\nu)}\psi_8\|I_r(t)-I_r^\star(t)\| + \frac{2\nu}{(2-\nu).\mathscr{M}(\nu)}\psi_8^t\|I_r(\rho)-I_r^\star(\rho)\|. \end{cases} \tag{46}$$

The solution techniques in (46) satisfy the following inequality.

$$\begin{cases} \|S_h(t)-S_h^\star(t)\|\left\{-\frac{2\psi_1}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|E_h(t)-E_h^\star(t)\|\left\{-\frac{2\psi_2}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|I_h(t)-I_h^\star(t)\|\left\{-\frac{2\psi_3}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|Q_h(t)-Q_h^\star(t)\|\left\{-\frac{2\psi_4}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|S_r(t)-S_r^\star(t)\|\left\{-\frac{2\psi_5}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|E_r(t)-E_r^\star(t)\|\left\{-\frac{2\psi_6}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0, \\ \|I_r(t)-I_r^\star(t)\|\left\{-\frac{2\psi_7}{(2-\nu).\mathscr{M}(\nu)}(-t\nu+1-\nu)+1\right\} \le 0. \end{cases} \tag{47}$$

The final equation results in the statement that

$$\begin{array}{l} S_h(t)=S_h^\star(t),\ E_h(t)=E_h^\star(t),\ I_h(t)=I_h^\star(t),\ Q_h(t)=Q_h^\star(t), \\ R_h(t)=R_h^\star(t),\ S_r(t)=S_r^\star(t),\ E_r(t)=E_r^\star(t),\ I_r(t)=I_r^\star(t). \end{array} \tag{48}$$

4 Stability Analysis of the Proposed System

The fractional Monkeypox model, the iterative Laplace transform method and stability standards we created for numerical solutions are covered in this section.

4.1 Laplace Transform Iterative Technique

Consider the Monkeypox model (6) combined with the starting conditions (7). We are able to analyze the system (6) using the Laplace transform.

$$\begin{cases} \frac{\varpi\mathscr{L}(S_h(t))-S_h(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - \mu_h S_h + \varphi Q_h\right), \\ \frac{\varpi\mathscr{L}(E_h(t))-E_h(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h\right), \\ \frac{\varpi\mathscr{L}(I_h(t))-I_h(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(E_h\alpha_1 - (\delta_h+\mu_h+\gamma)I_h\right), \\ \frac{\varpi\mathscr{L}(Q_h(t))-Q_h(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(E_h\alpha_2 - (\varphi+\delta_h+\mu_h+\tau)Q_h\right), \\ \frac{\varpi\mathscr{L}(R_h(t))-R_h(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\gamma I_h + \tau Q_h - \mu_h R_h\right), \\ \frac{\varpi\mathscr{L}(S_r(t))-S_r(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\theta_r - \frac{\beta_3 S_r I_r}{N_r} - \mu_r S_r\right), \\ \frac{\varpi\mathscr{L}(E_r(t))-E_r(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\frac{\beta_3 S_r I_r}{N_r} - (\mu_r+\alpha_3)E_r\right), \\ \frac{\varpi\mathscr{L}(I_r(t))-I_r(0)}{\varpi+\nu(1-\varpi)} = \mathscr{L}\left(\alpha_3 E_r - (\mu_r+\delta_r)I_r\right). \end{cases} \tag{49}$$

Rearranging, we get

$$\begin{cases} \mathscr{L}(S_h(t)) = \frac{S_h(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - \mu_h S_h + \varphi Q_h\right), \\ \mathscr{L}(E_h(t)) = \frac{E_h(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h\right), \\ \mathscr{L}(I_h(t)) = \frac{I_h(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(E_h\alpha_1 - (\delta_h+\mu_h+\gamma)I_h\right), \\ \mathscr{L}(Q_h(t)) = \frac{Q_h(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(E_h\alpha_2 - (\varphi+\delta_h+\tau+\mu_h)Q_h\right), \\ \mathscr{L}(R_h(t)) = \frac{R_h(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\gamma I_h + \tau Q_h - \mu_h R_h\right), \\ \mathscr{L}(S_r(t)) = \frac{S_r(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r I_r}{N_r} - \mu_r S_r\right), \\ \mathscr{L}(E_r(t)) = \frac{E_r(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r I_r}{N_r} - (\mu_r+\alpha_3)E_r\right), \\ \mathscr{L}(I_r(t)) = \frac{I_r(0)}{\varpi} + \frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(-(\mu_r+\delta_r)I_r + \alpha_3 E_r\right). \end{cases} \tag{50}$$

Moreover, equation (50)'s inverse Laplace transform results in

$$\begin{cases} S_h(t) = S_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - \mu_h S_h + \varphi Q_h\right)\right], \\ E_h(t) = E_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r+\beta_2 I_h)S_h}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h\right)\right], \\ I_h(t) = I_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_1 E_h - (\mu_h+\delta_h+\gamma)I_h\right)\right], \\ Q_h(t) = Q_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_2 E_h - (\varphi+\tau+\delta_h+\mu_h)Q_h\right)\right], \\ R_h(t) = R_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\gamma I_h + \tau Q_h - \mu_h R_h\right)\right], \\ S_r(t) = S_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r I_r}{N_r} - \mu_r S_r\right)\right], \\ E_r(t) = E_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r I_r}{N_r} - (\mu_r+\alpha_3)E_r\right)\right], \\ I_r(t) = I_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_3 E_r - (\mu_r+\delta_r)I_r\right)\right]. \end{cases} \tag{51}$$

The infinite series roots are achieved by this technique given as

$$\begin{aligned} S_h &= \sum_{n=0}^{\infty} S_h^n, \;\; E_h = \sum_{n=0}^{\infty} E_h^n, \;\; I_h = \sum_{n=0}^{\infty} I_h^n, \;\; Q_h = \sum_{n=0}^{\infty} Q_h^n, \\ R_h &= \sum_{n=0}^{\infty} R_h^n, \;\; S_r = \sum_{n=0}^{\infty} S_r^n, \;\; E_r = \sum_{n=0}^{\infty} E_r^n, \;\; I_r = \sum_{n=0}^{\infty} I_r^n. \end{aligned} \tag{52}$$

The non linearity S_hE_h, S_hI_h, S_hQ_h, S_hR_h, S_hS_r, S_hE_r and S_hI_r can be written as

$$\begin{aligned} S_hE_h &= \sum_{n=0}^{\infty} B^n, \;\; S_hI_h = \sum_{n=0}^{\infty} C^n, \;\; S_hQ_h = \sum_{n=0}^{\infty} E^n, \;\; S_hR_h = \sum_{n=0}^{\infty} F^n, \\ S_hS_r &= \sum_{n=0}^{\infty} G^n, \;\; S_hE_r = \sum_{n=0}^{\infty} H^n, \;\; S_hI_r = \sum_{n=0}^{\infty} I^n. \end{aligned} \tag{53}$$

where B^n, C^n, E^n, F^n, G^n, H^n and I^n are decomposed as follows

$$B^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} E_h^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} E_h^n$$

$$C^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} I_h^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} I_h^n$$

$$E^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} Q_h^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} Q_h^n$$

$$F^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} R_h^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} R_h^n$$

$$G^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} S_r^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} S_r^n$$

$$H^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} E_r^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} E_r^n$$

$$I^n = \sum_{i=0}^{n} S_h^n \sum_{i=0}^{n} I_r^n - \sum_{i=0}^{n-1} S_h^n \sum_{i=0}^{n-1} I_r^n$$

The following recursive formula is then obtained by using starting conditions.

$$\begin{cases} S_h^{n+1}(t) = S_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - \mu_h S_h^n + \varphi Q_h^n\right)\right], \\ E_h^{n+1}(t) = E_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h^n\right)\right], \\ I_h^{n+1}(t) = I_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_1 E_h^n - (\mu_h+\delta_h+\gamma)I_h^n\right)\right], \\ Q_h^{n+1}(t) = Q_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_2 E_h^n - (\varphi+\tau+\delta_h+\mu_h)Q_h^n\right)\right], \\ R_h^{n+1}(t) = R_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\gamma I_h^n + \tau Q_h^n - \mu_h R_h^n\right)\right], \\ S_r^{n+1}(t) = S_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r^n I_r^n}{N_r} - \mu_r S_r^n\right)\right], \\ E_r^{n+1}(t) = E_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r^n I_r^n}{N_r} - (\mu_r+\alpha_3)E_r^n\right)\right], \\ I_r^{n+1}(t) = I_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_3 E_r^n - (\mu_r+\delta_r)I_r^n\right)\right]. \end{cases} \tag{54}$$

5 Analysis of the Proposed System's Iteration Method

Assume of $\mathbb{U}$ as the self map of $\mathbb{Q}$ and the Banach space $(\mathbb{Q}, \|\cdot\|)$. Also represented by the notation $W_{n+1} = g(J, W_n)$ is a precise repeating process. Let's say J($\mathbb{U}$) denotes a fixed point established on $\mathbb{U}$. In addition, for W_n to converge to the point $i \in J(\mathbb{U})$, $\mathbb{U}$ must also provide at least one element. Consider that $\{q_n \in \mathbb{Q}\}$ and construct $K_n = \|q_{n+1} - g(\mathbb{U}, q_n)\|$. If $\lim_{n\to\infty} K^n = 0$ shows that $\lim_{n\to\infty} q^n = i$, the iteration approach $q_{n+1} = g(\mathbb{U}, q_n)$ is said to be $\mathbb{U}$ stable. The sequence q_n, in contrast, has an upper limit, we claim. This iteration is referred to as $Picard's$ iteration and is $\mathbb{U}$ stable if all of the conditions for $q_{n+1} = Yq_n$ are satisfied.

Theorem 4. *The Banach space* $(\mathbb{Q}, \|\cdot\|)$ *and described* $\mathbb{U}$ *as an acceptable self map on* $\mathbb{Q}$.

$$\|\mathbb{U}_x - \mathbb{U}_y\| \le B\|x - Y_x\| + b\|x - y\| \tag{55}$$

$\forall x, y \in \mathbb{Q}$, $0 \le B, 0 \le b < 1$.

Proof. Suppose that $\mathbb{U}$ is stable to Picard. Think about the connection between equations (54) and (6).

$$\begin{cases} S_h^{n+1}(t) = S_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - \mu_h S_h^n + \varphi Q_h^n\right)\right], \\ E_h^{n+1}(t) = E_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h^n\right)\right], \\ I_h^{n+1}(t) = I_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_1 E_h^n - (\mu_h+\delta_h+\gamma)I_h^n\right)\right], \\ Q_h^{n+1}(t) = Q_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_2 E_h^n - (\varphi+\tau+\delta_h+\mu_h)Q_h^n\right)\right], \\ R_h^{n+1}(t) = R_h^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\gamma I_h^n + \tau Q_h^n - \mu_h R_h^n\right)\right], \\ S_r^{n+1}(t) = S_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r^n I_r^n}{N_r} - \mu_r S_r^n\right)\right], \\ E_r^{n+1}(t) = E_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r^n I_r^n}{N_r} - (\mu_r+\alpha_3)E_r^n\right)\right], \\ I_r^{n+1}(t) = I_r^0 + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_3 E_r^n - (\mu_r+\delta_r)I_r^n\right)\right]. \end{cases} \tag{56}$$

where the fractional Lagrange multiplier is $\frac{\varpi+\nu(1-\varpi)}{\varpi}$.

Theorem 5. *Now self map $\mathbb{U}$ is defined as*

$$\begin{cases} \mathbb{U}(S_h^n(t)) = S_h^{n+1}(t) = S_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - \mu_h S_h^n + \varphi Q_h^n\right)\right], \\ \mathbb{U}(E_h^n(t)) = E_h^{n+1}(t) = E_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h^n\right)\right], \\ \mathbb{U}(I_h^n(t)) = I_h^{n+1}(t) = I_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_1 E_h^n - (\mu_h+\delta_h+\gamma)I_h^n\right)\right], \\ \mathbb{U}(Q_h^n(t)) = Q_h^{n+1}(t) = Q_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_2 E_h^n - (\varphi+\tau+\delta_h+\mu_h)Q_h^n\right)\right], \\ \mathbb{U}(R_h^n(t)) = R_h^{n+1}(t) = R_h(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\gamma I_h^n + \tau Q_h^n - \mu_h R_h^n\right)\right], \\ \mathbb{U}(S_r^n(t)) = S_r^{n+1}(t) = S_r(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r^n I_r^n}{N_r} - \mu_r S_r^n\right)\right], \\ \mathbb{U}(E_r^n(t)) = E_r^{n+1}(t) = E_r(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r^n I_r^n}{N_r} - (\mu_r+\alpha_3)E_r^n\right)\right], \\ \mathbb{U}(I_r^n(t)) = I_r^{n+1}(t) = I_r(0) + \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_3 E_r^n - (\mu_r+\delta_r)I_r^n\right)\right]. \end{cases} \tag{57}$$

is $\mathbb{U}$ stable in $L^1(a,b)$ if

$$\begin{cases} \left(\frac{\beta_1\Xi_3}{N_h}h_1(\nu) + \frac{\beta_1\Xi_1}{N_h}h_2(\nu) + \frac{\beta_2\Xi_4}{N_h}h_3(\nu) + \frac{\beta_2\Xi_1}{N_h}h_4(\nu) + \mu_h h_5(\nu) + \varphi h_6(\nu)\right) < 1, \\ \left(\frac{\beta_1\Xi_3}{N_h}h_7(\nu) + \frac{\beta_1\Xi_1}{N_h}h_8(\nu) + \frac{\beta_2\Xi_4}{N_h}h_9(\nu) + \frac{\beta_2\Xi_1}{N_h}h_{10}(\nu) + (\alpha_1+\alpha_2+\mu_h)h_{11}(\nu)\right) < 1, \\ \left(\alpha_1 h_{12}(\nu) + (\mu_h+\delta_h+\gamma)h_{13}(\nu)\right) < 1, \\ \left(\alpha_2 h_{14}(\nu) + (\varphi+\tau+\delta_h+\mu_h)h_{15}(\nu)\right) < 1, \\ \left(\gamma h_{16}(\nu) + \tau h_{17}(\nu) + \mu_h h_{18}(\nu)\right) < 1, \\ \left(\frac{\beta_3\Xi_2}{N_r}h_{19}(\nu) + \frac{\beta_3\Xi_5}{N_r}h_{20}(\nu) + \mu_r h_{21}(\nu)\right) < 1, \\ \left(\frac{\beta_3\Xi_2}{N_r}h_{22}(\nu) + \frac{\beta_3\Xi_5}{N_r}h_{23}(\nu) + (\mu_r+\alpha_3)h_{24}(\nu)\right) < 1, \\ \left(\alpha_3 h_{25}(\nu) + (\mu_r+\delta_r)h_{26}(\nu)\right) < 1. \end{cases} \tag{58}$$

Proof. Here, we will show that $\mathbb{U}$ has a fixed point. Hence, for all $(m,n) \in N \times N$ we evaluate the followings.

$$\begin{cases} \mathbb{U}(S_h^n) - \mathbb{U}(S_h^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - \mu_h S_h^n + \varphi Q_h^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\theta_h - \frac{(\beta_1 I_r^m + \beta_2 I_h^m)S_h^m}{N_h} - \mu_h S_h^m + \varphi Q_h^m\right)\right], \\ \mathbb{U}(E_h^n) - \mathbb{U}(E_h^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r^n + \beta_2 I_h^n)S_h^n}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\frac{(\beta_1 I_r^m + \beta_2 I_h^m)S_h^m}{N_h} - (\alpha_1+\alpha_2+\mu_h)E_h^m\right)\right], \\ \mathbb{U}(I_h^n) - \mathbb{U}(I_h^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_1 E_h^n - (\mu_h+\delta_h+\gamma)I_h^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\alpha_1 E_h^m - (\mu_h+\delta_h+\gamma)I_h^m\right)\right], \\ \mathbb{U}(Q_h^n) - \mathbb{U}(Q_h^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_2 E_h^n - (\varphi+\tau+\delta_h+\mu_h)Q_h^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\alpha_2 E_h^m - (\varphi+\tau+\delta_h+\mu_h)Q_h^m\right)\right], \\ \mathbb{U}(R_h^n) - \mathbb{U}(R_h^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\gamma I_h^n + \tau Q_h^n - \mu_h R_h^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\gamma I_h^m + \tau Q_h^m - \mu_h R_h^m\right)\right], \\ \mathbb{U}(S_r^n) - \mathbb{U}(S_r^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r^n I_r^n}{N_r} - \mu_r S_r^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\theta_r - \frac{\beta_3 S_r^m I_r^m}{N_r} - \mu_r S_r^m\right)\right], \\ \mathbb{U}(E_r^n) - \mathbb{U}(E_r^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r^n I_r^n}{N_r} - (\mu_r+\alpha_3)E_r^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\frac{\beta_3 S_r^m I_r^m}{N_r} - (\mu_r+\alpha_3)E_r^m\right)\right], \\ \mathbb{U}(I_r^n) - \mathbb{U}(I_r^m) = \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\alpha_3 E_r^n - (\mu_r+\delta_r)I_r^n\right)\right] \\ -\mathscr{L}^{-1}\left[\frac{\varpi+\nu(1-\varpi)}{\varpi}\mathscr{L}\left(\alpha_3 E_r^m - (\mu_r+\delta_r)I_r^m\right)\right]. \end{cases} \tag{59}$$

Without losing generality, and assuming the norm from (59), we obtain

$$\begin{cases} \left\|\mathbb{U}(S_h^n)-\mathbb{U}(S_h^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\theta_h-\frac{(\beta_1 I_r^n+\beta_2 I_h^n)S_h^n}{N_h}-\mu_h S_h^n+\varphi Q_h^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\theta_h-\frac{(\beta_1 I_r^m+\beta_2 I_h^m)S_h^m}{N_h}-\mu_h S_h^m+\varphi Q_h^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(E_h^n)-\mathbb{U}(E_h^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\frac{(\beta_1 I_r^n+\beta_2 I_h^n)S_h^n}{N_h}-(\alpha_1+\alpha_2+\mu_h)E_h^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\frac{(\beta_1 I_r^m+\beta_2 I_h^m)S_h^m}{N_h}-(\alpha_1+\alpha_2+\mu_h)E_h^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(I_h^n)-\mathbb{U}(I_h^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\alpha_1 E_h^n-(\mu_h+\delta_h+\gamma)I_h^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\alpha_1 E_h^m-(\mu_h+\delta_h+\gamma)I_h^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(Q_h^n)-\mathbb{U}(Q_h^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\alpha_2 E_h^n-(\varphi+\tau+\delta_h+\mu_h)Q_h^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\alpha_2 E_h^m-(\varphi+\tau+\delta_h+\mu_h)Q_h^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(R_h^n)-\mathbb{U}(R_h^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\gamma I_h^n+\tau Q_h^n-\mu_h R_h^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\gamma I_h^m+\tau Q_h^m-\mu_h R_h^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(S_r^n)-\mathbb{U}(S_r^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\theta_r-\frac{\beta_3 S_r^n I_r^n}{N_r}-\mu_r S_r^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\theta_r-\frac{\beta_3 S_r^m I_r^m}{N_r}-\mu_r S_r^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(E_r^n)-\mathbb{U}(E_r^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\frac{\beta_3 S_r^n I_r^n}{N_r}-(\mu_r+\alpha_3)E_r^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\frac{\beta_3 S_r^m I_r^m}{N_r}-(\mu_r+\alpha_3)E_r^m\Big)\Big]\Big\|, \\ \left\|\mathbb{U}(I_r^n)-\mathbb{U}(I_r^m)\right\| = \Big\|\mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\alpha_3 E_r^n-(\mu_r+\delta_r)I_r^n\Big)\Big] \\ -\mathscr{L}^{-1}\Big[\frac{\varpi+v(1-\varpi)}{\varpi}\mathscr{L}\Big(\alpha_3 E_r^m-(\mu_r+\delta_r)I_r^m\Big)\Big]\Big\|. \end{cases} \tag{60}$$

Triangular inequality combined with further simplification (60) results in

$$\begin{cases} \left\|\mathbb{U}(S_h^n(t))-\mathbb{U}(S_h^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|-\frac{\beta_1 I_r^n}{N_h}(S_h^n-S_h^m)\Big\|+\Big\|-\frac{\beta_1 S_h^m}{N_h}(I_r^n-I_r^m)\Big\| \\ +\Big\|-\frac{\beta_2 I_h^n}{N_h}(S_h^n-S_h^m)\Big\|+\Big\|-\frac{\beta_2 S_h^m}{N_h}(I_h^n-I_h^m)\Big\|+\Big\|-\mu_h(S_h^n-S_h^m)\Big\|+\Big\|\varphi(Q_h^n-Q_h^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(E_h^n(t))-\mathbb{U}(E_h^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\frac{\beta_1 I_r^n}{N_h}(S_h^n-S_h^m)\Big\|+\Big\|-\frac{\beta_1 S_h^m}{N_h}(I_r^n-I_r^m)\Big\| \\ +\Big\|\frac{\beta_2 I_h^n}{N_h}(S_h^n-S_h^m)\Big\|+\Big\|-\frac{\beta_2 S_h^m}{N_h}(I_h^n-I_h^m)\Big\|+\Big\|-(\alpha_1+\alpha_2+\mu_h)(E_h^n-E_h^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(I_h^n(t))-\mathbb{U}(I_h^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\alpha_1(E_h^n-E_h^m)\Big\|+\Big\|-(\mu_h+\delta_h+\gamma)(I_h^n-I_h^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(Q_h^n(t))-\mathbb{U}(Q_h^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\alpha_2(E_h^n-E_h^m)\Big\| \\ +\Big\|-(\varphi+\tau+\delta_h+\mu_h)(Q_h^n-Q_h^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(R_h^n(t))-\mathbb{U}(R_h^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\gamma(I_h^n-I_h^m)\Big\|+\Big\|\tau(Q_h^n-Q_h^m\Big\| \\ +\Big\|-\mu_h(R_h^n-R_h^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(S_r^n(t))-\mathbb{U}(S_r^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|-\frac{\beta_3 S_r^n}{N_r}(I_r^n-I_r^m)\Big\|+\Big\|-\frac{\beta_3 I_r^m}{N_r}(S_r^n-S_r^m)\Big\| \\ +\Big\|-\mu_r(S_r^n-S_r^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(E_r^n(t))-\mathbb{U}(E_r^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\frac{\beta_3 S_r^n}{N_r}(I_r^n-I_r^m)\Big\|+\Big\|\frac{\beta_3 I_r^m}{N_r}(S_r^n-S_r^m)\Big\| \\ +\Big\|-(\mu_r+\alpha_3)(E_r^n-E_r^m)\Big\|\Big)\Big], \\ \left\|\mathbb{U}(I_r^n(t))-\mathbb{U}(I_r^m(t))\right\| \le \mathscr{L}^{-1}\Big[\frac{(1-\varpi)v+\varpi}{\varpi}\mathscr{L}\Big(\Big\|\alpha_3(E_r^n-E_r^m\Big\|+\Big\|-(\mu_r+\delta_r)(I_r^n-I_r^m)\Big\|\Big)\Big]. \end{cases} \tag{61}$$

Given that the discovered solutions take on a comparable role, we assume that

$$\begin{aligned} &\|S_h^n(t)-S_h^m(t)\| = \|E_h^n(t)-E_h^m(t)\| = \|I_h^n(t)-I_h^m(t)\| = \|Q_h^n(t)-Q_h^m(t)\| \\ &= \|R_h^n(t)-R_h^m(t)\| = \|S_r^n(t)-S_r^m(t)\| = \|E_r^n(t)-E_r^m(t)\| = \|I_r^n(t)-I_r^m(t)\|. \end{aligned} \tag{62}$$

By changing this in (62), we get the connection shown below.

$$
\begin{cases}
\left\|\mathbb{U}(S_h^n(t))-\mathbb{U}(S_h^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|-\frac{\beta_1 I_r^n}{N_h}(S_h^n-S_h^m)\right\|+\left\|-\frac{\beta_1 S_h^m}{N_h}(S_h^n-S_h^m)\right\|\right.\right.\\
\left.\left.+\left\|-\frac{\beta_2 I_h^n}{N_h}(S_h^n-S_h^m)\right\|+\left\|-\frac{\beta_2 S_h^m}{N_h}(S_h^n-S_h^m)\right\|+\left\|-\mu_h(S_h^n-S_h^m)\right\|+\left\|\varphi(S_h^n-S_h^m)\right\|\right)\right],\\
\left\|\mathbb{U}(E_h^n(t))-\mathbb{U}(E_h^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\frac{\beta_1 I_r^n}{N_h}(E_h^n-E_h^m)\right\|+\left\|-\frac{\beta_1 S_h^m}{N_h}(E_h^n-E_h^m)\right\|\right.\right.\\
\left.\left.+\left\|\frac{\beta_2 I_h^n}{N_h}(E_h^n-E_h^m)\right\|+\left\|-\frac{\beta_2 S_h^m}{N_h}(E_h^n-E_h^m)\right\|+\left\|-(\alpha_1+\alpha_2+\mu_h)(E_h^n-E_h^m)\right\|\right)\right],\\
\left\|\mathbb{U}(I_h^n(t))-\mathbb{U}(I_h^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\alpha_1(I_h^n-I_h^m)\right\|+\left\|-(\mu_h+\delta_h+\gamma)(I_h^n-I_h^m)\right\|\right)\right],\\
\left\|\mathbb{U}(Q_h^n(t))-\mathbb{U}(Q_h^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\alpha_2(Q_h^n-Q_h^m)\right\|\right.\right.\\
\left.\left.+\left\|-(\varphi+\tau+\delta_h+\mu_h)(Q_h^n-Q_h^m)\right\|\right)\right],\\
\left\|\mathbb{U}(R_h^n(t))-\mathbb{U}(R_h^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\gamma(R_h^n-R_h^m)\right\|+\left\|\tau(R_h^n-R_h^m\right\|\right.\right.\\
\left.\left.+\left\|-\mu_h(R_h^n-R_h^m)\right\|\right)\right],\\
\left\|\mathbb{U}(S_r^n(t))-\mathbb{U}(S_r^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|-\frac{\beta_3 S_r^n}{N_r}(S_r^n-S_r^m)\right\|+\left\|-\frac{\beta_3 I_r^m}{N_r}(S_r^n-S_r^m)\right\|\right.\right.\\
\left.\left.+\left\|-\mu_r(S_r^n-S_r^m)\right\|\right)\right],\\
\left\|\mathbb{U}(E_r^n(t))-\mathbb{U}(E_r^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\frac{\beta_3 S_r^n}{N_r}(E_r^n-E_r^m)\right\|+\left\|\frac{\beta_3 I_r^m}{N_r}(E_r^n-E_r^m)\right\|\right.\right.\\
\left.\left.+\left\|-(\mu_r+\alpha_3)(E_r^n-E_r^m)\right\|\right)\right],\\
\left\|\mathbb{U}(I_r^n(t))-\mathbb{U}(I_r^m(t))\right\| \leq \mathscr{L}^{-1}\left[\frac{(1-\varpi)\nu+\varpi}{\varpi}\mathscr{L}\left(\left\|\alpha_3(I_r^n-I_r^m\right\|+\left\|-(\mu_r+\delta_r)(I_r^n-I_r^m)\right\|\right)\right].
\end{cases}
\tag{63}
$$

Also S_h^m, S_r^n, I_r^n, I_h^n and I_r^m are convergent sequence, We can acquire three different positive constants since they are bounded Ξ_1, Ξ_2, Ξ_3, Ξ_4 and Ξ_5 for all t such as

$$
\|S_h^m\|<\Xi_1,\ \ \|S_r^n\|<\Xi_2,\ \ \|I_r^n\|<\Xi_3, \|I_h^n\|<\Xi_4,\ \ \|I_r^m\|<\Xi_5. \tag{64}
$$

Next consider Equations (63) and (64), we get

$$
\begin{cases}
\left\|\mathbb{U}(S_h^n(t))-\mathbb{U}(S_h^m(t))\right\| \leq \left(\frac{\beta_1\Xi_3}{N_h}h_1(\nu)+\frac{\beta_1\Xi_1}{N_h}h_2(\nu)+\frac{\beta_2\Xi_4}{N_h}h_3(\nu)+\frac{\beta_2\Xi_1}{N_h}h_4(\nu)\right.\\
\left.+\mu_h h_5(\nu)+\varphi h_6(\nu)\right),\\
\left\|\mathbb{U}(E_h^n(t))-\mathbb{U}(E_h^m(t))\right\| \leq \left(\frac{\beta_1\Xi_3}{N_h}h_7(\nu)+\frac{\beta_1\Xi_1}{N_h}h_8(\nu)+\frac{\beta_2\Xi_4}{N_h}h_9(\nu)+\frac{\beta_2\Xi_1}{N_h}h_{10}(\nu)\right.\\
\left.+(\alpha_1+\alpha_2+\mu_h)h_{11}(\nu)\right),\\
\left\|\mathbb{U}(I_h^n(t))-\mathbb{U}(I_h^m(t))\right\| \leq \left(\alpha_1 h_{12}(\nu)+(\mu_h+\delta_h+\gamma)h_{13}(\nu)\right),\\
\left\|\mathbb{U}(Q_h^n(t))-\mathbb{U}(Q_h^m(t))\right\| \leq \left(\alpha_2 h_{14}(\nu)+(\varphi+\tau+\delta_h+\mu_h)h_{15}(\nu)\right),\\
\left\|\mathbb{U}(R_h^n(t))-\mathbb{U}(R_h^m(t))\right\| \leq \left(\gamma h_{16}(\nu)+\tau h_{17}(\nu)+\mu_h h_{18}(\nu)\right),\\
\|\mathbb{U}(S_r^n(t))-\mathbb{U}(S_r^m(t))\| \leq \left(\frac{\beta_3\Xi_2}{N_r}h_{19}(\nu)+\frac{\beta_3\Xi_5}{N_r}h_{20}(\nu)+\mu_r h_{21}(\nu)\right),\\
\|\mathbb{U}(E_r^n(t))-\mathbb{U}(E_r^m(t))\| \leq \left(\frac{\beta_3\Xi_2}{N_r}h_{22}(\nu)+\frac{\beta_3\Xi_5}{N_r}h_{23}(\nu)+(\mu_r+\alpha_3)h_{24}(\nu)\right),\\
\|\mathbb{U}(I_r^n(t))-\mathbb{U}(I_r^m(t))\| \leq \left((\delta_r+\mu_r)h_{26}(\nu)+h_{25}(\nu)\alpha_3\right).
\end{cases}
\tag{65}
$$

where $h_i(\nu)$, $i=1,2,3,\cdots,26$ are functions from $\mathscr{L}^{-1}\left[\mathscr{L}\frac{(1-\varpi)\nu+\varpi}{\varpi}\right]$.

As a result, the mapping Y has a fixed point. We then show that Y satisfies each condition in the aforementioned Theorem 4. Provided that equations (64) and (65) are correct.

$$
\aleph=(0,0,0,0,0,0,0,0,0,0),
$$

$$\aleph = \begin{cases} \left(\frac{\beta_1 \Xi_3}{N_h} h_1(\nu) + \frac{\beta_1 \Xi_1}{N_h} h_2(\nu) + \frac{\beta_2 \Xi_4}{N_h} h_3(\nu) + \frac{\beta_2 \Xi_1}{N_h} h_4(\nu) + \mu_h h_5(\nu) + \varphi h_6(\nu)\right) < 1, \\ \left(\frac{\beta_1 \Xi_3}{N_h} h_7(\nu) + \frac{\beta_1 \Xi_1}{N_h} h_8(\nu) + \frac{\beta_2 \Xi_4}{N_h} h_9(\nu) + \frac{\beta_2 \Xi_1}{N_h} h_{10}(\nu) + (\alpha_1 + \alpha_2 + \mu_h) h_{11}(\nu)\right) < 1, \\ \left(\alpha_1 h_{12}(\nu) + (\mu_h + \delta_h + \gamma) h_{13}(\nu)\right) < 1, \\ \left(\alpha_2 h_{14}(\nu) + (\varphi + \tau + \delta_h + \mu_h) h_{15}(\nu)\right) < 1, \\ \left(\gamma h_{16}(\nu) + \tau h_{17}(\nu) + \mu_h h_{18}(\nu)\right) < 1, \\ \left(\frac{\beta_3 \Xi_2}{N_r} h_{19}(\nu) + \frac{\beta_3 \Xi_5}{N_r} h_{20}(\nu) + \mu_r h_{21}(\nu)\right) < 1, \\ \left(\frac{\beta_3 \Xi_2}{N_r} h_{22}(\nu) + \frac{\beta_3 \Xi_5}{N_r} h_{23}(\nu) \right. \\ \left. + (\mu_r + \alpha_3) h_{24}(\nu)\right) < 1, \\ \left(\alpha_3 h_{25}(\nu) + (\mu_r + \delta_r) h_{26}(\nu)\right) < 1. \end{cases} \tag{66}$$

all the conditions in Theorem 5 are satisfied by $\mathbb{U}$. Therefore, $\mathbb{U}$ is Picard $\mathbb{U}$ stable.

6 Results and discussion

The Caputo Fabrizio Monkeypox model (6) is numerically simulated in the current part for various values of the fractional order $\nu \in (0,1)$. The pertinent physical parameters are listed along with their values. Figures (1-8) display the numerical simulations for the vulnerable individuals category. $S_h(t)$ for the population that exposed humans, see $E_h(t)$ for the population that was infected by humans, see $I_h(t)$. $Q_h(t)$ represents the collective of the separated persons, Using the Iterative Laplace transform method of the fractional Monkeypox model, symptom free values of $R_h(t)$ for the group of recovered humans, $S_r(t)$ for the group of susceptible rodents, $E_r(t)$ for the exposed rodents, and $I_r(t)$ for the infected rodents group have been obtained. This shows that the approach of Iterative Laplace Transform can forecast stated behaviour. The simulations show that the dynamics of the model are influenced by esteem differences. Non-integer order barely affects the Monkeypox model's transmission dynamics.

As shown in Figure 1, the vulnerable human population that is unvaccinated $S_h(t)$ first exhibits some expansion before experiencing a sharp decline at different fractional levels. Figure 2's graph of $I_h(t)$ for the population exposed to humans demonstrates that it grows quickly in the first few years and subsequently dramatically drops at various fractional levels. According to figure 3, the population of $I_h(t)$ for the infected population is declining with time and growing as the equilibrium point converges to integer and noninteger values of ν. Figure 4 demonstrates that for both integer and non-integer values of ν, the population of $Q_h(t)$, which represents the group of solitary monkeypox patients who are asymptomatic, rises with equilibrium point convergence to non zero. Figure 5 demonstrates that for both integer and non integer values of order ν, the population of $R_h(t)$ for the set of people who have recovered rises as the equilibrium point converges to non-zero. Figure 6 demonstrates that for both integer and non-integer values of ν, the population of $S_r(t)$ for the group of sensitive rats with severe symptoms rises as the equilibrium point converges to non-zero. The population of $E_r(t)$ increases for exposed rodent circumstances as the equilibrium point approaches non-zero for both integer and non-integer values of ν, as shown by Figure 7. Figure 8 shows that the population of $I_r(t)$ the infected rodent cases increases as the equilibrium point converges to non zero for both integer and non-integer values of ν. This demonstrates that a certain fractional operator, such as the CF operator, adds to precise forecasts and is less noisy. As an added bonus, Caputo Fabrizio's hybrid features enable him to accurately capture complex patterns and make insightful forecasts.

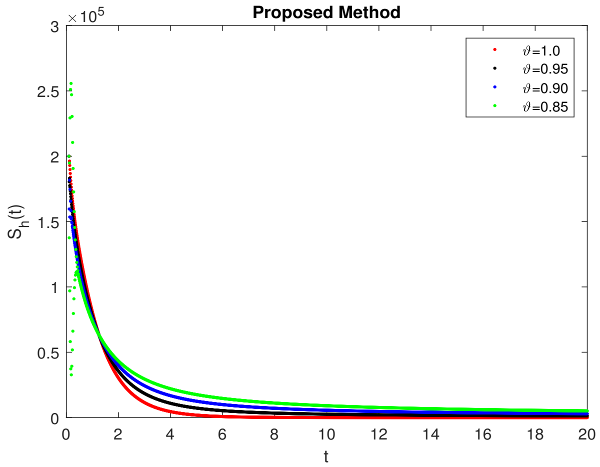

Fig. 1: Simulation $S_h(t)$ population at different fractional order values.

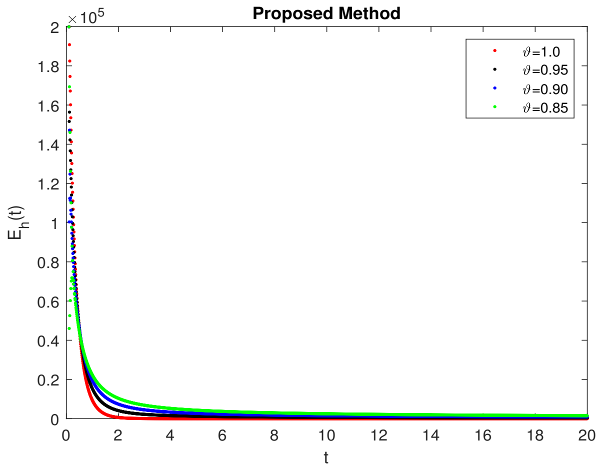

Fig. 2: Simulation $E_h(t)$ population at different fractional order values.

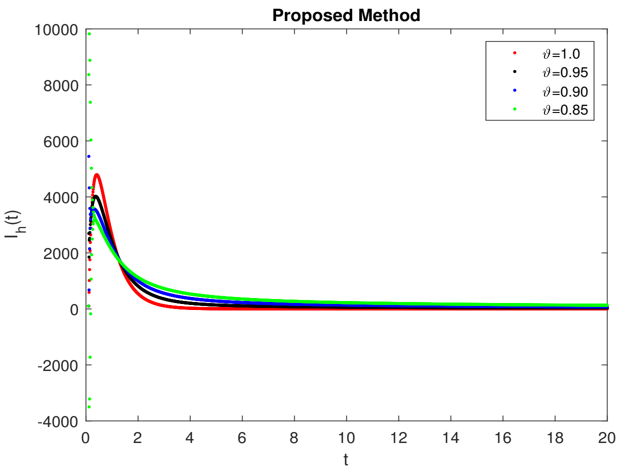

Fig. 3: Simulation $I_h(t)$ population at different fractional order values.

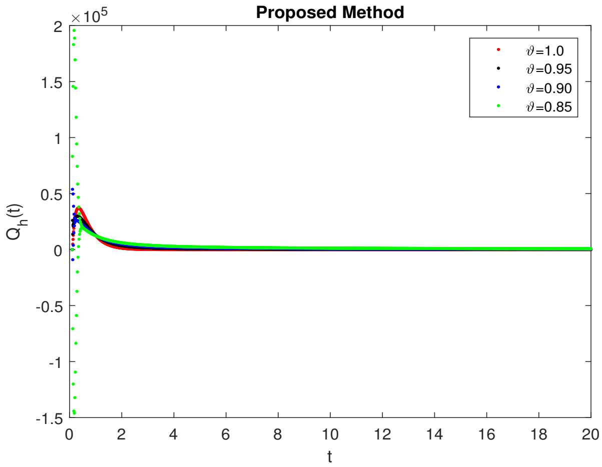

Fig. 4: Simulation $Q_h(t)$ population at different fractional order values.

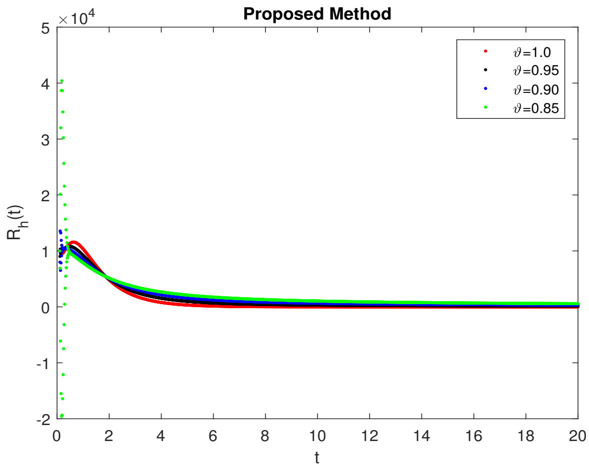

Fig. 5: Simulation $R_h(t)$ population at different fractional order values.

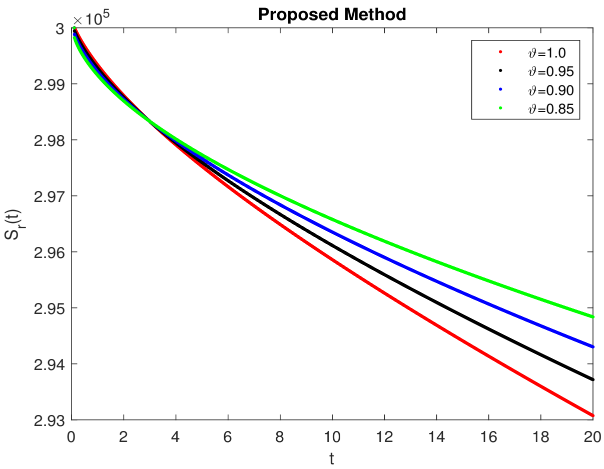

Fig. 6: Simulation $S_r(t)$ population at different fractional order values.

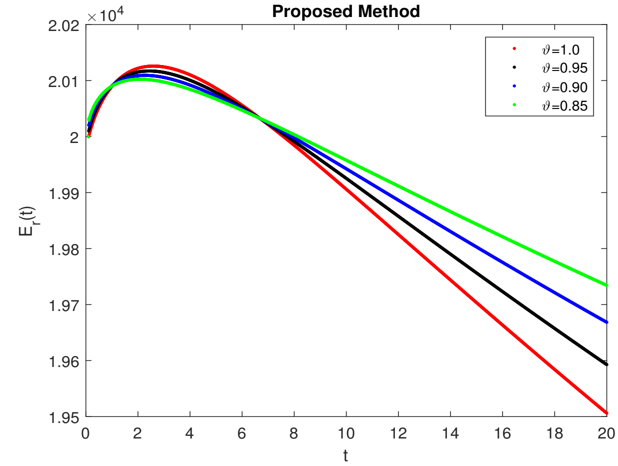

Fig. 7: Simulation $E_r(t)$ population at different fractional order values.

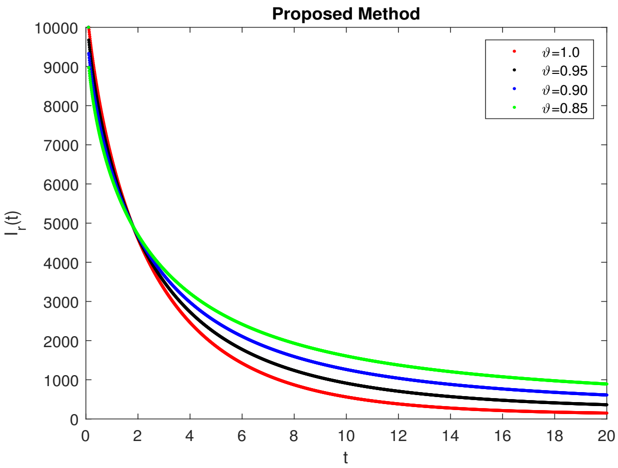

Fig. 8: Simulation $I_r(t)$ population at different fractional order values.

7 Conclusion

In this research, we observe the transmission Monkeypox virus in society by using the Caputo Fabrizio fractional operator. The Banach theory results are used for existence, singularity and stability for steady solutions. The progression of outcomes produced by this effective strategy demonstrates a accurate agreement to restrict the terrible consequences of Mokeypox for the various time period in specific time. It is predicted that reducing the fractional values rather than the classical derivative would lead to a more successful solution since the behavior in all figures is predicted to be close to a steady state by using laplace transform with Adomian decomposition method. These simulations demonstrate how variations in value have an effect on the model's behaviour having better convergence approach for such epidemic model. Future simulations with different parameter combinations may be utilized to produce a sample of possible dynamical framework behaviour. The techniques outlined in this article ought to be applicable to epidemic models other than the Monkeypox model as well. The simulations' findings support the accuracy and effectiveness of the Caputo Fabrizio non-integer derivative in estimating the dynamics of the Brucellosis illness and other issues of a similar nature.

Acknowledgements: Research Supporting Project number (RSP2023R167), King Saud University, Riyadh, Saudi Arabia.

Funding: This Project is funded by King Saud University, Riyadh, Saudi Arabia.
Data availability: All data generated or analysed during this study are included in this published article. No human data involved in this study

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